

PARTICULAR TRACE DECOMPOSITIONS AND
APPLICATIONS OF TRACE DECOMPOSITION
TO ALMOST PROJECTIVE INVARIANTS

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Abstract. First, by using the formulae of Krupka, the trace decomposition for some particular classes of tensors of types (1, 2) and (1, 3) is obtained. Second, it is proved that the traceless part of a tensor is an almost projective invariant of weight 1. We apply this result to Weyl curvature tensors.

Keywords: traceless tensor, trace decomposition, almost projective invariant

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INTRODUCTION

Let E be a real n -dimensional linear space, $n \geq 2$, $T_q^p E$ the linear space of tensors of type (p, q) on E . A tensor is said to be *traceless* if its traces are all zeros. By [2], [3, p. 303] *the trace decomposition problem* consists in finding a decomposition of a tensor in which the first term is traceless and the other terms are linear combinations of Kronecker's δ -tensors.

In [2], [3, p. 309] the following results are proved:

Theorem 0.1. *If $A = (A_k^i) \in T_1^1 E$, then there exists a unique traceless tensor $B = (B_k^i) \in T_1^1 E$ and a unique scalar C such that*

$$(0.1) \quad A_k^i = B_k^i + \delta_k^i C.$$

We have $C = \frac{1}{n} A_s^s$ and $B_k^i = A_k^i - \frac{1}{n} \delta_k^i A_s^s$.

Theorem 0.2. If $A = (A_{kl}^i) \in T_2^1 E$, then there exists a unique traceless tensor $B = (B_{kl}^i) \in T_2^1 E$ and unique tensors $C = (C_k), D = (D_k) \in T_1^0 E$ such that

$$(0.2) \quad A_{kl}^i = B_{kl}^i + \delta_k^i C_l + \delta_l^i D_k.$$

These tensors are defined by

$$\begin{aligned} C_l &= \varrho(n, 2)(nA_{tl}^t - A_{lt}^t) \\ D_k &= \varrho(n, 2)(-A_{tk}^t + nA_{kt}^t) \\ B_{kl}^i &= A_{kl}^i - \varrho(n, 2)[\delta_k^i(nA_{tl}^t - A_{lt}^t) + \delta_l^i(-A_{tk}^t + nA_{kt}^t)] \end{aligned}$$

where $\varrho(n, 2) = \frac{1}{n^2-1}$.

Theorem 0.3. If $n \geq 3$ and $A = (A_{klm}^i) \in T_3^1 E$, then there exists a unique traceless tensor $B = (B_{klm}^i) \in T_3^1 E$ and unique tensors $C = (C_{lm}), D = (D_{km}), E = (E_{kl}) \in T_2^0 E$ such that

$$(0.3) \quad A_{klm}^i = B_{klm}^i + \delta_k^i C_{lm} + \delta_l^i D_{km} + \delta_m^i E_{kl}.$$

These tensors are defined by:

$$\begin{aligned} C_{kl} &= \varrho(n, 3) \\ & [n(n^2-3)A_{tkl}^t + (-n^2+2)A_{ktl}^t + nA_{klt}^t - 2A_{tlk}^t + nA_{ltk}^t + (-n^2+2)A_{lkt}^t], \\ D_{kl} &= \varrho(n, 3) \\ & [(-n^2+2)A_{tkl}^t + n(n^2-3)A_{ktl}^t + (-n^2+2)A_{klt}^t + nA_{tlk}^t - 2A_{ltk}^t + nA_{lkt}^t], \\ E_{kl} &= \varrho(n, 3) \\ & [nA_{tkl}^t + (-n^2+2)A_{ktl}^t + n(n^2-3)A_{klt}^t + (-n^2+2)A_{tlk}^t + nA_{ltk}^t - 2A_{lkt}^t], \\ B_{klm}^i &= A_{klm}^i - \delta_k^i C_{lm} - \delta_l^i D_{km} - \delta_m^i E_{kl}, \end{aligned}$$

where $\varrho(n, 3) = \frac{1}{(n^2-1)(n^2-4)}$.

Remark 0.4. (i) For $n = 2$ and $A \in T_3^1 E$ the trace decomposition of A is not unique (theorem 2 (a) of [2], [3, p. 309]).

(ii) In [3, p. 305] it is proved that the traced decomposition problem has a solution for every $A \in T_q^p(M)$ and for every p, q with $p \leq q$. Moreover, the traceless part is unique.

(iii) For the generalization of the trace decomposition problem to spaces with complex structure see [4] and for spaces with quaternionic structure see [5].

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1. THE TRACE DECOMPOSITION OF SOME PARTICULAR TENSORS
OF TYPES (1, 2) AND (1, 3)

In this section we restrict our attention to tensors of types (1, 2) and (1, 3) because the most important tensors given by a linear connection, namely the torsion tensor and the curvature tensor, are of these types. After a straightforward computation we obtain

Proposition 1.1. *If $A = (A_{kl}^i) \in T_2^1 E$ has the form*

$$(1.1) \quad A_{kl}^i = E_{\{k}^i F_{l\}} = E_k^i F_l + E_l^i F_k$$

then the trace decomposition of A is

$$(1.2) \quad A_{kl}^i = B_{kl}^i + \delta_k^i C_l + \delta_l^i C_k$$

with $C_l = \frac{1}{n+1} A_{tl}^t$ and $B_{kl}^i = A_{kl}^i - \frac{1}{n+1} (\delta_k^i A_{tl}^t + \delta_l^i A_{tk}^t)$.

Corollary 1.2. *If $A \in T_2^1 E$ has the form (1.1) with $F = \delta$, i.e.*

$$(1.3) \quad A_{kl}^i = \delta_k^i G_l + \delta_l^i G_k,$$

then in the trace decomposition (1.2) of A we have $C_k = G_k, B_{kl}^i = 0$.

Proposition 1.3. *If $A = (A_{kl}^i) \in T_2^1 E$ has the form*

$$(1.4) \quad A_{kl}^i = E_{[k}^i F_{l]} = E_k^i F_l - E_l^i F_k$$

then the trace decomposition of A is

$$(1.5) \quad A_{kl}^i = B_{kl}^i + \delta_k^i C_l - \delta_l^i C_k$$

with $C_l = \frac{1}{n-1} A_{tl}^t$ and $B_{kl}^i = A_{kl}^i - \frac{1}{n-1} (\delta_k^i A_{tl}^t + \delta_l^i A_{kt}^t)$.

Corollary 1.4. *If $A \in T_2^1 E$ has the form (1.4) with $F = \delta$, i.e.*

$$(1.6) \quad A_{kl}^i = \delta_k^i G_l - \delta_l^i G_k,$$

then in the trace decomposition (1.5) of A we have $C_l = G_l, B_{kl}^i = 0$.

Example 1.5.

(i) Let $\nabla, \tilde{\nabla}$ be two linear connections on a smooth n -dimensional manifold M . Then $A = \tilde{\nabla} - \nabla \in T_2^1(M)$. H. Weyl proved that ∇ and $\tilde{\nabla}$ have the same autoparallel curves if and only if $\exists \psi \in T_1^0(M)$ such that

$$(1.7) \quad A = \delta \otimes \psi + \psi \otimes \delta,$$

i.e. (1.3), and he called the transformation $\nabla \rightarrow \tilde{\nabla}$ given by (1.7) a *projective transformation* ([6, p. 178]) or sometimes a *geodesic mapping*. In particular, if $\nabla, \tilde{\nabla}$ are the Levi-Civita connections for the Riemannian metrics g, \tilde{g} on M and A is given by (1.7) then the Riemannian spaces $(M, g), (M, \tilde{g})$ are said to be *in geodesic representation* (or *correspondence*) ([6, p. 322]).

A generalization of (1.7) is

$$(1.8) \quad A = \delta \otimes \psi_1 + \psi_2 \otimes \delta$$

with $\psi_1, \psi_2 \in T_1^0(M)$, which is called an *almost projective transformation*. In the next section we define almost projective transformations for tensors of type (p, q) , $p \leq q$.

(ii) If ∇ is a linear connection on a smooth n -dimensional manifold M then the torsion T of ∇ is in $T_2^1(M)$. ∇ is called *semisymmetric* if $\exists t \in T_1^0(M)$ such that

$$(1.8') \quad T = \delta \otimes t - t \otimes \delta$$

i.e. (1.6) holds ([7, p. 194]).

Proposition 1.6. *If $n \geq 3$ and $A = (A_{klm}^i) \in T_3^1 E$ has the form*

$$(1.9) \quad A_{klm}^i = F_{\{k}^i G_{lm\}} = F_k^i G_{lm} + F_l^i G_{mk} + F_m^i G_{kl}$$

then in the trace decomposition of A (see Th. 0.3) we have

$$C_{kl} = E_{kl} = \frac{1}{n^2 - 4}(nA_{tkl}^t - 2A_{ktl}^t),$$

$$D_{kl} = \frac{1}{n^2 - 4}(-2A_{tkl}^t + nA_{ktl}^t).$$

Corollary 1.7. *If $n \geq 3$ and $A \in T_3^1 E$ has the form (1.9) with $F = \delta$, i.e.*

$$(1.10) \quad A_{klm}^i = \delta_k^i G_{lm} + \delta_l^i G_{mk} + \delta_m^i G_{kl},$$

then in the trace decomposition of A we have $C_{kl} = E_{kl} = G_{kl}, D_{kl} = G_{lk}, B_{klm}^i = 0$.

Proposition 1.8. *If $n \geq 3$ and $A = (A_{klm}^i) \in T_3^1 E$ has the form:*

$$(1.11) \quad A_{klm}^i = F_{[k} G_{lm]} = F_k^i G_{lm} - F_l^i G_{mk} + F_m^i G_{kl}$$

then in the trace decomposition of A (see Th. 0.3) we have

$$\begin{aligned} \frac{C_{kl}}{\varrho(n, 3)} &= (n^3 - 3n)F_t^t G_{kl} + (5n - n^3)F_k^t G_{lt} \\ &\quad + (n^3 - 3n)F_l^t G_{tk} - 2F_k^t G_{tl} - 2F_t^t G_{lk} + (2 - 2n^2)F_l^t G_{kt}, \\ \frac{D_{kl}}{\varrho(n, 3)} &= (2 - 2n^2)F_t^t G_{kl} - 2F_k^t G_{lt} - 2F_l^t G_{tk} + (n^3 - 3n)F_k^t G_{tl} \\ &\quad + (5n - n^3)F_t^t G_{lk} + (n^3 - 3n)F_l^t G_{kt}, \\ \frac{E_{kl}}{\varrho(n, 3)} &= (n^3 - 3n)F_t^t G_{kl} + (n^3 - 3n)F_k^t G_{lt} \\ &\quad + (5n - n^3)F_l^t G_{tk} + (6 - 2n^2)F_k^t G_{tl} - 2F_t^t G_{lk} - 2F_l^t G_{kt}. \end{aligned}$$

Corollary 1.9. *If $n \geq 3$ and $A \in T_3^1 E$ has the form (1.11) with $F = \delta$, i.e.*

$$A_{klm}^i = \delta_k^i G_{lm} - \delta_l^i G_{mk} + \delta_m^i G_{kl}$$

then in the trace decomposition of A we have $C_{kl} = E_{kl} = G_{kl}, D_{kl} = -G_{lk}, B_{klm}^i = 0$.

2. APPLICATIONS TO ALMOST PROJECTIVE INVARIANTS

Let M be a smooth n -dimensional manifold, $C^\infty(M)$ the ring of real-valued functions on M , $T_q^p(M)$ the vector space of tensors of type (p, q) on M , $\mathcal{T}(M)$ the tensorial algebra of M .

For $p \leq q$ consider the subspace $\delta(T_q^p(M))$ of $T_q^p(M)$ generated by the tensors fields $X = (X_{j_1 \dots j_q}^{i_1 \dots i_p})$ of the form ([3, p. 306])

$$\begin{aligned} X_{j_1 \dots j_q}^{i_1 \dots i_p} &= \delta_{j_1}^{i_1} X_{(1)j_2 \dots j_q}^{(1)i_2 \dots i_p} + \delta_{j_2}^{i_1} X_{(2)j_1 j_3 \dots j_q}^{(1)i_2 \dots i_p} + \dots + \delta_{j_q}^{i_1} X_{(q)j_1 \dots j_{q-1}}^{(1)i_2 \dots i_p} \\ &\quad + \delta_{j_1}^{i_2} X_{(1)j_2 \dots j_q}^{(2)i_1 i_3 \dots i_p} + \delta_{j_2}^{i_2} X_{(2)j_1 j_3 \dots j_q}^{(2)i_1 i_3 \dots i_p} + \dots + \delta_{j_q}^{i_2} X_{(q)j_1 \dots j_{q-1}}^{(2)i_1 i_3 \dots i_p} \\ &\quad \dots \\ &\quad + \delta_{j_1}^{i_p} X_{(1)j_2 \dots j_q}^{(p)i_1 \dots i_{p-1}} + \delta_{j_2}^{i_p} X_{(2)j_1 j_3 \dots j_q}^{(p)i_1 \dots i_{p-1}} + \dots + \delta_{j_q}^{i_p} X_{(q)j_1 \dots j_{q-1}}^{(p)i_1 \dots i_{p-1}}. \end{aligned}$$

Definition 2.1. (i) If $A \in T_q^p(M)$, a transformation

$$(2.1) \quad \tilde{A} = \varrho A + X$$

with $\varrho \in C^\infty(M)$ and $X \in \delta(T_q^p(M))$ is called an *almost projective transformation*.

(ii) With respect to (2.1) a tensor field $N \in \mathcal{T}(M)$ derived from A by any means is called an *almost projective invariant of weight k* (a natural number) if

$$(2.2) \quad \tilde{N} = \varrho^k N$$

where \tilde{N} is derived from \tilde{A} in the same manner as N is derived from A .

Proposition 2.2. *If $A \in T_q^p(M)$ then the traceless part of A is an almost projective invariant of weight 1.*

Proof. Let $g = (g_{ij})$ be a Riemannian metric on M . Then g admits a lift, denoted by \langle, \rangle , to $T_q^p(M)$. It is a simple computation that $B \in T_q^p(M)$ is traceless if and only if B is \langle, \rangle -orthogonal to $\delta(T_q^p(M))$. Then A has the decomposition $A = B + X(A)$ where B is the traceless part of A (which is unique!) and $X(A) \in \delta(T_q^p(M))$.

Returning to (2.1) we get

$$\tilde{A} = \varrho A + X = \varrho(B + X(A)) + X = \varrho B + (\varrho X(A) + X).$$

Obviously $\varrho X(A) + X \in \delta(T_q^p(M))$ and the uniqueness of the traceless part yields that the traceless part of \tilde{A} is

$$\tilde{B} = \varrho B.$$

□

Remark 2.3. For the case $p = 1$, $q = 2$ this result appears in [1].

3. APPLICATIONS TO WEYL CURVATURE TENSORS

Let $g = (g_{ij})$ be a semi-Riemannian metric on M and $R = (R_{jkl}^i) \in T_3^1(M)$ the curvature tensor of g . In [3, p. 314] it is proved that the traceless part of R is exactly *the Weyl projective curvature tensor*, and if we consider the tensor $R_{kl}^{ij} = g^{js} R_{skl}^i \in T_2^2(M)$ then the traceless part of (R_{kl}^{ij}) is exactly *the Weyl conformal curvature tensor*. Applying the result of the previous section we get

Proposition 3.1. *For a semi-Riemannian metric the Weyl projective curvature tensor and the Weyl conformal curvature tensor are almost projective invariants of weight 1.*

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