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Trace Decomposition and Recurrency

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Abstract

Some applications of trace decomposition in recurrence problems are pointed. The main result of this paper establish that the traceless part of a k-recurrent tensor field is also recurrent with the same order and form of recurrence. We apply this fact to Weyl curvature tensors and Einstein tensor.

Key words: Traceless tensor, trace decomposition, recurrent tensor field.

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1 Trace decompositions of tensor fields

Let E be a real n-dimensional linear space, $n \geq 2$ and $T^p_q E$ the linear space of tensors of type (p, q) on E. By fixing a basis on E, and therefore, by extension, on $T^p_q E$, a given tensor $A \in T^p_q E$ is identified with its components $A = \left(A^{i_1 \dots i_p}_{j_1 \dots j_q}\right)$. A tensor is said to be *traceless* if its traces are all zeros. After [3], [4, p. 303] *the trace decomposition problem* consists in finding a decomposition of a given tensor in which the first term is traceless and the other terms are linear combinations of Kronecker's δ-tensors.

The following theorem of Krupka gives the solution ([4]):

Theorem 1 *Let p, q, n positive integers, $p \leq q$ and $A = \left(A^{i_1 \cdots i_p}_{j_1 \cdots j_q}\right) \in T^p_q E$. There exist a traceless tensor $B = \left(B^{i_1 \cdots i_p}_{j_1 \cdots j_q}\right) \in T^p_q E$ and tensors $B^{(r)}_{(s)} = \left(B^{(r) i_1 i_2 \cdots i_{p-1}}_{(s) j_1 j_2 \cdots j_{q-1}}\right) \in T^{p-1}_{q-1} E$, where $1 \leq r \leq p$, $1 \leq s \leq q$, such that:*

$$\begin{aligned} A^{i_1 \cdots i_p}_{j_1 \cdots j_q} = {} & B^{i_1 \cdots i_p}_{j_1 \cdots j_q} + \delta^{i_1}_{j_1} B^{(1) i_2 \cdots i_p}_{(1) j_2 \cdots j_q} + \delta^{i_1}_{j_2} B^{(1) i_2 \cdots i_p}_{(2) j_1 j_3 \cdots j_q} + \ldots + \delta^{i_1}_{j_q} B^{(1) i_2 \cdots i_p}_{(q) j_1 \cdots j_{q-1}} \\ & + \delta^{i_2}_{j_1} B^{(2) i_1 i_3 \cdots i_p}_{(1) j_2 \cdots j_q} + \delta^{i_2}_{j_2} B^{(2) i_1 i_3 \cdots i_p}_{(2) j_1 j_3 \cdots j_q} + \ldots + \delta^{i_2}_{j_q} B^{(2) i_1 i_3 \cdots i_p}_{(q) j_1 \cdots j_{q-1}} \\ & \cdots \\ & + \delta^{i_p}_{j_1} B^{(p) i_1 \cdots i_{p-1}}_{(1) j_2 \cdots j_q} + \delta^{i_p}_{j_2} B^{(p) i_1 \cdots i_{p-1}}_{(2) j_1 j_3 \cdots j_q} + \ldots + \delta^{i_p}_{j_q} B^{(p) i_1 \cdots i_{p-1}}_{(q) j_1 \cdots j_{q-1}}. \end{aligned}$$

The tensor B is unique.

Let us note that Krupka's results are generalized by J. Mikeš in [5], [6].

In the following let us restrict to the case $p = 1$; let us remark that this fact does not restricts the generalization because, usually, we work with a fixed scalar product on E (see the demonstration of the theorem 1 in [4, p. 306]) and then we low supplimentary indices with musical isomorphisms(see also the example were we work on a fixed Riemannian manifold). For this case the relation above becomes:

$$A^i_{j_1 \cdots j_q} = B^i_{j_1 \cdots j_q} + \delta^i_{j_1} B_{(1) j_2 \cdots j_q} + \ldots + \delta^i_{j_q} B_{(q) j_1 \cdots j_{q-1}}. \tag{1}$$

If we make the contraction $(1, s)$, $1 \leq s \leq q$ in (1), using the traceless of B it results:

$$\begin{aligned} A^a_{j_1 \cdots j_{s-1} a j_{s+1} \cdots j_q} = {} & B_{(1) j_2 \cdots j_{s-1} j_1 j_{s+1} \cdots j_q} + \ldots + B_{(s-1) j_1 \cdots j_{s-2} j_{s-1} j_{s+1} \cdots j_q} \\ & + n B_{(s) j_1 \cdots j_{s-1} j_{s+1} \cdots j_q} + B_{(s+1) j_1 \cdots j_{s-1} j_{s+1} j_{s+2} \cdots j_q} + \ldots + B_{(q) j_1 \cdots j_{s-1} j_q j_{s+1} \cdots j_{q-1}} \end{aligned} \tag{2}$$

i.e. we obtain a linear system in unknowns $B_{(s)}$. Then we have:

Proposition 1 *The tensors $B_{(s)}$, $1 \leq s \leq q$, are linear combinations of the contractions of A.*

2 Trace decomposition and k-recurrent spaces

Our next framework consists in a pair (M, ∇) where M is a smooth n-dimensional manifold and ∇ is a linear connection on M. Let us denotes $C^\infty(M)$ the ring of real-valued functions on M, $T^p_q(M)$ the linear space of tensor fields of type (p, q) on M, $\Omega^k(M)$ the $C^\infty(M)$-module of k-differential forms on M.

Recall that for a natural number k, $1 \leq k \leq n$, a tensor field $A \in T^p_q(M)$ is called *k-recurrent with respect to ∇* (if A is a Riemannian tensor then see [2]) if there exists $\omega \in \Omega^k(M)$ such that:

$$\nabla_{X_k} \cdots \nabla_{X_1} A = \omega(X_1, \ldots, X_k) \cdot A \tag{3}$$

for all $X_1, \ldots, X_k \in T_0^1(M) = \mathcal{X}(M)$=the $C^\infty(M)$-module of vector fields on M. In a local chart (3) reads:

$$A^{i_1 \ldots i_p}_{j_1 \ldots j_q\ , l_1 \ldots l_k} = \omega_{l_1 \ldots l_k} A^{i_1 \ldots i_p}_{j_1 \ldots j_q} \tag{4}$$

where "," denotes the covariant derivative with respect to ∇. We call ω the *k-form of recurrency* for A. If in (4) we make the contraction (r, s) then:

$$A^{i_1 \ldots i_{r-1} a i_{r+1} \ldots i_p}_{j_1 \ldots j_{s-1} a j_{s+1} \ldots j_q\ , \ l_1 \ldots l_k} = \omega_{l_1 \ldots l_k} A^{i_1 \ldots i_{r-1} a i_{r+1} \ldots i_p}_{j_1 \ldots j_{s-1} a j_{s+1} \ldots j_q} \tag{5}$$

i.e. it follows:

Proposition 2 *If A is k-recurrent then every contraction of A is k-recurrent with the same form of recurrence.*

Then propositions 1 and 2 yields:

Proposition 3 *Let M be a n-dimensional manifold and $A \in T_q^1(M)$ with $q \leq n$. If A is k-recurrent then the tensors $B_{(s)}$ from* (1) *are k-recurrent with the same form of recurrence.*

Because the recurrency is preserved by sum and obviously the Kronecker tensor is parallel (so k-recurrent with $\omega = 0$) we obtain the main result of the paper:

Proposition 4 *Let M be a n-dimensional manifold and $A \in T_q^1(M)$ with $q \leq n$. If A is k-recurrent then the traceless part of A is k-recurrent with the same form of recurrence.*

Applications Let $g = (g_{ij})$ be a Riemannian metric on M and $R = \left(R^i_{jkl}\right) \in T_3^1(M)$ the curvature tensor of g. The Riemannian space (M, g) is called *k-recurrent space* if R is k-recurrent and is called *k-symmetric space* if R is k-recurrent with $\omega \equiv 0$ (see [2]). In [4, p. 314] it is proved that the traceless part of R is exactly *the Weyl projective curvature tensor* and the traceless part of $R^{ij}_{kl} = g^{js} R^i_{skl}$ is exactly *the Weyl conformal curvature tensor*. Applying the proposition 4 we get:

Proposition 5 *In a k-recurrent (particularly k-symmetric) space the Weyl projective curvature tensor and the Weyl conformal curvature tensor are k-recurrent (particularly k-symmetric) with the same form of recurrence as the curvature tensor.*

In [5, p. 50] it is proved that the traceless part of the Ricci tensor is exactly *the Einstein tensor*. Also, is it used the notion of Ricci k-recurrent space as a Riemannian space with the Ricci tensor k-recurrent. Therefore:

Proposition 6 *In a Ricci-recurrent space the Einstein tensor is k-recurrent with the same form of recurrence as the Ricci tensor.*

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