Conserved quantities for dynamical systems with relative invariant form

Mircea Crâşmăreanu

Memoriile Sect. Stiint. Acad. Romana, 21(1998), 91-97. MR 2004d:70024

Paper presented by Radu Miron, full member of the Romanian Academy, at the Session of the Section of Mathematical Sciences, March 1999

On y présente deux méthodes pour obtenir les intégrales premières. La première méthode a comme point de départ la notion de "pseudosymétrie" et la deuxième méthode utilisée le calcul avec les formes volumes. Nous pouvons appliquer ces résultates pour les systèmes dynamiques plus généraux que ceux Lagrangiens. A titre de conclusion, on donne l'oscillateur harmonique de dimension 2.

Subject Classification: 58F35, 58F25, 58F05, 58G35, 70H05, 70H33, 34A26.
Key words and phrases: conservation law, symmetry, pseudosymmetry, absolute invariant form, relative invariant form.

Introduction

The very well-known way to obtain conservation laws for a system of differential equations given by a variational principle is Noether theorem ([1]) which associates to every symmetry a conservation law.

In this paper, we present two methods which not require that the differential system is of Euler-Lagrange type.

1 Pseudosymmetries and invariant forms

Let M be a smooth, n-dimensional manifold, $C^{\infty}(M)$ the ring of real-valued smooth functions, $\mathcal{X}(M)$ the Lie algebra of vector fields and $\Omega^{p}(M)$ the $C^{\infty}(M)$-module of p-differential forms, $1 \leq p \leq n$.

For $X \in \mathcal{X}(I\!R \times M)$ with local expression $X = \frac{\partial}{\partial t} + X^i(t,x)\frac{\partial}{\partial x^i}$ one consider the system of differential equations which give the flow of X:

$$(1.1)\quad \dot{x}^i(t) = \frac{dx^i}{dt}(t) = X^i\left(t, x^1(t), \ldots, x^n(t)\right), \qquad 1 \leq i \leq n.$$

A solution of (1.1) is called *integral curve* of X.

Definition 1.1 A function $f \in C^\infty(I\!R \times M)$ is called *conservation law*(or *first integral*, or *constant of motion*, or *invariant function*) for X or (1.1) if f is constant along the solutions of (1.1) that is $\frac{d(f \circ c)}{dt}(t) = 0$ for every integral curve $c(t)$ of X.

Because for $f \in C^\infty(I\!R \times M)$ its rate of change along (1.1) is:
$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}\dot{x}^i = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}X^i = \mathcal{L}_X f$ where the right-hand side means *the Lie derivative of fwith respect to* X we get:

Proposition 1.2 *$f \in C^\infty(I\!R \times M)$ is conservation law for (1.1) if and only if:*

$$(1.2)\quad \mathcal{L}_X f = 0.$$

For our approach is necessary the following:

Definition 1.3 (i) $Y \in \mathcal{X}(I\!R \times M)$ is called *symmetry for X* if:

$$(1.3)\quad \mathcal{L}_X Y = 0.$$

(ii) If $Y \in \mathcal{X}(I\!R \times M)$ is fixed then $Z \in \mathcal{X}(I\!R \times M)$ is called *Y-pseudosymmetry for X* if there exists $f \in C^\infty(I\!R \times M)$ such that:

$$(1.4)\quad \mathcal{L}_X Z = fY.$$

(iii) $\omega \in \Omega^p(I\!R \times M)$ is called *absolute invariant form for X* if:

$$(1.5)\quad \mathcal{L}_X \omega = 0.$$

(iv) $\theta \in \Omega^p(I\!R \times M)$ is called *relative invariant form* for X if:

$$(1.6)\quad i_X d\theta = 0.$$

Remark 1.4 (i) A 0-pseudosymmetry is obviously a symmetry.
(ii) A X-pseudosymmetry for X is called *pseudosymmetry for X* in [3, p. 1055] and *Lie point symmetry* in [7, p. 25].
(iii) If in (1.4) f is constant then $\mathcal{L}_X Z$ is symmetry for X.
(iv) If in (1.4) f is not constant then $\mathcal{L}_X Z$ is symmetry for X if and only if f is conservation law for X.
(v) If f is conservation law for X and Y is pseudsymmetry for X then $Y(f)$ is conservation law for X.
(v) Following the terminology of [4, p. 25] $\theta \in \Omega^1(I\!R \times M)$ is relative invariant form for X if and only if X belongs to *the characteristic distribution* of $d\theta$.
(vi) If θ is relative invariant form then $d\theta$ is absolute invariant form. This terminology appears from H. Poincare, cf. [8].

2 The first method

The result which give the association between pseudosymmetries and conservation laws is:

Theorem 2.1 *Let* $X \in \mathcal{X}(I\!R \times M)$ *be fixed and* $\theta \in \Omega^{p-1}(I\!R \times M)$ *be a relative invariant* $(p-1)$*-form for* X. *If* $Y \in \mathcal{X}(I\!R \times M)$ *is symmetry for* X *and* $S_1, \ldots, S_{p-1} \in \mathcal{X}(I\!R \times M)$ *are* $(p-1)$ Y*-pseudosymmetries for* X *then:*

$$(2.1) \quad \phi = d\theta(S_1, \ldots, S_{p-1}, Y)$$

or locally:

$$(2.2) \quad \phi = S_1^{i_1} \ldots S_{p-1}^{i_{p-1}} Y^{i_p} \omega_{i_1 i_2 \ldots i_p}$$

is a conservation law for X *where* $\omega = d\theta$. *Particularly, if* $Y, S_1, \ldots, S_{p-1}$ *are symmetries for* X *then* ϕ *given by* (2.1) *is conservation law.*

Proof Applying the properties of Lie derivatives one have:

$$\mathcal{L}_X\phi = (\mathcal{L}_X S_1)^{i_1} \ldots + S_1^{i_1} (\mathcal{L}_X S_2)^{i_2} \ldots + \ldots + \ldots (\mathcal{L}_X Y)^{i_p} \omega_{\ldots} + \ldots Y^{i_p} (\mathcal{L}_X \omega)_{i_1 \ldots i_p}.$$

In this relation each of the first $p-1$ terms has a factor of the form:

$$\mathcal{L}_X S_j = \lambda_j Y$$

so that ω is contracted with two factors of Y and then each term vanishes by the antisymmetry of ω. The p-th term and $p+1$-th term vanishes since (1.3) and (1.5). □

Remark 2.2 (i) If the pseudosymmetries are linearly dependent then $\phi = 0$ by the antisymmetry of ω.
(ii) For $Y = X$ one obtain the main result of G. L. Jones([3, p. 1056]).
(iii) If $p = 1$ one obtain theorem 2.5.10 of ten Eikelder([2, p. 48]).
(iv) The fact that the pseudosymmetries (1.4) with f =constant can be used to integrate planar($n = 2$) vector fields can be found in [9, p. 37-38, relation 8.13].

3 The second method

In this section we present a method based on volume forms.

Let $X \in \mathcal{X}(I\!R \times M)$ be a fixed vector field which have a relative invariant 1-form $\theta \in \Omega^1(I\!R \times M)$ and a symmetry $Y \in \mathcal{X}(I\!R \times M)$. Let us define the 1-form:

$$(3.1) \quad \theta_Y = \mathcal{L}_Y \theta.$$

Proposition 3.1 $d\theta_Y$ *is absolute invariant form for* X.

Proof We have:

$$\mathcal{L}_X d\theta_Y = \mathcal{L}_X d\left(\mathcal{L}_Y \theta\right) = \mathcal{L}_X \mathcal{L}_Y d\theta = \left(\mathcal{L}_{[X,Y]} + \mathcal{L}_Y \mathcal{L}_X\right) d\theta = \mathcal{L}_Y \left(\mathcal{L}_X d\theta\right) = 0.$$

Therefore, the following $n+1$ volume forms for $I\!R \times TM$:

$$(3.2) \quad \Omega_Y^k = dt \wedge d\theta_Y \wedge \ldots \wedge d\theta_Y \wedge d\theta \wedge \ldots \wedge d\theta, \ 0 \leq k \leq n$$

are absolute invariant forms for X, where Ω_Y^k contains k factors of $d\theta_Y$ and $n-k$ factors of $d\theta$. Because $I\!R \times TM$ is orientable, each Ω_Y^k can be expressed with respect to the canonical volume forms of $I\!R \times TM$:

$$(3.3) \quad \Omega_Y^k = \rho_Y^k dt \wedge dx^1 \wedge \ldots \wedge dx^n \wedge d\dot{x}^1 \wedge \ldots \wedge d\dot{x}^n$$

with $\rho_Y^k \in C^\infty\left(I\!R \times M\right)$, $0 \leq k \leq n$. This yields the proportionality of every two Ω_Y^k, Ω_Y^l:

$$(3.4) \quad \Omega_Y^k = \sigma_Y^{kl} \Omega_Y^l$$

with:

$$(3.5) \quad \sigma_Y^{kl} = \rho_Y^k / \rho_Y^l.$$

The main result of this section is:

Theorem 3.2 σ_Y^{kl} *is conservation law for* X.

Proof From (3.4) it results:

$$\mathcal{L}_X \Omega_Y^k = 0 = X\left(\sigma_Y^{kl}\right) \cdot \Omega_Y^l.$$

4 A remarkable field of applications: the Lagrangian mechanics

Recall the framework of Lagrangian dynamics:

Definition 4.1 If M is a $2n$-dimensional manifold then on the manifold $E = I\!R \times M$ the pair (dt, ω) with $\omega \in \Omega^2(E)$ is called *cosymplectic structure* if the $(2n+1)$-form $dt \wedge \omega$ is volume form on E.

The following result is classical:

Theorem 4.2(Reeb) *For a cosymplectic structure* (dt, ω) *there exists a unique* $R \in \mathcal{X}(E)$*, called the Reeb vector field, such that:*

$$(4.1a) \quad i_R dt = 1$$

$$(4.1b) \quad i_R \omega = 0.$$

Suppose that we have a Lagrangian system with n degree of freedom and with *the configuration space* $I\!R \times Q$ i.e. the evolution of the system is characterized by the Lagrangian $L : I\!R \times TQ \to I\!R$. If we denote the local coordinates on TQ by $(q^i, \dot{q}^i)_{1 \le i \le n}$ then the associated *Cartan* 1-*form* of L is $\theta_L \in \Omega^1 (I\!R \times TQ)$:

$$(4.2) \quad \theta_L = \frac{\partial L}{\partial \dot{q}^i} dq^i - H dt$$

where H is *the energy* of L:

$$(4.3) \quad H = \frac{\partial L}{\partial \dot{q}^i} \dot{q}^i - L.$$

A straightforward computation give the equivalence of the following two assertions:
(i) $(dt, d\theta_L)$ is cosymplectic structure on $E = I\!R \times TQ$
(ii) the Hessian of L, that is the matrix with entries $g_{ij} = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}$ is nondegenerate.

Then we must define:

Definition 4.3 The Lagrangian L is called *regular* if the Hessian of L is invertible.

Therefore, if L is regular there exists a unique $R_L \in \mathcal{X}(I\!R \times TQ)$ such that θ_L is relative invariant form for R_L. The flow of R_L is exactly *the Euler-Lagrange system* which give the extremals of L. Consequently, the conservation laws of L are the first integrals of R_L and the methods given here can be applied to analitycal mechanics. So, theorem 2.1 generalize the method of [3] and theorem 3.2 generalize the method of [5].

5 An example

Let the 2-dimensional isotropic harmonic oscillator:

$$\begin{array}{lrcl} (5.1a) & \ddot{q}^1 + \omega^2 q^1 & = & 0 \\ (5.1b) & \ddot{q}^2 + \omega^2 q^2 & = & 0 \end{array}$$

a toy model for many methods to finding conservation laws.

The Lagrangian is:

$$(5.2) \quad L = \frac{1}{2} \left[\left(\dot{q}^1\right)^2 + \left(\dot{q}^2\right)^2 \right] - \frac{\omega^2}{2} \left[\left(q^1\right)^2 + \left(q^2\right)^2 \right]$$

and then applying the conservation of energy H(L is time-independent) we have two conservation laws:

$$\begin{aligned} (5.3a) \quad \phi_1 &= \left(\dot{q}^1\right)^2 + \omega^2 \left(q^1\right)^2 \\ (5.3b) \quad \phi_2 &= \left(\dot{q}^2\right)^2 + \omega^2 \left(q^2\right)^2 . \end{aligned}$$

A straightforward computation give the Noetherian conservation law([1, p. 192]):

$$(5.4) \quad \phi_3 = q^2\dot{q}^1 - q^1\dot{q}^2.$$

But we can obtain a nonnoetherian conservation law with symmetries. The Reeb vector field of (5.1) is:

$$(5.5) \quad R_L = \dot{q}^1\frac{\partial}{\partial q^1} + \dot{q}^2\frac{\partial}{\partial q^2} - \omega^2 q^1\frac{\partial}{\partial \dot{q}^1} - \omega^2 q^2\frac{\partial}{\partial \dot{q}^2}$$

and another calculus give that:

$$(5.6) \quad Y = \dot{q}^2\frac{\partial}{\partial q^1} + \dot{q}^1\frac{\partial}{\partial q^2} - \omega^2 q^2\frac{\partial}{\partial \dot{q}^1} - \omega^2 q^1\frac{\partial}{\partial \dot{q}^2}$$

is a symmetry for R_L. Also, because R_L is total 1-homogeneous, i.e. with respect to all variables $(q, \dot{q})$ it result that:

$$(5.7) \quad Z = q^1\frac{\partial}{\partial q^1} + q^2\frac{\partial}{\partial q^2} + \dot{q}^1\frac{\partial}{\partial \dot{q}^1} + \dot{q}^2\frac{\partial}{\partial \dot{q}^2}$$

is symmetry for R_L. We have:

$$(5.8) \quad \theta_L = \dot{q}^1 dq^1 + \dot{q}^2 dq^2$$

$$(5.9) \quad \omega_L = d\theta_L = d\dot{q}^1 \wedge dq^1 + d\dot{q}^2 \wedge dq^2$$

and $\phi = \omega_L\left(R_L, Y\right) = 0, \quad \phi = \omega_L\left(R_L, Z\right) = 2H$ i.e. we not obtain new conservation law. But:

$$(5.10) \quad \phi_4 = \omega_L\left(Y, Z\right) = \dot{q}^1\dot{q}^2 + \omega^2 q^1 q^2$$

is a new conservation law given by proposition 2.2. We remark that ϕ_4 represent the energy of a new Lagrangian of (5.1), that is:

$$(5.11) \quad L^* = \dot{q}^1\dot{q}^2 - \omega^2 q^1 q^2$$

a result very important from the point of view of Inverse Problem of Analytical Mechanics([6]). Our Lagrangian L^* appear in [6, p. 122].

Applying the second method, we get:

$$(5.12) \quad \mathcal{L}_Y \theta_L = -d\phi_4$$

$$(5.13) \quad \mathcal{L}_Z \theta_L = 0$$

and then:

$$(5.15) \quad d\theta_Y = \mathcal{L}_Y d\theta_L = 0$$

$$(5.16) \quad d\theta_Z = \mathcal{L}_Z d\theta_L = 0$$

i.e. we not have new conservation laws.

REFERENCES

1. Bocaletti, D., Pucacco, G., *Killing equations in classical mechanics*, Il Nuovo Cimento, Vol. 112(1997), N. 2-3, 181-212.
2. ten Eikelder, H.M.M., *Symmetries for dynamical and Hamiltonian systems*, CWI Tract, no. 17, Amsterdam, 1985.
3. Jones, G.L., *Symmetry and conservation laws of differential equations*, Il Nuovo Cimento, 112(1997), N. 7, 1053-1059.
4. Krupkova, O., *The Geometry of Ordinary Variational Equations*, Springer-Verlag, LNM, no. 1678, 1997.
5. Lutzky, M., *Conserved quantities from non-Noether symmetries without alternative Lagrangians*, International Journal of Non-Linear Mechanics, 34(1999), 387-390.
6. Santilli, R. M., *Foundations of Theoretical Mechanics I: The Inverse Problem in Newtonian Mechanics*, Texts and Monographs in Physics, Springer-Verlag, N.Y., 1978.
7. Stephani, H., *Differential equations: Their solution using symmetries*(edited by MacCallum, M.), Cambridge Univ. Press, 1989.
8. Talpaert, Y., *Lecons et Applications de Géométrie Différentielle et de Mécanique Analytique*, Cépaduès Éditions, 1993.
9. Taylor, M.E., *Partial differential equations*(vol. I- "Basic Theory"), Texts in applied Mathematics, no. 23, Springer, 1996.

Faculty of Mathematics
University "Al. I. Cuza"
Iaşi, 6600, Romania
E-mail: mcrasm@uaic.ro