Finslerian fibre bundles: a first step to a Finslerian Kaluza-Klein theory

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Abstract

A sufficient condition for a map between two Finsler manifolds were a fibre bundle is given by using a theorem of Blumenthal. This result is related to the construction of a Finslerian generalization of usually Kaluza-Klein theories which uses Riemannian metrics.

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Introduction

Only suggested by B. Riemann in his "Habilitationvortrag"(1854) and rediscovered by P. Finsler in 1918, the Finsler manifolds become recently a very much alive research domain in geometry([1], [2], [3], [7], [11]). For example, among the notions introduced and studied by Professor Radu Miron, very interesting and useful for applications in physics and biology, there are some generalizations of Finsler spaces, namely Lagrange and generalized Lagrange spaces, and more recently higher-order Finsler and higher-order Lagrange spaces([12], [13]). It is important to note that the Romanian school of geometry has remarkable contributions to the study of this theories.

In this paper we are concerning with the following:

Question *Decide when a mapping between two Finslerian spaces* $\pi : \left(\tilde{M}, \tilde{L}\right) \rightarrow (M, L)$ *defines a fibre bundle, that is, for every* $p \in M$ *there exists a neighborhood* $p \in U$ *such that* $\pi^{-1}(U)$ *is diffemorphic to* $U \times \pi^{-1}(p)$*. In this case* $\left(\tilde{M}, \pi, M\right)$ *is called Finslerian fibre bundle.*

The motivation of such a result comes from physical oriented theories. First, the notion of Finsler fundamental function has a variational, more precisely Lagrangian origin(see the next section). Second, the treatment by means of fibre bundles appears to be fruitful in order to obtain some remarkable geometrical models for gauge theories. Only an example: in Kaluza-Klein's attempts([8], [9]) for a unified theory the space-time is the base space of a fibre bundle(more exactly a principal fibre bundle). A main ingredient of Kaluza-Klein theory is a Riemannian metric and a generalization of Riemannian metrics are Finsler metrics([11]). Therefore it seems that a Finslerian Kaluza-Klein theory can be constructed. This paper is intented in this direction.

The first who pointed the posibility of a Finslerian Kaluza-Klein theory was R. G. Beil in [4] and [5]. For other several applications of Finsler metrics see the cited books.

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1 Finsler geometry revisited

Let $M = (x^i)$ be a smooth, finite-dimensional manifold with $TM = (x^i, y^i)$ and T^*M the tangent and cotangent bundle. If $L : TM \rightarrow I\!R$ is a smooth function, usualy called *Lagrangian*, let $FL : TM \rightarrow T^*M$ be *the fiber derivative* of L([10, p. 26]):

$$FL(v) \cdot w = \frac{d}{d\varepsilon} |_{\varepsilon=0} L(v + \varepsilon w) \tag{1.1}$$

for $v, w \in T_pM, p \in M$. If Ω denotes the canonical symplectic structure on T^*M let $\Omega_L = (FL)^* \Omega$ be the pullback on TM.

Definition 1.1([10], [11])

(i) The Lagrangian L is called *regular* if Ω_L is a symplectic structure on TM.

(ii) *The energy* of L is $\mathcal{E} : TM \to I\!R$ given by:

$$\mathcal{E}(v) = FL(v) \cdot v. \tag{1.2}$$

If L is a regular Lagrangian by using the nondegeneracy of the symplectic form Ω_L it result that on TM there exists a unique vector field $S_L \in \mathcal{X}(TM)$ such that:

$$i_{S_L}\Omega_L = d\mathcal{E} \tag{1.3}$$

where i_Z denotes the interior product with respect to the vector field Z. S_L is called the *Euler-Lagrange vector field* of L because (1.3) are the well-known *Euler-Lagrange equations* of L.

Definition 1.2([11])

(i) A vector field $S \in \mathcal{X}(TM)$ is called *semispray* or *second order differential equation* if:

$$T\tau \circ S = 1_{TM} \tag{1.4}$$

where $T\tau$ is the differential of tangent bundle projection $\tau : TM \to M$ and 1_{TM} is the identity of TM.

(ii) A semispray $S \in \mathcal{X}(TM)$ is called *spray* if S is positive-homogeneous of order 1 with respect to velocity, that is:

$$S(av) = a(\mu_a)_*(S(v)) \tag{1.5}$$

where $a \in I\!R$ and $\mu_a : TM \to TM$ is the fibre multiplication(i.e. homotety) by a.

A first remarkable result is:

Proposition 1.3([10], [11]) *If L is a regular Lagrangian then the associated Euler-Lagrange vector field is a semispray.*

In order to obtain exactly a spray we need:

Definition 1.4([11]) A regular Lagrangian L is called *Finsler fundamental function* or *Finslerian* if satisfy:

(i) the following 2-homogeneous condition:

$$L \circ \mu_a = a^2 L \tag{1.6}$$

(ii) L is smooth on T_0M and continuous on $TM \setminus T_0M$, where T_0M is the subset of nonnull tangent vectors.

It result from (1.2) that the energy $\mathcal{E}$ is 2-homogeneous and applying (1.3) one obtain the main result of this section:

Proposition 1.5([11]) *If L is Finslerian then S_L is spray.*

An example: if $g = (g_{ij}(x))$ is a Riemannian metric on M then the *kinetic energy* of g:

$$E(g) = \frac{1}{2} g_{ij} y^i y^j \tag{1.7}$$

is a Finslerian function and S_L is the usual *geodesic spray* who has as projections of integral curves exactly the geodesics of g.

2 Finslerian bundles

In order to give an answer to the question of introduction we will use the following fibration theorem due to Robert A. Blumenthal:

Theorem 2.1([6]) *Let $\left(\tilde{M}, \tilde{S}\right)$ and (M, S) be connected manifolds with sprays and let $\pi : \tilde{M} \to M$ be a submersion. Let $E \subset T\tilde{M}$ be the kernel of $T\pi$ and suppose that exists $Q \subset T\tilde{M}$ be a complementary subbundle of $T\tilde{M}$(i.e. $T\tilde{M} = E \oplus Q$) such that:*

(i) *Q is a union of integral curves of $\tilde{S}$(in the words of cited paper Q is totally geodesic)*

(ii) *$\tilde{S} \mid_Q$ is $T\pi$-related to S.*

If $\tilde{S} \mid_Q$ is complete then π is onto, π is a locally trivial fibre bundle and S is complete.

Therefore we are able to give the main result of this paper:

Theorem 2.2 *Let $\left(\tilde{M}, \tilde{L}\right)$ and (M, L) be connected Finsler manifolds and let $\pi : \tilde{M} \to M$ be a submersion. Let $E \subset T\tilde{M}$ be the kernel of $T\pi$ and suppose that exists $Q \subset T\tilde{M}$ a complementary subbundle with (i) and (ii) from the previous theorem. If $S_{\tilde{L}} \mid_Q$ is complete then π is onto, π is locally trivial fibre bundle and S_L is complete.*

In the case of Riemannian spaces the Blumenthal theorem reduces to the well-known:

Theorem 2.3(R. Hermann, [6]) *Let $\tilde{M}$ and M be connected Riemannian manifolds and let $\pi : \tilde{M} \to M$ be a Riemannian submersion. If $\tilde{M}$ is complete then π is fibre bundle and M is complete.*

We end with the fact that Hermann's result led to following:

Question *Which is the natural generalization to Finsler geometry of the notion of Riemannian submersion?*

This question appears in the list of open problems of [1].

3 Local expressions

Let M be a m-dimensional manifold with $x = (x^i)_{1 \le i \le m}$ a local chart and let $(x, y) = (x^i, y^i)$ the adapted chart on TM. A semispray S have the expression([11]):

$$S = y^i \frac{\partial}{\partial x^i} - S^i(x, y) \frac{\partial}{\partial y^i} \tag{3.1}$$

and S is spray if and only if $S^i(x, \lambda y) = \lambda^2 S^i(x, y)$ for every i.

Let $\pi : \tilde{M} \to M$ be a submersion between a $(m+n)$-dimensional manifold and a m-dimensional manifold. Then $\pi : (x^i, \tilde{x}^a)\, 1 \le i \le m$
$1 \le a \le n \to (x^i)$ and $T\pi : (x^i, \tilde{x}^a, y^i, \tilde{y}^a) \to (x^i, y^i)$. If S is a spray on M given by (3.1) and $\tilde{S}$ is a spray on $\tilde{M}$ then condition (ii) from theorem 2.1 means:

$$\tilde{S} = y^i \frac{\partial}{\partial x^i} + \tilde{y}^a \frac{\partial}{\partial \tilde{x}^a} - S^i \frac{\partial}{\partial y^i} - \tilde{S}^a \frac{\partial}{\partial \tilde{y}^a} \tag{3.2}$$

The kernel of $T\pi$ is $E = span\left(\frac{\partial}{\partial \tilde{x}^a}\right)$and let $\left(\frac{\delta}{\delta x^i}\right)$ be a basis on the complementary subbundle Q which satisfy theorem 2.1. From $T\pi\left(\frac{\delta}{\delta x^i}\right) = \frac{\partial}{\partial x^i}$ it results:

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - B_i^a \frac{\partial}{\partial \tilde{x}^a} \tag{3.3}$$

that is $Q = (x^i, \tilde{x}^a, y^i, -B_i^a y^i)$. The condition (i) of theorem 2.1 imply the following form of theorem 2.2:

Theorem 3.1 *Let* $\left(\tilde{M}, \tilde{L}\right)$ *and* (M, L) *be connected Finsler manifolds and* $\pi : \tilde{M} \to M$ *be a submersion. Suppose that exists a complementary subbundle* Q *of* TM *such that in each pair of adapted charts* $(x^i), (x^i, \tilde{x}^a)$ *we have:*

(i) Q *is spanned by* $\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - B_i^a \frac{\partial}{\partial \tilde{x}^a}$

(ii) *the canonical sprays are given by* (3.1) *and* (3.2) *with:*

$$\tilde{S}^a(x(t), \tilde{x}(t)) = \frac{d}{dt}\left[B_i^a\left(x(t), \frac{dx}{dt}(t), \tilde{x}(t), \frac{d\tilde{x}}{dt}(t)\right)\frac{dx^i}{dt}\right]. \tag{3.4}$$

If $S_{\tilde{L}} |_Q$ is complete then π is onto, π is locally trivial fibre bundle and S_L is complete.

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