Nonholonomic Lagrangians of third order: Equations of motion for the constrained Lagrangian

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Abstract

The equations of motion for the associated Lagrangian to a nonholonomic Lagrangian of third order are computed and an example is given.

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Introduction

In the last years there is an increasing interest in nonholonomic mechanics especially from a geometrical point of view. Following the methodology of [2], where are treated nonholonomic Lagrangians of first order, we obtain the equations of motion in terms of the associated constrained Lagrangian of a nonholonomic Lagrangian involving velocities of third order. In [3] this problem is solved for Lagrangians of second order and the spinning particle is given as example.
This paper is dedicated to the memory of Romanian Academician Gheorghe Vrânceanu (1900-1979) who introduces in 1926 the notion of nonholonomic spaces, in order to give a geometrical approach to nonholonomic mechanics ([7], [8]). Note that the Romanian school of mathematics has an important contribution to this subject ([4], [7], [8], [9]).

1 Equations of motion

The starting point is a configuration-space given by a \( n \)-dimensional manifold \( Q \), for which we consider the tangent bundle of order three \( T^3 Q \) ([5], [6]). For coordinates \( (q^i)_{1 \leq i \leq n} \) on \( Q \) we have the induced coordinates
\[
(q^i, q^{(1)i} = \frac{dq^i}{dt}, q^{(2)i} = \frac{d^2q^i}{dt^2}, q^{(3)i} = \frac{d^3q^i}{dt^3})
\]
on \( T^3 Q \).

Let us suppose that the evolution of the considered system is described by the following objects:

1. a third-order Lagrangian, that is a smooth map \( L : T^3 Q \rightarrow \mathbb{R} \) ([5], [6])
2. a set of \( p \) independent one-forms \( (\omega^a(q))_{1 \leq a \leq p} \) whose vanishing gives the constraints of the system.

This 1-forms defines an \( (n - p) \)-dimensional distribution \( D \) on \( Q \) i.e. \( (\omega^a(q)) \) is a local basis of the annihilator \( D^0 \) of \( D \). Also, this constraints means that the only allowable velocities are the tangent vectors belonging to \( D \) or in other words the motion is constrained to the submanifold \( D \).

The Lagrangian \( L \) gives the Euler-Lagrange equations of order three ([5], [6]):
\[
\delta L = (EL)^{free}_i \delta q^i = 0 \quad (1.1a)
\]
with:
\[
(EL)^{free}_i = \frac{\partial L}{\partial q^i} - \frac{d}{dt} \left( \frac{\partial L}{\partial q^{(1)i}} \right) + \frac{d^2}{dt^2} \left( \frac{\partial L}{\partial q^{(2)i}} \right) - \frac{d^3}{dt^3} \left( \frac{\partial L}{\partial q^{(3)i}} \right) \quad (1.1b)
\]
and supposing that the constraints are nonholonomic, we can choose a local coordinate chart and a local basis for the constraints such that ([2, p. 31]):
\[
\omega^a(q) = ds^a + A^a_n (r, s) dr^a, \quad 1 \leq a \leq p \quad (1.2)
\]
where \( q = (r, s) \in \mathbb{R}^{n-p} \times \mathbb{R}^p \).
From (1.2) it results that:

$$\delta s^a + \frac{1}{A_a} \delta r^a = 0$$  \hspace{1cm} (1.3)

which, by substitution into (1.1) yields:

$$\frac{\partial L}{\partial r^a} - \frac{d}{dt} \left( \frac{\partial L}{\partial r^{(1)a}} \right) - \frac{d^2}{dt^2} \left( \frac{\partial L}{\partial r^{(2)a}} \right) - \frac{d^3}{dt^3} \left( \frac{\partial L}{\partial r^{(3)a}} \right) =$$

$$= A_a \left[ \frac{\partial L}{\partial s^a} - \frac{d}{dt} \left( \frac{\partial L}{\partial s^{(1)a}} \right) + \frac{d^2}{dt^2} \left( \frac{\partial L}{\partial s^{(2)a}} \right) - \frac{d^3}{dt^3} \left( \frac{\partial L}{\partial s^{(3)a}} \right) \right].$$  \hspace{1cm} (1.4)

Equations (1.4) combined with the constraint equations:

$$s^{(1)a} = - \frac{1}{A_a} r^{(1)a}$$  \hspace{1cm} (1.5a)

$$s^{(2)a} = - \frac{d}{dt} \left( \frac{1}{A_a} r^{(1)a} \right) - \frac{1}{A_a} r^{(2)a}$$  \hspace{1cm} (1.5b)

$$s^{(3)a} = - \frac{d^2}{dt^2} \left( \frac{1}{A_a} r^{(1)a} \right) - 2 \frac{d}{dt} \left( \frac{1}{A_a} r^{(2)a} \right) - \frac{1}{A_a} r^{(3)a}$$  \hspace{1cm} (1.5c)

gives a complete description of the equations of motion. Remark that another form for (1.5b) is:

$$s^{(2)a} = \frac{2}{A_{a\beta}} r^{(1)a} r^{(1)\beta} - \frac{1}{A_a} r^{(2)a}$$  \hspace{1cm} (1.5b')

where:

$$\frac{2}{A_{a\beta}} (r, s) = \frac{\partial}{\partial s^b} \frac{A_{b\beta}}{A_a} - \frac{\partial}{\partial r^b} \frac{A_{b\beta}}{A_a}$$  \hspace{1cm} (1.6)

and another form of (1.5c) is:

$$s^{(3)a} = \frac{3}{A_{a\beta\gamma}} r^{(1)a} r^{(1)\beta} r^{(1)\gamma} + \frac{2}{A_{a\beta}} r^{(2)a} r^{(1)\beta} - \frac{1}{A_a} r^{(3)a}$$  \hspace{1cm} (1.5c')

where:

$$\frac{3}{A_{a\beta\gamma}} (r, s) = \frac{\partial}{\partial r^\gamma} \frac{2}{A_{a\beta}} - \frac{\partial}{\partial s^b} \frac{2}{A_{a\beta}} \frac{1}{A_\gamma}.$$

Following [2, p. 31] we define an associated \textit{constrained} Lagrangian $L_c$ by substituting the constraints (1.5) into the Lagrangian $L$:

$$L_c \left( r^\alpha, s^a, r^{(1)\alpha}, r^{(2)\alpha}, r^{(3)\alpha} \right) \overset{def.}{=}$$

$$3$$
\[
L(r^\alpha, s^\alpha, r^{(1)\alpha}, -A^a_\alpha r^{(1)\alpha}, r^{(2)\alpha}, A^a_\alpha r^{(1)\alpha} r^{(1)\beta} - A^a_\alpha r^{(2)\alpha}, r^{(3)\alpha}, A^a_\alpha \beta r^{(1)\alpha} r^{(1)\beta} r^{(1)\gamma} + (2 A^a_\alpha + 2 A^a_\beta) r^{(2)\alpha} r^{(1)\beta} - A^a_\alpha r^{(3)\alpha}).
\]

A direct coordinates calculation shows:

\[
\frac{\partial L}{\partial r^\alpha} = \frac{\partial L}{\partial s^\alpha} \partial A^b_\beta \partial r^{(1)\beta} + \frac{\partial L}{\partial s^a} \left( \frac{\partial A^b_\beta}{\partial r^\alpha} r^{(1)\beta} r^{(1)\gamma} - \frac{\partial A^b_\beta}{\partial r^\alpha} r^{(2)\beta} r^{(1)\gamma} - \frac{\partial A^b_\beta}{\partial r^\alpha} r^{(3)\beta} \right) + \frac{\partial L}{\partial s^a} \left( \frac{\partial A^b_\beta}{\partial r^\alpha} r^{(1)\beta} r^{(1)\gamma} + (2 \frac{\partial A^b_\beta}{\partial r^\alpha} + \frac{\partial A^b_\beta}{\partial r^\alpha} r^{(2)\beta} r^{(1)\gamma} - \frac{\partial A^b_\beta}{\partial r^\alpha} r^{(3)\beta} \right) + (1.9a)
\]

\[
\frac{\partial L}{\partial s^a} = \frac{\partial L}{\partial s^a} - \frac{\partial L}{\partial s^a} \partial A^b_\beta \partial r^{(1)\beta} + \frac{\partial L}{\partial s^a} \left( \frac{\partial A^b_\beta}{\partial s^a} r^{(1)\beta} r^{(1)\gamma} - \frac{\partial A^b_\beta}{\partial s^a} r^{(2)\beta} r^{(1)\gamma} - \frac{\partial A^b_\beta}{\partial s^a} r^{(3)\beta} \right) + (1.9b)
\]

A long, but straightforward computation which uses the formulae:

\[
\frac{d}{dt} \frac{1}{A^a_\beta} = -2 A^b_\alpha r^{(1)\beta} \quad (1.10a)
\]

\[
\frac{d}{dt} A^b_\alpha = A^b_\alpha \gamma r^{(1)\gamma} \quad (1.10b)
\]
gives the equations of motion for \( L_c \):

\[
(EL)_{\alpha}^{\text{constraints}} = \left[ \frac{\partial L}{\partial s^{(1)b}} - \frac{d}{dt} \left( \frac{\partial L}{\partial s^{(2)b}} \right) + \frac{d^2}{dt^2} \left( \frac{\partial L}{\partial s^{(3)b}} \right) \right] b^b_{\alpha\beta} r^{(1)\beta} + \\
+ \left( \frac{\partial L}{\partial s^{(2)b}} - \frac{d}{dt} \left( \frac{\partial L}{\partial s^{(3)b}} \right) \right) b^b_{\alpha\beta\gamma} r^{(1)\beta} r^{(1)\gamma} + \\
+ \frac{\partial L}{\partial s^{(3)b}} \left( b^b_{\alpha\beta\gamma\delta} r^{(1)\beta} r^{(1)\gamma} r^{(1)\delta} + b^b_{\alpha\beta\gamma} r^{(2)\beta} r^{(1)\gamma} \right)
\]

(1.11)

where:

\[
(EL)_{\alpha}^{\text{constraints def.}} = \frac{\partial L_c}{\partial r^\alpha} - \frac{d}{dt} \left( \frac{\partial L_c}{\partial r^{(1)\alpha}} \right) + \frac{d^2}{dt^2} \left( \frac{\partial L_c}{\partial r^{(2)\alpha}} \right) - \frac{d^3}{dt^3} \left( \frac{\partial L_c}{\partial r^{(3)\alpha}} \right) - \frac{1}{A_\alpha} \frac{\partial L_c}{\partial s^a}
\]

(1.12)

\[
B^b_{\alpha\beta} = \frac{1}{A_\alpha} - \frac{1}{A_\beta}
\]

(1.13a)

\[
B^b_{\alpha\beta\gamma} = \frac{3}{A_\alpha} - \frac{3}{A_\beta - A_\gamma}
\]

(1.13b)

\[
B^b_{\alpha\beta\gamma\delta} = \frac{3}{A_\alpha} - \frac{3}{A_\beta - A_\gamma - A_\delta}
\]

(1.13c)

Remark that the coefficients \( B \) does not depend of Lagrangian but only of constraints and in the following we give another expression. So, \( D = \text{span}\left\{ \frac{\delta}{\delta r^\alpha} \right\} \) where:

\[
\frac{\delta}{\delta r^\alpha} = \frac{\partial}{\partial r^\alpha} - A_\alpha \frac{\partial}{\partial s^a}
\]

(3.14)

and then:

\[
A^a_{\alpha\beta} = \frac{1}{\partial r^\beta} A_\alpha
\]

(3.15a)

\[
A^a_{\alpha\beta\gamma} = \frac{1}{\partial r^\gamma} A^a_{\alpha\beta} = \frac{1}{\partial r^\beta} A^a_{\alpha\gamma}
\]

(3.15b)

With respect to coefficients \( B \) it results:

\[
\left[ \frac{\delta}{\delta r^\alpha}, \frac{\delta}{\delta r^\beta} \right] = b^b_{\alpha\beta} \frac{\partial}{\partial s^b}
\]

(3.16a)

\[
\tilde{B}^b_{\alpha\beta\gamma} = \left[ \frac{\delta}{\delta r^\gamma}, \frac{\delta}{\delta r^\alpha} \right] A^b_{\beta}
\]

(3.16b)
\[ B_{\alpha\beta\gamma\delta} = \left[ \frac{\delta}{\delta x^\alpha}, \frac{\delta}{\delta x^\gamma} \right] A^b_{\beta\gamma} \]

and then: \[ B_{\alpha\alpha} = B_{\alpha\beta\alpha} = B_{\alpha\beta\gamma\alpha} = 0 \] for every \( \alpha \).

2. An example

Let us recall that on \( M = \mathbb{R}^3 \) we have:

(i) the free particle is described by the Lagrangian of first order:
\[ L (q^{(1)}) = \frac{1}{2} \sum_{i=1}^{3} (q^{(1)i})^2 \]

(ii) the elastic beam is described by the Lagrangian of second order([5])
\[ L (q^{(2)}) = \frac{1}{2} \sum_{i=1}^{3} (q^{(2)i})^2. \]

Therefore, it seems naturally to consider the next Lagrangian of third order:
\[ L (q^{(3)}) = \frac{1}{2} \sum_{i=1}^{3} (q^{(3)i})^2 \]

with the associated Euler-Lagrange equations:
\[ (EL)_i^{\text{free}} = \frac{d^6 q^j}{dt^6} = 0, \quad 1 \leq i \leq 3. \]

Consider the nonholonomic constraint of Rosenberg-Bates-Sniatycki type ([1], [2, p. 84]):
\[ z^{(1)} = x^{(1)} y \]
which gives:
\[ z^{(2)} = x^{(2)} y + x^{(1)} y^{(1)} \] \hspace{1cm} (2.4a)
\[ z^{(3)} = x^{(3)} y + 2x^{(2)} y + x^{(1)} y^{(2)} \] \hspace{1cm} (2.4b)
which means that \( p = 1, s^1 = z, r^1 = x, r^2 = y \) and:
\[ A_{11} = \frac{1}{1}, \quad A_{12} = 0, \quad A_{21} = \frac{1}{2}, \quad A_{22} = \frac{1}{2} \] \hspace{1cm} (2.5a)
\[ A_{11} = \frac{1}{2}, \quad A_{22} = \frac{1}{2}, \quad A_{21} = \frac{1}{2} \] \hspace{1cm} (2.5b)
\[ A_{13} = 0. \] \hspace{1cm} (2.5c)
The constrained Lagrangian is:

\[ L_c = \frac{1}{2} \left[ (x^{(3)})^2 + (y^{(3)})^2 + (x^{(3)}y + 2x^{(2)}y^{(1)} + x^{(1)}y^{(2)})^2 \right] \]  \hspace{1cm} (2.6)

and:

\[ \frac{\partial L_c}{\partial y} = x^{(3)}z^{(3)} \hspace{1cm} (2.7a) \]

\[ \frac{\partial L_c}{\partial x^{(1)}} = y^{(2)}z^{(3)} \hspace{1cm} (2.7b) \]

\[ \frac{\partial L_c}{\partial x^{(2)}} = 2y^{(1)}z^{(3)} \hspace{1cm} (2.7c) \]

\[ \frac{\partial L_c}{\partial x^{(3)}} = z^{(3)} + yz^{(3)} \hspace{1cm} (2.7d) \]

where \( z^{(3)} \) is given by (2.4b).

Therefore:

\[ (EL)_{1\text{constraints}} = -\frac{d}{dt} \left( y^{(2)}z^{(3)} \right) + 2 \frac{d^2}{dt^2} \left( y^{(1)}z^{(3)} \right) - \frac{d^3}{dt^3} \left( x^{(3)} + yz^{(3)} \right) \] \hspace{1cm} (2.8a)

\[ (EL)_{2\text{constraints}} = x^{(3)}z^{(3)} - 2 \frac{d}{dt} \left( x^{(2)}z^{(3)} \right) + \frac{d^2}{dt^2} \left( x^{(1)}z^{(3)} \right) - \frac{d^3}{dt^3} \left( y^{(3)} \right) \] \hspace{1cm} (2.8b)

which get:

\[ (EL)_{1\text{constraints}} = -x^{(6)} - yz^{(6)} - y^{(1)}z^{(5)} + y^{(2)}z^{(4)} \] \hspace{1cm} (2.9a)

\[ (EL)_{2\text{constraints}} = x^{(1)}z^{(5)} - y^{(6)}. \] \hspace{1cm} (2.9b)

From (2.5) the only nonzero \( B \) is \( B_{12} = -1 \) and then the right hand side of (1.11) is:

\[ (EL)_{1\text{constraints}} = -y^{(1)}z^{(5)} \] \hspace{1cm} (2.10a)

\[ (EL)_{2\text{constraints}} = x^{(1)}z^{(5)} \] \hspace{1cm} (2.10b)

and, in conclusion we have:

\[ (EL)_{1\text{constraints}} : x^{(6)} + yz^{(6)} - y^{(2)}z^{(4)} = 0 \] \hspace{1cm} (2.11a)

\[ (EL)_{2\text{constraints}} : y^{(6)} = 0. \] \hspace{1cm} (2.11b)
References


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