Nonholonomic Lagrangians of third order:
Equations of motion for the constrained Lagrangian

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Abstract

The equations of motion for the associated Lagrangian to a nonholonomic Lagrangian of third order are computed and an example is given.

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Introduction

In the last years there is an increasing interest in nonholonomic mechanics especially from a geometrical point of view. Following the methodology of [2], where are treated nonholonomic Lagrangians of first order, we obtain the equations of motion in terms of the associated constrained Lagrangian of a nonholonomic Lagrangian involving velocities of third order. In [3] this problem is solved for Lagrangians of second order and the spinning particle is given as example.

This paper is dedicated to the memory of Romanian Academician Gheorghe Vrânceanu(1900-1979) who introduces in 1926 the notion of *nonholonomic spaces*, in order to give a geometrical approach to nonholonomic mechanics([7],[8]). Note that the Romanian school of mathematics has an important contribution to this subject([4], [7], [8], [9]).

1 Equations of motion

The starting point is a configuration-space given by a n-dimensional manifold Q, for which we consider the tangent bundle of order three T^3Q([5], [6]). For coordinates $(q^i)_{1\leq i\leq n}$ on Q we have the induced coordinates $\left(q^i, q^{(1)i} = \frac{dq^i}{dt}, q^{(2)i} = \frac{d^2q^i}{dt^2}, q^{(3)i} = \frac{d^3q^i}{dt^3}\right)$ on T^3Q.

Let us suppose that the evolution of the considered system is described by the following objects:

1. a third-order Lagrangian, that is a smooth map $L : T^3Q \longrightarrow I\!R$([5], [6])

2. a set of p independent one-forms $(\omega^a(q))_{1\leq a\leq p}$ whose vanishing gives the constraints of the system.

This 1-forms defines an $(n-p)$-dimensional distribution D on Q i.e. $(\omega^a(q))$ is a local basis of the annihilator D^0 of D. Also, this constraints means that the only allowable velocities are the tangent vectors belonging to D or in other words the motion is constrained to the submanifold D.

The Lagrangian L gives the Euler-Lagrange equations of order three ([5], [6]):

$$\delta L = (EL)_i^{free}\,\delta q^i = 0 \tag{1.1a}$$

with:

$$(EL)_i^{free} = \frac{\partial L}{\partial q^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial q^{(1)i}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial q^{(2)i}}\right) - \frac{d^3}{dt^3}\left(\frac{\partial L}{\partial q^{(3)i}}\right) \tag{1.1b}$$

and supposing that the constraints are nonholonomic, we can choose a local coordinate chart and a local basis for the constraints such that([2, p. 31]):

$$\omega^a(q) = ds^a + \overset{1}{A}{}^a_\alpha(r,s)\,dr^\alpha,\ 1\leq a\leq p \tag{1.2}$$

where $q = (r,s) \in I\!R^{n-p} \times I\!R^p$.

From (1.2) it results that:

$$\delta s^a + \overset{1}{A}{}^a_\alpha \, \delta r^\alpha = 0 \tag{1.3}$$

which, by substitution into (1.1) yields:

$$\frac{\partial L}{\partial r^\alpha} - \frac{d}{dt}\left(\frac{\partial L}{\partial r^{(1)\alpha}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial r^{(2)\alpha}}\right) - \frac{d^3}{dt^3}\left(\frac{\partial L}{\partial r^{(3)\alpha}}\right) =$$

$$= \overset{1}{A}{}^a_\alpha \left[\frac{\partial L}{\partial s^a} - \frac{d}{dt}\left(\frac{\partial L}{\partial s^{(1)a}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial s^{(2)a}}\right) - \frac{d^3}{dt^3}\left(\frac{\partial L}{\partial s^{(3)a}}\right)\right]. \tag{1.4}$$

Equations (1.4) combined with the constraint equations:

$$s^{(1)a} = - \overset{1}{A}{}^a_\alpha \, r^{(1)\alpha} \tag{1.5a}$$

$$s^{(2)a} = -\frac{d}{dt}(\overset{1}{A}{}^a_\alpha) r^{(1)\alpha} - \overset{1}{A}{}^a_\alpha \, r^{(2)\alpha} \tag{1.5b}$$

$$s^{(3)a} = -\frac{d^2}{dt^2}\left(\overset{1}{A}{}^a_\alpha\right) r^{(1)\alpha} - 2\frac{d}{dt}\left(\overset{1}{A}{}^a_\alpha\right) r^{(2)\alpha} - \overset{1}{A}{}^a_\alpha \, r^{(3)\alpha} \tag{1.5c}$$

gives a complete description of the equations of motion. Remark that another form for (1.5*b*) is:

$$s^{(2)a} = \overset{2}{A}{}^a_{\alpha\beta} \, r^{(1)\alpha} r^{(1)\beta} - \overset{1}{A}{}^a_\alpha \, r^{(2)\alpha} \tag{1.5b'}$$

where:

$$\overset{2}{A}{}^a_{\alpha\beta}(r,s) = \frac{\partial \overset{1}{A}{}^a_\alpha}{\partial s^b} \overset{1}{A}{}^b_\beta - \frac{\partial \overset{1}{A}{}^a_\alpha}{\partial r^\beta} \tag{1.6}$$

and another form of (1.5*c*) is:

$$s^{(3)a} = \overset{3}{A}{}^a_{\alpha\beta\gamma} \, r^{(1)\alpha} r^{(1)\beta} r^{(1)\gamma} + (2\overset{2}{A}{}^a_{\alpha\beta} + \overset{2}{A}{}^a_{\beta\alpha}) r^{(2)\alpha} r^{(1)\beta} - \overset{1}{A}{}^a_\alpha \, r^{(3)\alpha} \tag{1.5c'}$$

where:

$$\overset{3}{A}{}^a_{\alpha\beta\gamma} = \frac{\partial \overset{2}{A}{}^a_{\alpha\beta}}{\partial r^\gamma} - \frac{\partial \overset{2}{A}{}^a_{\alpha\beta}}{\partial s^b} \overset{1}{A}{}^b_\gamma \, . \tag{1.7}$$

Following [2, p. 31] we define an associated *constrained* Lagrangian L_c by substituting the constraints (1.5) into the Lagrangian L:

$$L_c\left(r^\alpha, s^a, r^{(1)\alpha}, r^{(2)\alpha}, r^{(3)\alpha}\right) \overset{def.}{=} \tag{1.8}$$

$$= \quad L(r^\alpha, s^a, r^{(1)\alpha}, -\overset{1}{A}{}^a_\alpha\, r^{(1)\alpha}, r^{(2)\alpha}, \overset{2}{A}{}^a_{\alpha\beta}\, r^{(1)\alpha} r^{(1)\beta} - \overset{1}{A}{}^a_\alpha\, r^{(2)\alpha}, r^{(3)\alpha},$$
$$, \overset{3}{A}{}^a_{\alpha\beta\gamma}\, r^{(1)\alpha} r^{(1)\beta} r^{(1)\gamma} + (2\,\overset{2}{A}{}^a_{\alpha\beta} + \overset{2}{A}{}^a_{\beta\alpha}) r^{(2)\alpha} r^{(1)\beta} - \overset{1}{A}{}^a_\alpha\, r^{(3)\alpha}).$$

A direct coordinates calculation shows:

$$\frac{\partial L_c}{\partial r^\alpha} = \frac{\partial L}{\partial r^\alpha} - \frac{\partial L}{\partial s^{(1)b}} \frac{\partial \overset{1}{A}{}^b_\beta}{\partial r^\alpha} r^{(1)\beta} + \frac{\partial L}{\partial s^{(2)b}} \left(\frac{\partial \overset{2}{A}{}^b_{\beta\gamma}}{\partial r^\alpha} r^{(1)\beta} r^{(1)\gamma} - \frac{\partial \overset{1}{A}{}^b_\beta}{\partial r^\alpha} r^{(2)\beta} \right) +$$
$$+ \frac{\partial L}{\partial s^{(3)b}} \left(\frac{\partial \overset{3}{A}{}^b_{\beta\gamma\delta}}{\partial r^\alpha} r^{(1)\beta} r^{(1)\gamma} r^{(1)\delta} + (2\frac{\partial \overset{2}{A}{}^b_{\beta\gamma}}{\partial r^\alpha} + \frac{\partial \overset{2}{A}{}^b_{\gamma\beta}}{\partial r^\alpha}) r^{(2)\beta} r^{(1)\gamma} - \frac{\partial \overset{1}{A}{}^b_\beta}{\partial r^\alpha} r^{(3)\beta} \right) \tag{1.9a}$$

$$\frac{\partial L_c}{\partial s^a} = \frac{\partial L}{\partial s^a} - \frac{\partial L}{\partial s^{(1)b}} \frac{\partial \overset{1}{A}{}^b_\beta}{\partial s^a} r^{(1)\beta} + \frac{\partial L}{\partial s^{(2)b}} \left(\frac{\partial \overset{2}{A}{}^b_{\beta\gamma}}{\partial s^a} r^{(1)\beta} r^{(1)\gamma} - \frac{\partial \overset{1}{A}{}^b_\beta}{\partial s^a} r^{(2)\beta} \right) +$$
$$+ \frac{\partial L}{\partial s^{(3)b}} \left(\frac{\partial \overset{3}{A}{}^b_{\beta\gamma\delta}}{\partial s^a} r^{(1)\beta} r^{(1)\gamma} r^{(1)\delta} + (2\frac{\partial \overset{2}{A}{}^b_{\beta\gamma}}{\partial s^a} + \frac{\partial \overset{2}{A}{}^b_{\gamma\beta}}{\partial s^a}) r^{(2)\beta} r^{(1)\gamma} - \frac{\partial \overset{1}{A}{}^b_\beta}{\partial s^a} r^{(3)\beta} \right) \tag{1.9b}$$

$$\frac{\partial L_c}{\partial r^{(1)\alpha}} = \frac{\partial L}{\partial r^{(1)\alpha}} - \frac{\partial L}{\partial s^{(1)b}}\, \overset{1}{A}{}^b_\alpha + \frac{\partial L}{\partial s^{(2)b}} (\overset{2}{A}{}^b_{\alpha\beta} + \overset{2}{A}{}^b_{\beta\alpha}) r^{(1)\beta} +$$
$$+ \frac{\partial L}{\partial s^{(3)b}} \left((\overset{3}{A}{}^b_{\alpha\beta\gamma} + \overset{3}{A}{}^b_{\beta\alpha\gamma} + \overset{3}{A}{}^b_{\beta\gamma\alpha}) r^{(1)\beta} r^{(1)\gamma} + (2\,\overset{2}{A}{}^b_{\beta\alpha} + \overset{2}{A}{}^b_{\alpha\beta}) r^{(2)\beta} \right) \tag{1.9c}$$

$$\frac{\partial L_c}{\partial r^{(2)\alpha}} = \frac{\partial L}{\partial r^{(2)\alpha}} - \frac{\partial L}{\partial s^{(2)b}}\, \overset{1}{A}{}^b_\alpha + \frac{\partial L}{\partial s^{(3)b}} (2\,\overset{2}{A}{}^b_{\alpha\beta} + \overset{2}{A}{}^b_{\beta\alpha}) r^{(1)\beta} \tag{1.9d}$$

$$\frac{\partial L_c}{\partial r^{(3)\alpha}} = \frac{\partial L}{\partial r^{(3)\alpha}} - \frac{\partial L}{\partial s^{(3)b}}\, \overset{1}{A}{}^b_\alpha\ . \tag{1.9e}$$

A long, but straightforward computation which uses the formulae:

$$\frac{d}{dt}\, \overset{1}{A}{}^b_\alpha = -\, \overset{2}{A}{}^b_{\alpha\beta}\, r^{(1)\beta} \tag{1.10a}$$

$$\frac{d}{dt}\, \overset{2}{A}{}^b_{\alpha\beta} = \overset{3}{A}{}^b_{\alpha\beta\gamma}\, r^{(1)\gamma} \tag{1.10b}$$

gives the equations of motion for L_c:

$$(EL)_\alpha^{constraints} = \left[\frac{\partial L}{\partial s^{(1)b}} - \frac{d}{dt}\left(\frac{\partial L}{\partial s^{(2)b}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial s^{(3)b}}\right)\right] \overset{1}{B}{}^b_{\alpha\beta}\, r^{(1)\beta} +$$

$$+\left(\frac{\partial L}{\partial s^{(2)b}} - \frac{d}{dt}\left(\frac{\partial L}{\partial s^{(3)b}}\right)\right) \overset{2}{B}{}^b_{\alpha\beta\gamma}\, r^{(1)\beta} r^{(1)\gamma} +$$

$$+\frac{\partial L}{\partial s^{(3)b}}\left(\overset{3}{B}{}^b_{\alpha\beta\gamma\delta}\, r^{(1)\beta} r^{(1)\gamma} r^{(1)\delta} + \overset{2}{B}{}^b_{\alpha\beta\gamma}\, r^{(2)\beta} r^{(1)\gamma}\right) \tag{1.11}$$

where:

$$(EL)_\alpha^{constraints} \overset{def.}{=} \frac{\partial L_c}{\partial r^\alpha} - \frac{d}{dt}\left(\frac{\partial L_c}{\partial r^{(1)\alpha}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L_c}{\partial r^{(2)\alpha}}\right) - \frac{d^3}{dt^3}\left(\frac{\partial L_c}{\partial r^{(3)\alpha}}\right) - \overset{1}{A}{}^a_\alpha \frac{\partial L_c}{\partial s^a} \tag{1.12}$$

$$\overset{1}{B}{}^b_{\alpha\beta} = \overset{2}{A}{}^b_{\beta\alpha} - \overset{2}{A}{}^b_{\alpha\beta} \tag{1.13a}$$

$$\overset{2}{B}{}^b_{\alpha\beta\gamma} = \overset{3}{A}{}^b_{\beta\gamma\alpha} - \overset{3}{B}{}^b_{\beta\alpha\gamma} \tag{1.13b}$$

$$\overset{3}{B}{}^b_{\alpha\beta\gamma\delta} = \frac{\partial \overset{3}{A}{}^b_{\beta\gamma\delta}}{\partial r^\alpha} - \frac{\partial \overset{3}{A}{}^b_{\beta\gamma\alpha}}{\partial r^\delta} + \overset{1}{A}{}^a_\delta \frac{\partial \overset{3}{A}{}^b_{\beta\gamma\alpha}}{\partial s^a} - \overset{1}{A}{}^a_\alpha \frac{\partial \overset{3}{A}{}^b_{\beta\gamma\delta}}{\partial s^a}. \tag{1.13c}$$

Remark that the coefficients B does not depend of Lagrangian but only of constraints and in the following we give another expression. So, $D = span\{\frac{\delta}{\delta r^\alpha}\}$ where:

$$\frac{\delta}{\delta r^\alpha} = \frac{\partial}{\partial r^\alpha} - \overset{1}{A}{}^a_\alpha \frac{\partial}{\partial s^a} \tag{3.14}$$

and then:

$$\overset{2}{A}{}^a_{\alpha\beta} = -\frac{\delta}{\delta r^\beta} \overset{1}{A}{}^a_\alpha \tag{3.15a}$$

$$\overset{3}{A}{}^a_{\alpha\beta\gamma} = \frac{\delta}{\delta r^\gamma} \overset{2}{A}{}^a_{\alpha\beta} = -\frac{\delta}{\delta r^\gamma}\frac{\delta}{\delta r^\beta} \overset{1}{A}{}^a_\alpha\ . \tag{3.15b}$$

With respect to coefficients B it results:

$$\left[\frac{\delta}{\delta r^\alpha}, \frac{\delta}{\delta r^\beta}\right] = \overset{1}{B}{}^b_{\alpha\beta} \frac{\partial}{\partial s^b} \tag{3.16a}$$

$$\overset{2}{B}{}^b_{\alpha\beta\gamma} = \left[\frac{\delta}{\delta r^\gamma}, \frac{\delta}{\delta r^\alpha}\right] \overset{1}{A}{}^b_\beta \tag{3.16b}$$

$$\overset{3}{B}{}^{b}_{\alpha\beta\gamma\delta}=\left[\frac{\delta}{\delta r^{\alpha}},\frac{\delta}{\delta r^{\delta}}\right]\overset{2}{A}{}^{b}_{\beta\gamma} \tag{3.16c}$$

and then: $\overset{1}{B}{}^{b}_{\alpha\alpha}=\overset{2}{B}{}^{b}_{\alpha\beta\alpha}=\overset{3}{B}{}^{b}_{\alpha\beta\gamma\alpha}=0$ for every α.

2 An example

Let us recall that on $M=I\!R^3$ we have:

(i) the free particle is described by the Lagrangian of first order: $L\left(q^{(1)}\right)=\frac{1}{2}\sum\limits_{i=1}^{3}\left(q^{(1)i}\right)^2$

(ii) the elastic beam is described by the Lagrangian of second order([5]) $L\left(q^{(2)}\right)=\frac{1}{2}\sum\limits_{i=1}^{3}\left(q^{(2)i}\right)^2$.

Therefore, it seems naturally to consider the next Lagrangian of third order:

$$L\left(q^{(3)}\right)=\frac{1}{2}\sum_{i=1}^{3}\left(q^{(3)i}\right)^2 \tag{2.1}$$

with the associated Euler-Lagrange equations:

$$(EL)_i^{free}=\frac{d^6q^i}{dt^6}=0,\quad 1\leq i\leq 3. \tag{2.2}$$

Consider the nonholonomic constraint of Rosenberg-Bates-Sniatycki type ([1], [2, p. 84]):

$$z^{(1)}=x^{(1)}y \tag{2.3}$$

which gives:

$$z^{(2)}=x^{(2)}y+x^{(1)}y^{(1)} \tag{2.4a}$$

$$z^{(3)}=x^{(3)}y+2x^{(2)}y^{(1)}+x^{(1)}y^{(2)} \tag{2.4b}$$

which means that $p=1, s^1=z, r^1=x, r^2=y$ and:

$$\overset{1}{A}{}^{1}_{1}=-y,\quad \overset{1}{A}{}^{1}_{2}=0 \tag{2.5a}$$

$$\overset{2}{A}{}^{1}_{11}=\overset{2}{A}{}^{1}_{22}=\overset{2}{A}{}^{1}_{21}=0,\quad \overset{2}{A}{}^{1}_{12}=-1 \tag{2.5b}$$

$$\overset{3}{A}{}^{1}_{\alpha\beta\gamma}=0. \tag{2.5c}$$

The constrained Lagrangian is:

$$L_c = \frac{1}{2}\left[\left(x^{(3)}\right)^2 + \left(y^{(3)}\right)^2 + \left(x^{(3)}y + 2x^{(2)}y^{(1)} + x^{(1)}y^{(2)}\right)^2\right] \tag{2.6}$$

and:

$$\frac{\partial L_c}{\partial y} = x^{(3)}z^{(3)} \tag{2.7a}$$

$$\frac{\partial L_c}{\partial x^{(1)}} = y^{(2)}z^{(3)}, \quad \frac{\partial L_c}{\partial y^{(1)}} = 2x^{(2)}z^{(3)} \tag{2.7b}$$

$$\frac{\partial L_c}{\partial x^{(2)}} = 2y^{(1)}z^{(3)}, \quad \frac{\partial L_c}{\partial y^{(2)}} = x^{(1)}z^{(3)} \tag{2.7c}$$

$$\frac{\partial L_c}{\partial x^{(3)}} = x^{(3)} + yz^{(3)}, \quad \frac{\partial L_c}{\partial y^{(3)}} = y^{(3)} \tag{2.7d}$$

where $z^{(3)}$ is given by (2.4b).

Therefore:

$$(EL)_1^{constraints} = -\frac{d}{dt}\left(y^{(2)}z^{(3)}\right) + 2\frac{d^2}{dt^2}\left(y^{(1)}z^{(3)}\right) - \frac{d^3}{dt^3}\left(x^{(3)} + yz^{(3)}\right) \tag{2.8a}$$

$$(EL)_2^{constraints} = x^{(3)}z^{(3)} - 2\frac{d}{dt}\left(x^{(2)}z^{(3)}\right) + \frac{d^2}{dt^2}\left(x^{(1)}z^{(3)}\right) - \frac{d^3}{dt^3}\left(y^{(3)}\right) \tag{2.8b}$$

which get:

$$(EL)_1^{constraints} = -x^{(6)} - yz^{(6)} - y^{(1)}z^{(5)} + y^{(2)}z^{(4)} \tag{2.9a}$$

$$(EL)_2^{constraints} = x^{(1)}z^{(5)} - y^{(6)}. \tag{2.9b}$$

From (2.5) the only nonzero B is $\overset{1\,1}{B}_{12} = -1$ and then the right hand side of (1.11) is:

$$(EL)_1^{constraints} = -y^{(1)}z^{(5)} \tag{2.10a}$$

$$(EL)_2^{constraints} = x^{(1)}z^{(5)} \tag{2.10b}$$

and, in conclusion we have:

$$(EL)_1^{constraints} : x^{(6)} + yz^{(6)} - y^{(2)}z^{(4)} = 0 \tag{2.11a}$$

$$(EL)_2^{constraints} : y^{(6)} = 0. \tag{2.11b}$$

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