

A NEW METHOD TO OBTAIN PYTHAGOREAN TRIPLE PRESERVING MATRICES

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Abstract. Another method to obtain Pythagorean Triple Preserving Matrices is proposed and a singular case is put in evidence. Also, a possible connection with physics is sketched by proving that the set of these matrices is a group. In the last section, we generalize our method to Weighted Pythagorean Triple Preserving Matrices. An interesting open problem is generated by the fact that this type of matrix appears as a product of two matrices of order 4 with a form suggesting quaternions.

1. Pythagorean Triple Preserving Matrices. In [2] Palmer, Ahuja, and Tikoo obtained all matrices which convert a Pythagorean Triple into another Pythagorean Triple. In this paper we give a second method which uses the matrix equation of a quadric in real 3-dimensional space.

Recall that a Pythagorean Triple (PT) is a triple (a, b, c) of natural numbers such that $a^2 + b^2 = c^2$ and recall that the general expression of a PT is

$$(a, b, c) = (m^2 - n^2, 2mn, m^2 + n^2)$$

where m and n are two integers. So, a PT represents the coordinates of a point $X \in \mathbb{R}^3$ which belongs to the quadric $\Gamma : x^2 + y^2 - z^2 = 0$. The matrix equation of this quadric is $\Gamma : X^t \cdot S \cdot X = 0$ where

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Using [2], define a Pythagorean Triple Preserving Matrix (PTPM)

$$A = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix}.$$

That is, if $X \in \Gamma$ then $A \cdot X \in \Gamma$. Therefore, $(AX)^t \cdot S \cdot (AX) = 0$ which means $X^t \cdot (A^t SA) \cdot X = 0$. In conclusion, A is a PTPM if and only if there exists a real number ρ such that

$$A^t SA = \rho S. \quad (1.1)$$

A straightforward computation leads to the following form of (1.1).

$$\begin{cases} \alpha_1^2 + \beta_1^2 - \gamma_1^2 = \rho \\ \alpha_2^2 + \beta_2^2 - \gamma_2^2 = \rho \\ \alpha_3^2 + \beta_3^2 - \gamma_3^2 = -\rho \\ \alpha_1\alpha_2 + \beta_1\beta_2 - \gamma_1\gamma_2 = 0 \\ \alpha_2\alpha_3 + \beta_2\beta_3 - \gamma_2\gamma_3 = 0 \\ \alpha_3\alpha_1 + \beta_3\beta_1 - \gamma_3\gamma_1 = 0. \end{cases} \quad (1.2)$$

If we make exactly the choice of [2], namely

$$\begin{cases} r^2 = \frac{\alpha_1 + \alpha_3 + \gamma_1 + \gamma_3}{2}, & s^2 = \frac{\alpha_3 - \alpha_1 + \gamma_3 - \gamma_1}{2} \\ t^2 = \frac{\gamma_1 + \gamma_3 - (\alpha_1 + \alpha_3)}{2}, & u^2 = \frac{\gamma_3 - \gamma_1 - (\alpha_3 - \alpha_1)}{2}. \end{cases} \quad (1.3)$$

then, from (1.2₁), (1.2₃) and (1.2₆) it follows that

$$\begin{cases} \beta_1 + \beta_3 = 2rt \\ -\beta_1 + \beta_3 = 2su \end{cases}$$

which gives

$$\begin{cases} \beta_1 = rt - su \\ \beta_3 = rt + su. \end{cases} \quad (1.4)$$

From (1.3) we have, exactly as in [2], that

$$\begin{cases} \alpha_1 = \frac{(r^2 - t^2) - (s^2 - u^2)}{2}, & \alpha_3 = \frac{(r^2 - t^2) + (s^2 - u^2)}{2} \\ \gamma_1 = \frac{(r^2 + t^2) - (s^2 + u^2)}{2}, & \gamma_3 = \frac{(r^2 + t^2) + (s^2 + u^2)}{2} \end{cases} \quad (1.5)$$

and then, from (1.2₁) it follows that

$$\rho = (ru - st)^2. \quad (1.6)$$

Equations (1.2₂), (1.2₄) and (1.2₅) yield

$$\begin{cases} \alpha_2 = rs - tu \\ \beta_2 = ru + st \\ \gamma_2 = rs + tu. \end{cases} \quad (1.7)$$

In conclusion, from (1.4), (1.5) and (1.7), it follows that the general form of a PTPM is

$$A(r, s, t, u) = \begin{pmatrix} \frac{1}{2}(r^2 - t^2 - s^2 + u^2) & rs - tu & \frac{1}{2}(r^2 - t^2 + s^2 - u^2) \\ rt - su & ru + st & rt + su \\ \frac{1}{2}(r^2 + t^2 - s^2 - u^2) & rs + tu & \frac{1}{2}(r^2 + t^2 + s^2 + u^2) \end{pmatrix} \quad (1.8)$$

which is exactly the expression given in [2].

A first advantage of the present method (which is of geometrical nature, like PT) is that it uses only 10 variables, namely $(\alpha_i), (\beta_i), (\gamma_i)$ and ρ , instead of 11 variables $(\alpha_i), (\beta_i), (\gamma_i), M, N$ as in [2]. A second advantage is that given in the singular case $\rho = 0$ for relation (1.1) which we will discuss below. A third advantage is that it offers a very quick proof that the set of PTPM, considered with rational entries, is a group with respect to multiplication (see section 3).

We can obtain the pair $(A(r, s, t, u), \rho)$ from the product of two matrices of order 4. Considering

$$\Phi_1 = \begin{pmatrix} r & -s & -t & u \\ t & -u & r & -s \\ r & -s & t & -u \\ t & -u & -r & s \end{pmatrix} \quad \text{and} \quad \Phi_2 = \begin{pmatrix} r & s & r & s \\ s & -r & -s & r \\ t & u & t & u \\ u & -t & -u & t \end{pmatrix} \quad (1.9)$$

we obtain

$$\begin{aligned} \frac{1}{2}\Phi_1 \cdot \Phi_2 &= \frac{1}{2} \begin{pmatrix} r^2 - s^2 - t^2 + u^2 & 2(rs - tu) & r^2 + s^2 + t^2 + u^2 & 0 \\ 2(rt - su) & 2(ru + ts) & 2(rt + su) & 0 \\ r^2 - s^2 + t^2 - u^2 & 2(rs + tu) & r^2 + s^2 + t^2 + u^2 & 0 \\ 0 & 0 & 0 & -2(ru - st) \end{pmatrix} \\ &= \begin{pmatrix} A(r, s, t, u) & 0 \\ 0 & -\sqrt{\rho} \end{pmatrix} \end{aligned} \quad (1.10)$$

and this fact, using the expression of Φ_1 and Φ_2 yields the following problem.

Open problem. Does there exist a connection between PTPM and the algebra of quaternions?

As a possible answer, let us note that the matrix (1.8) is close to the matrix from [4] representing the rotations in \mathbb{R}^3 .

2. The Singular Case. For relation (1.1) the case $\rho = 0$ appears as a singular case. From relation (1.6) we have $ru = st$.

Case I. Suppose that one of r or u is zero. Then one of t and s is zero. We make the choice $r = s = 0$ and then it follows that

$$\begin{aligned} A(0, 0, t, u) &= \begin{pmatrix} \frac{u^2 - t^2}{2} & -tu & \frac{-u^2 - t^2}{2} \\ 0 & 0 & 0 \\ \frac{t^2 - u^2}{2} & tu & \frac{t^2 + u^2}{2} \end{pmatrix} = \frac{t^2}{2} \begin{pmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \\ &+ \frac{u^2}{2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} + tu \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\ &= t^2 A(0, 0, 1, 0) + u^2 A(0, 0, 0, 1) + tu \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \end{aligned} \quad (2.1)$$

We have

$$A(0, 0, 1, 0) \begin{pmatrix} m^2 - n^2 \\ 2mn \\ m^2 + n^2 \end{pmatrix} = \begin{pmatrix} -m^2 \\ 0 \\ m^2 \end{pmatrix} \text{ and } A(0, 0, 0, 1) \begin{pmatrix} m^2 - n^2 \\ 2mn \\ m^2 + n^2 \end{pmatrix} = \begin{pmatrix} -n^2 \\ 0 \\ n^2 \end{pmatrix}.$$

That is, we obtain the “singular” PT $(-1, 0, 1)$.

Case II. Suppose that $r, u \neq 0$. Then $s, t \neq 0$. From the relation $s = \frac{ru}{t}$ it follows that

$$A\left(r, \frac{ru}{t}, t, u\right) = \begin{pmatrix} \frac{1}{2} \left(r^2 - t^2 - \frac{r^2 u^2}{t^2} + u^2 \right) & \frac{r^2 u}{t} - tu & \frac{1}{2} \left(r^2 - t^2 + \frac{r^2 u^2}{t^2} - u^2 \right) \\ rt - \frac{ru^2}{t} & 2ru & rt + \frac{ru^2}{t} \\ \frac{1}{2} \left(r^2 + t^2 - \frac{r^2 u^2}{t^2} - u^2 \right) & \frac{r^2 u}{t} + tu & \frac{1}{2} \left(r^2 + t^2 + \frac{r^2 u^2}{t^2} + u^2 \right) \end{pmatrix}. \quad (2.2)$$

For example,

$$A(r, ru, 1, u) = \begin{pmatrix} \frac{1}{2} (r^2 - 1 - r^2 u^2 + u^2) & u (r^2 - 1) & \frac{1}{2} (r^2 - 1 + r^2 u^2 - u^2) \\ r (1 - u^2) & 2ru & r (1 + u^2) \\ \frac{1}{2} (r^2 + 1 - r^2 u^2 - u^2) & u (r^2 + 1) & \frac{1}{2} (r^2 + 1 + r^2 u^2 + u^2) \end{pmatrix} \quad (2.3)$$

and then

$$A(r, r, 1, 1) = \begin{pmatrix} 0 & r^2 - 1 & r^2 - 1 \\ 0 & 2r & 2r \\ 0 & r^2 + 1 & r^2 + 1 \end{pmatrix} \quad (2.4)$$

which gives

$$A(r, r, 1, 1) \begin{pmatrix} m^2 - n^2 \\ 2mn \\ m^2 + n^2 \end{pmatrix} = \begin{pmatrix} (r^2 - 1) (m + n)^2 \\ 2r (m + n)^2 \\ (r^2 + 1) (m + n)^2 \end{pmatrix}.$$

So,

$$A(1, 1, 1, 1) \begin{pmatrix} m^2 - n^2 \\ 2mn \\ m^2 + n^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2(m+n)^2 \\ 2(m+n)^2 \end{pmatrix}$$

i.e. the “singular” PT $(0, 1, 1)$. Comparing the results of this section with a proposition from [2]: “no specific conditions on the nature of r, s, t and u are imposed”. In conclusion, relation (1.1) characterizes PTPM yielding “non-singular” PT only for the case $\rho \neq 0$.

3. Connections With Physics. A field of possible applications for the previous results is the 2 + 1 Theory of Relativity. Consider \mathbb{R}^3 with the Lorentzian metric ([5])

$$\langle \vec{A}, \vec{B} \rangle = a_1 b_1 + a_2 b_2 - a_3 b_3 \quad (3.1)$$

for $\vec{A} = (a_1, a_2, a_3)$, $\vec{B} = (b_1, b_2, b_3) \in \mathbb{R}^3$. In [5], the pair $E^{3,1} = (\mathbb{R}^3, \langle, \rangle)$ is called the *Minkowski 3-space*. In this space-time the quadric $\Gamma : x^2 + y^2 - z^2 = 0$ is exactly the *set of null vectors* ([5]). More precisely, Γ is the *null cone* of $E^{3,1}$ because if $\vec{A} \in \Gamma$ then $\lambda \vec{A} \in \Gamma$ for all real λ .

Therefore, a PT represents a point in the null cone, with natural coordinates and then a PTPM is a linear transformation of $E^{3,1}$ which preserves the points of natural coordinates from the null cone of $E^{3,1}$.

Using (1.1) results in the fact that the set of PTPM with rational entries is a group with respect to multiplication. Indeed, A is the unit matrix for $r = u = 1$, $s = t = 0$ and if A_1 and A_2 are PTPM with corresponding ρ_1 and ρ_2 , then (1.1) yields

$$(A_1 A_2)^t S (A_1 A_2) = A_2^t (A_1^t S A_1) A_2 = \rho_1 A_2^t S A_2 = \rho_1 \rho_2 S$$

which means that $A_1 A_2$ is a PTPM with corresponding $\rho_1 \rho_2$. With *MAPLE* it is easy to obtain the relation

$$\begin{aligned} & A(r_1, s_1, t_1, u_1) \cdot A(r_2, s_2, t_2, u_2) \\ &= A(r_1 r_2 + t_2 s_1, r_1 s_2 + u_2 s_1, r_2 t_1 + t_2 u_1, t_1 s_2 + u_1 u_2) \end{aligned} \quad (3.2)$$

which implies

$$A^2(r, s, t, u) = A(r^2 + ts, (r + u)s, (r + u)t, ts + u^2) \quad (3.3)$$

$$A^{-1}(r, s, t, u) = A\left(\frac{u}{ru - st}, \frac{-s}{ru - st}, \frac{-t}{ru - st}, \frac{r}{ru - st}\right) \quad (3.4)$$

for $ru \neq st$ (see the previous section). Other properties of $A(r, s, t, u)$ which are obtained with *MAPLE* are

(i) The trace is

$$\text{Tr}A = r^2 + u^2 + ru + st. \quad (3.5)$$

(ii) The eigenvalues are

$$\lambda_1 = ru - st \quad (3.6a)$$

$$\lambda_2 = \frac{1}{2}(r^2 + u^2) + ts + \frac{1}{2}\sqrt{r^4 - 2r^2u^2 + 4r^2st + u^4 + 4u^2st + 8rstu} \quad (3.6b)$$

$$\lambda_3 = \frac{1}{2}(r^2 + u^2) + ts - \frac{1}{2}\sqrt{r^4 - 2r^2u^2 + 4r^2st + u^4 + 4u^2st + 8rstu}. \quad (3.6c)$$

4. Weighted Pythagorean Triple Preserving Matrices. A Weighted Pythagorean Triple (WPT) is a triple (x, y, z) of natural numbers such that

$$p^2x^2 + q^2y^2 = p^2q^2z^2 \quad (4.1)$$

where p and q are two natural numbers. So, a WPT represents the coordinates of a point $X \in \mathbb{R}^3$ which belongs to the quadric $\Gamma : p^2x^2 + q^2y^2 - p^2q^2z^2 = 0$. The matrix equation of this quadric is $\Gamma : X^t \cdot S(p, q) \cdot X = 0$ where

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } S(p, q) = \begin{pmatrix} p^2 & 0 & 0 \\ 0 & q^2 & 0 \\ 0 & 0 & -p^2q^2 \end{pmatrix}.$$

In this section we find the general form of a Weighted Pythagorean Triple Preserving Matrix (WPTPM) A , i.e., if $X \in \Gamma$ then $A \cdot X \in \Gamma$. Using the same

argument as in the first section results in the fact that A is a WPTPM if and only if there exists a real number ρ such that

$$A^t \cdot S(p, q) \cdot A = \rho S(p, q). \quad (4.2)$$

A straightforward computation leads to the following form of (4.2).

$$\begin{cases} p^2 \alpha_1^2 + q^2 \beta_1^2 - p^2 q^2 \gamma_1^2 = \rho p^2 \\ p^2 \alpha_2^2 + q^2 \beta_2^2 - p^2 q^2 \gamma_2^2 = \rho q^2 \\ p^2 \alpha_3^2 + q^2 \beta_3^2 - p^2 q^2 \gamma_3^2 = -\rho p^2 q^2 \\ p^2 \alpha_1 \alpha_2 + q^2 \beta_1 \beta_2 - p^2 q^2 \gamma_1 \gamma_2 = 0 \\ p^2 \alpha_2 \alpha_3 + q^2 \beta_2 \beta_3 - p^2 q^2 \gamma_2 \gamma_3 = 0 \\ p^2 \alpha_3 \alpha_1 + q^2 \beta_3 \beta_1 - p^2 q^2 \gamma_3 \gamma_1 = 0. \end{cases} \quad (4.3)$$

With the choice

$$\begin{cases} r^2 = \frac{1}{2q^2} [pq(\gamma_3 + q\gamma_1) + p(\alpha_3 + q\alpha_1)], s^2 = \frac{1}{2q^2} [pq(\gamma_3 - q\gamma_1) + p(\alpha_3 - q\alpha_1)] \\ t^2 = \frac{1}{2q^2} [pq(\gamma_3 + q\gamma_1) - p(\alpha_3 + q\alpha_1)], u^2 = \frac{1}{2q^2} [pq(\gamma_3 - q\gamma_1) - p(\alpha_3 - q\alpha_1)] \end{cases} \quad (4.4)$$

it follows that the solution

$$A(r, s, t, u) = \begin{pmatrix} \frac{q}{2p} (r^2 - t^2 - s^2 + u^2) & \frac{q^2}{p^2} (rs - tu) & \frac{q^2}{2p} (r^2 - t^2 + s^2 - u^2) \\ rt - su & \frac{q}{p} (ru + st) & q(rt + su) \\ \frac{1}{2p} (r^2 + t^2 - s^2 - u^2) & \frac{q}{p^2} (rs + tu) & \frac{q}{2p} (r^2 + t^2 + s^2 + u^2) \end{pmatrix}. \quad (4.5)$$

Also,

$$\rho = \frac{q^2}{p^2} (ru - st)^2. \quad (4.6)$$

Returning to (4.1) with $x = qa$ and $y = pb$ results in $a^2 + b^2 = z^2$, i.e. (a, b, z) is a PT and therefore, we have the general form of a WPT.

$$(x, y, z) = (q(m^2 - n^2), 2pmn, m^2 + n^2). \quad (4.7)$$

Finally, consider the system

$$A(r, s, t, u) \cdot \begin{pmatrix} q(m^2 - n^2) \\ 2pmn \\ m^2 + n^2 \end{pmatrix} = \begin{pmatrix} q(M^2 - N^2) \\ 2pMN \\ M^2 + N^2 \end{pmatrix} \quad (4.8)$$

with solution

$$M^2 = \frac{q}{p}(mr + ns)^2, \quad N^2 = \frac{q}{p}(mt + ns)^2. \quad (4.9)$$

This yields the following proposition.

Proposition. The results of this section are true only for the case

$$q = p \cdot \alpha^2 \quad (4.10)$$

with α a natural number. Then (4.1) becomes

$$x^2 + \alpha^4 y^2 = p^2 \alpha^4 z^2. \quad (4.11)$$

Obviously, for $p = q = 1$ we reobtain the results of the first section.

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