Nambu-Poisson description for a generalization of Volterra model

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Abstract

A Nambu-Poisson description of order three for a generalization of Volterra system is point out.

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The *Lotka-Volterra system*, or for short, the Volterra system, has been one of the most used model in nonlinear science because of its wide applicability. After a preliminary study of Lotka([4]), this model was introduced in 1931 by Volterra([7]) in order to describe population dynamics of competing species in biological science.

In [1], [2] the authors consider the following differential system:

$$\begin{cases} \dot{x}^i = \gamma^i \sum\limits_{m=1}^{p} \left(e^{x^{i+m}} - e^{x^{i-m}}\right) \\ 1 \leq i \leq n, 1 \leq p \leq \left[\frac{n-1}{2}\right], 3 \leq n \\ x^{i+n} = x^i \end{cases} \tag{1}$$

which admits some remarkable particular cases:

1) *Volterra system* for $\gamma_i = 1, p = 1$ and $x^i = \ln v^i$:

$$\dot{v}^i = v^i \left(v^{i+1} - v^{i-1}\right) \tag{2}$$

2) *Toda lattice system* for $\gamma_i = 1, p = 1, x^i = y^i - y^{i-1}$:

$$\dot{y}^i = e^{y^{i+1}-y^i} + e^{y^i - y^{i-1}} \tag{3}$$

3) *Bogoyavlensky lattice system* for $\gamma_i = 1, p \geq 1, x^i = \ln v^i$:

$$\dot{v}^i = v^i \sum_{m=1}^{p} \left(v^{i+m} - v^{i-m} \right). \tag{4}$$

The search of [1], [2] is for a Nambu-Poisson description for (1) with $p = 1$, more precisely for a Nambu-Poisson description of order n. Recall that an autonomous differential system:

$$\dot{x}^i = f^i(x) \tag{5}$$

admits a *Nambu-Poisson description of order* k if there exists $(k-1)$ first integrals $H_1, \ldots, H_{k-1}$ such that (5) can be write:

$$\dot{x}^i = \Lambda^{ii_1 \ldots i_{k-1}} \frac{\partial H_1}{\partial x^{i_1}} \cdots \frac{\partial H_{i_{k-1}}}{\partial x^{i_{k-1}}} \tag{6}$$

with $\Lambda = (\Lambda^{i_1 \ldots i_k})$ a k-vector field.

The authors does not give a general answer but point out that (1) with $p = 1$ has two first integrals:

$$\begin{cases} H_1 = \sum_{i=1}^{n} \frac{e^{x^i}}{\gamma_i} \\ H_2 = \sum_{i=1}^{n} \frac{x^i}{\gamma_i} \end{cases} \tag{7}$$

and for even n a third first integral:

$$H_3 = \frac{1}{2} \sum_{i=1}^{n} \frac{(-1)^i x^i}{\gamma_i}. \tag{8}$$

Let us remark that is known that Volterra model has a bi-Hamiltonian structure and for even case a tri-Hamiltonian structure is studied in [6].

In [5] is stated that if (5) admits two first integrals then (5) admits a Nambu-Poisson description of order three.

Therefore, we can conclude that always the system (1) with $p = 1$ admits a Nambu-Poisson description of order three. More precisely, we have that (1) with $p = 1$ can be write:

$$\dot{x}^i = \gamma_{i-1}\gamma_i\gamma_{i+1} \left(\nabla H_1\right)^t \cdot \overset{i}{\Lambda} \cdot \left(\nabla H_2\right) \tag{9}$$

where ∇H denotes the gradient of function H, $(\nabla H)^t$ denotes the transposition of matrix (∇H) and $\overset{i}{\Lambda}$ is the $n \times n$-matrix with only nonvanishing entries $\left(\overset{i}{\Lambda}\right)^{i-1}_{i+1} = -1, \left(\overset{i}{\Lambda}\right)^{i+1}_{i-1} = +1$ where the superscript denotes the row and subscript denotes the column.

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