Certain properties of a special foliation

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Abstract

Some properties of plaques for a special type of foliation are given.

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Let (M, F) be a foliated manifold; let us note that we use the notations and definitions of [3] but similar notions can be found in all books cited. The following result is obvious:

Lemma. *If α is a plaque defined by the foliated chart (U, φ) and β is a plaque defined by the foliated chart (V, ψ) such that $\alpha \cap \beta \neq \varnothing$, then $\alpha \cap \beta \subseteq \alpha \cap V$ and $\alpha \cap \beta \subseteq \beta \cap U$.*

The aim of this note is to study the properties related to plaques for a foliation which satisfy the "strong" condition:

$$\text{(C)} \qquad \alpha \cap \beta = \alpha \cap V = \beta \cap U.$$

A first result is:

Proposition 1. *If α and β are plaques defined by the foliated charts (U, φ), (U, ψ), that is $U = V$, then $\alpha = \beta$ or $\alpha \cap \beta = \varnothing$.*

Proof. If we suppose $\alpha \cap \beta \neq \varnothing$ then applying lemma it results $\alpha \cap \beta = \alpha \cap U = \beta \cap U$ and obviously $\alpha \cap U = \alpha, \beta \cap U = \beta$. □

Corollary. If α, β are plaques defined by the same foliated chart (U, φ), then $\alpha = \beta$ or $\alpha \cap \beta = \varnothing$.

The following result is motivated by the fact that a manifold with a global chart admits a foliation:

Proposition 2. *If the manifold M has a global foliated chart, then the leaf L_p passing through the point $p \in M$ is exactly the plaque α_p of p defined by the global chart.*

Proof. The inclusion $\alpha_p \subseteq L_p$ is obvious. Let $q \in L_p$. Therefore, there exists a chain of plaques $(\alpha_1, \dots, \alpha_k)$ from p to q. Because $p \in \alpha_p \cap \alpha_1$, applying the Corollary it results that $\alpha_1 = \alpha_p$. Because $\alpha_1 \cap \alpha_2 \neq \varnothing$, applying the same argument it follows

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$\alpha_2 = \alpha_p$. Then, at the end $\alpha_k = \alpha_p$ and therefore $q \in \alpha_k = \alpha_p$, which means $L_p \subseteq \alpha_p$. □

Let L be a leaf on (M, F), let $p, q \in L$ and let $(\alpha_1, \ldots, \alpha_k)$ be the *minimal* (i.e. $\alpha_i \neq \alpha_j$) chain of plaques from p to q with α_i defined by the foliated chart U_i. From $\alpha_i \cap \alpha_{i+1} \neq \varnothing$ it follows $U_i \cap U_{i+1} \neq \varnothing$. Let us suppose that there exists a natural number $s < k$ and $i \in \{1, \ldots, k-s\}$ such that $U_i = U_{i+s}$.

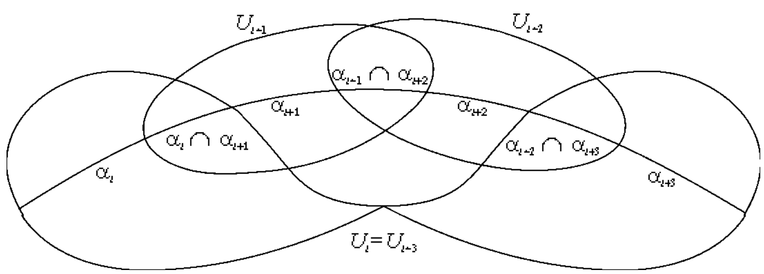
Proposition 3. *With above conditions fulfilled, one has* $s > 2$.

Proof. Case I. $s = 1$. Then for α_i, α_{i+1} we have $U_i = U_{i+1}$ and because $\alpha_i \cap \alpha_{i+1} \neq \varnothing$, applying the Corollary, it results $\alpha_i = \alpha_{i+1}$, contradiction with the minimality of the chain.

Case II. $s = 2$. From $U_i = U_{i+2}$ and the condition (C), it follows:

$$\alpha_i \cap \alpha_{i+1} = \alpha_{i+1} \cap U_i, \quad \alpha_{i+1} \cap \alpha_{i+2} = \alpha_{i+1} \cap U_{i+2} = \alpha_{i+1} \cap U_i,$$

and then $\alpha_i \cap \alpha_{i+1} = \alpha_{i+1} \cap \alpha_{i+2}$ which yields $\alpha_i \cap \alpha_{i+2} \neq \varnothing$. From the last relation and $U_i = U_{i+2}$, applying the Corollary we have $\alpha_i = \alpha_{i+2}$, contradiction with the minimality of the chain. □



Consider the fixed point x_0 in the leaf L and a natural number $k \neq 0$. Following [3, p. 11], let $A_{i_1 \ldots i_k}$ be the subset of L consisting of the points which can be joined to x_0 by a chain of plaques of $(\alpha_1, \ldots, \alpha_k)$ type. Suppose that α_i is defined by the foliated chart U_i. It is straightforward that

$$A_{i_1} = \begin{cases} \varnothing, & if \ x_0 \notin U_{i_1} \\ \alpha(x_0, i_1), & if \ x_0 \in U_{i_1}, \end{cases}$$

where $\alpha(x_0, i_1)$ is the plaque of $x_0 \in U_{i_1}$. Also, one can can prove, by induction, that $A_{i_1 \ldots i_k}$ is the set of plaques defined by U_{i_k} which meet the set $A_{i_1 \ldots i_{k-1}}$. In [3, p. 11] it is proved that $A_{i_1 \ldots i_k}$ is a union of plaques. In our framework we have

Proposition 4. *If* $A_{i_1 \ldots i_k} \neq \varnothing$, *then* $A_{i_1 \ldots i_k}$ *is a unique plaque.*

Proof. By induction after k. For $k = 1$ we have above the expression of A_{i_1}. Suppose that $A_{i_1 \ldots i_{k-1}}$ is exactly the plaque $\alpha_{i_{k-1}}$ and let $\alpha_k, \tilde{\alpha}_k$ be plaques in $A_{i_1 \ldots i_k}$. Then

$$\alpha_{i_{k-1}} \cap \alpha_{i_k} = \alpha_{i_{k-1}} \cap U_{i_k}, \quad \alpha_{i_{k-1}} \cap \tilde{\alpha}_k = \alpha_{i_{k-1}} \cap U_{i_k},$$

and then $\alpha_{i_{k-1}} \cap \alpha_{i_k} = \alpha_{i_{k-1}} \cap \tilde{\alpha}_{i_k}$, which yields $\alpha_{i_k} \cap \tilde{\alpha}_{i_k} \neq \varnothing$. Applying the Corollary, it results that $\alpha_{i_k} = \tilde{\alpha}_{i_k}$. □

References

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