Conservation laws generated by pseudosymmetries with applications to variational dynamical systems

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Abstract

In this paper we extend a result of Gerald L. Jones which give conservation laws for ordinary differential equations. Applications to Hamiltonian and Lagrangian systems are given.

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Introduction

The very well-known way to obtain conservation laws for a system of differential equations is Noether theorem which associates to every symmetry a conservation law. G. L. Jones gives in [3] another method based on a weaker generalization of notion of symmetry, namely *pseudosymmetries.* The advantages of this method are that it does not require any integration(if there are associated some natural invariants, see the Hamiltonian and Lagrangian case below) and does not require, as Noether theorem, that the differential equations follow from a variational principle.

In this paper we present a generalization of Jones result, in the first section, and applications to a special type of divergence-free(i.e. solenoidal) vector fields and to Hamiltonian and Lagrangian systems in the following two sections. As example, the 2-dimensional isotropic harmonic oscillator is treated.

1 From pseudosymmetries to conservation laws

Let M be a smooth, n-dimensional manifold, $C^{\infty}(M)$ the ring of real-valued smooth functions, $\mathcal{X}(M)$ the Lie algebra of vector fields and $\Omega^{p}(M)$ the $C^{\infty}(M)$-module of p-differential forms, $1 \leq p \leq n$.

For $X \in \mathcal{X}(M)$ with local expression

$$(1.1) \qquad X = X^{i}(x) \frac{\partial}{\partial x^{i}}$$

one consider the system of differential equations which give the flow of X

$$(1.2) \qquad \dot{x}^{i}(t) = \frac{dx^{i}}{dt}(t) = X^{i}\left(x^{1}(t), \ldots, x^{n}(t)\right), \qquad i = 1, \ldots, n.$$

A solution of (1.2) is called *integral curve* of X.

Definition 1.1 A function $f \in C^{\infty}(M)$ is called *conservation law*(or *first integral,* or *constant of motion,* or *invariant function*) for X or (1.2) if f is constant along the solutions of (1.2) that is

$$(1.3) \qquad \frac{d(f \circ c)}{dt}(t) = 0$$

for every integral curve $c(t)$ of X.

Because for $f \in C^{\infty}(M)$ its rate of change along (1.2) is

$$(1.4) \qquad \frac{df}{dt} = \frac{\partial f}{\partial x^{i}} \dot{x}^{i} = \frac{\partial f}{\partial x^{i}} X^{i} = \mathcal{L}_{X} f$$

where the right-hand side means *the Lie derivative of f with respect to X* we get

Proposition 1.2 *$f \in C^{\infty}(M)$ is conservation law for (1.2) if and only if*

$$(1.5) \qquad \mathcal{L}_{X} f = 0.$$

For our approach is necessary the following

Definition 1.3 (i) $Y \in \mathcal{X}(M)$ is called *symmetry for X* if

$$(1.6) \qquad \mathcal{L}_{X} Y = 0.$$

(ii) If $Y \in \mathcal{X}(M)$ is fixed then $Z \in \mathcal{X}(M)$ is called *Y-pseudosymmetry for X* if there exists $f \in C^{\infty}(M)$ such that

$$(1.7) \qquad \mathcal{L}_{X} Z = fY.$$

(iii) $\omega \in \Omega^p(M)$ is called *invariant form for* X if

$$(1.8) \quad \mathcal{L}_X \omega = 0.$$

Remark 1.4 (i) A 0-pseudosymmetry is obviously a symmetry.
(ii) A X-pseudosymmetry for X is called *pseudosymmetry for* X in ([3, p. 1055]).
(iii) If in (1.7) f is constant then $\mathcal{L}_X Z$ is symmetry for X.
(iv) If in (1.7) f is not constant then $\mathcal{L}_X Z$ is symmetry for X if and only if f is conservation law for X.

The result which give the association between pseudosymmetries and conservation laws is

Theorem 1.5 *Let* $X \in \mathcal{X}(M)$ *be a fixed vector field and* $\omega \in \Omega^p(M)$ *be a* p*-form invariant for* X*. If* $Y \in \mathcal{X}(M)$ *is symmetry for* X *and* $S_1, \ldots, S_{p-1} \in \mathcal{X}(M)$ *are* $(p-1)$ Y*-pseudosymmetries for* Y *then*

$$(1.9) \quad \phi = \omega(S_1, \ldots, S_{p-1}, Y)$$

or locally

$$(1.10) \quad \phi = S_1^{i_1} \ldots S_{p-1}^{i_{p-1}} Y^{i_p} \omega_{i_1 i_2 \ldots i_p}$$

is a conservation law for X*. Particularly, if* $Y, S_1, \ldots, S_{p-1}$ *are symmetries for* X *then* ϕ *given by* (1.9) *is conservation law.*

Proof Applying the properties of Lie derivatives one have

$$\mathcal{L}_X \phi = (\mathcal{L}_X S_1)^{i_1} \ldots + S_1^{i_1} (\mathcal{L}_X S_2)^{i_2} \ldots + \ldots + \ldots (\mathcal{L}_X Y)^{i_p} \omega_{\ldots} + \ldots Y^{i_p} (\mathcal{L}_X \omega)_{i_1 \ldots i_p}.$$

In this relation each of the first $p-1$ terms has a factor of the form

$$\mathcal{L}_X S_j = \lambda_j Y$$

so that ω is contracted with two factors of Y and then each term vanishes by the antisymmetry of ω. The p-th term and $p+1$-th term vanishes since (1.6) and (1.8). □

Remark 1.6 (i) If the pseudosymmetries are linearly dependent then

$$(1.11) \quad \phi = 0$$

by the antisymmetry of ω.
(ii) For $Y = X$ one obtain the main result of G. L. Jones([3, p. 1056]).
(iii) If $p = 1$ one obtain theorem 2.5.10 of ten Eikelder([2, p. 48]).
(iv) The fact that the pseudosymmetries (1.7) with f =constant can be used to integrate planar($n = 2$) vector fields can be found in [7, p. 37-38, relation 8.13].

2 Applications to Hamiltonian systems

2.1 Vector fields with a special invariant 2-form

Let $M = \mathbf{R}^{2m} = \{(x,y) = (x^i, y^i)_{i=1,\ldots,m}\}$ that is $n = 2m$, and let X with the form

$$(2.1.1) \qquad X = X^i(x,y)\frac{\partial}{\partial x^i} + \tilde{X}^i(x,y)\frac{\partial}{\partial y^i}.$$

Let us consider the 2-form $\omega = (\omega_{ij})$ given by

$$(2.1.2) \qquad \omega = \begin{pmatrix} 0_m & 1_m \\ -1_m & 0_m \end{pmatrix}$$

where 0_m is the null matrix and 1_m is the identity matrix of order m.

A straightforward computation give

$$(2.1.3a) \qquad (\mathcal{L}_X\omega)\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \frac{\partial \tilde{X}^i}{\partial x^j} - \frac{\partial \tilde{X}^j}{\partial x^i}$$

$$(2.1.3b) \qquad (\mathcal{L}_X\omega)\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = \frac{\partial X^j}{\partial y^i} - \frac{\partial X^i}{\partial y^j}$$

$$(2.1.3c) \qquad (\mathcal{L}_X\omega)\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j}\right) = \frac{\partial X^j}{\partial x^i} + \frac{\partial \tilde{X}^i}{\partial y^j}$$

$$(2.1.3d) \qquad (\mathcal{L}_X\omega)\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial x^j}\right) = -\frac{\partial X^i}{\partial x^j} - \frac{\partial \tilde{X}^j}{\partial y^i}.$$

As consequence we obtain

Proposition 2.1.1 *If X with expression* (2.1.1) *satisfy*

$$(2.1.4a) \qquad \frac{\partial X^i}{\partial y^j} = \frac{\partial X^j}{\partial y^i}$$

$$(2.1.4b) \qquad \frac{\partial \tilde{X}^i}{\partial x^j} = \frac{\partial \tilde{X}^j}{\partial x^i}$$

$$(2.1.4c) \qquad \frac{\partial X^j}{\partial x^i} + \frac{\partial \tilde{X}^i}{\partial y^j} = 0$$

for all $i, j = 1, \ldots, m$ then ω is 2-form invariant for X.

Applying the theorem 1.5 we have

Proposition 2.1.2 *If X satisfy* $(2.1.4a)-(2.1.4c)$, $Y\in\mathcal{X}(M)$ *is symmetry for X and $S\in\mathcal{X}(M)$ is Y-pseudosymmetry for X then*

$$(2.1.5) \quad \phi=\omega(S,Y)$$

is a conservation law for X where ω is given by (2.1.2). *Particularly if Y,S are symmetries for X then ϕ given by* (2.1.5) *is conservation law.*

If $Y=Y^i\frac{\partial}{\partial x^i}+\tilde{Y}^i\frac{\partial}{\partial y^i}$ and $S=S^i\frac{\partial}{\partial x^i}+\tilde{S}^i\frac{\partial}{\partial y^i}$ then

$$\phi=\left(S,\tilde{S}\right)\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}Y\\\tilde{Y}\end{pmatrix}=\left(-\tilde{S},S\right)\begin{pmatrix}Y\\\tilde{Y}\end{pmatrix}=S\tilde{Y}-\tilde{S}Y$$

that is

$$(2.1.6) \quad \phi=S^i\tilde{Y}^i-\tilde{S}^iY^i$$

with summation after $i=1,\dots,m$.

Corollary 2.1.3 *If X satisfy* $(2.1.4a)-(2.1.4c)$ *and $S\in\mathcal{X}(M)$ is a pseudosymmetry for X then*

$$(2.1.7) \quad \phi=\omega(S,X)=S^i\tilde{X}^i-\tilde{S}^iX^i$$

is a conservation law for X.

Remark 2.1.4 (i) If in relation $(2.1.4c)$ one makes $i=j$ and take the sum after $i=1,\dots,m$ then *the divergence of X* vanishes

$$(2.1.8) \quad divX:=\sum_{i=1}^{m}\left(\frac{\partial X^i}{\partial x^i}+\frac{\partial\tilde{X}^i}{\partial y^i}\right)=0$$

that is X is *divergence-free(*or *solenoidal* or *source-free*) vector field.

(ii) Every solenoidal vector field in dimension $n=2$, that is $m=1$, satisfy $(2.1.4a)-(2.1.4c)$. Applications for divergenceless vector fields in a odd dimension, namely $n=3$, are given in [3, p. 1056].

(iii) If one consider the tensor field J of type $(1,1)$ with the same matrix as ω then

$$(2.1.9) \quad J^2=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}0&1\\-1&0\end{pmatrix}=\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$$

that is J is *an almost complex structure on M*.

2.2 Applications to Hamiltonian systems in $\mathbf{R}^{2m}$

Let X be given by (2.1.1).

Definition 2.2.1 X is called *Hamiltonian vector field* if there exists $H \in C^{\infty}(M)$, usually called *Hamiltonian,* such that the flow system (1.2) of X is exactly the system of Hamilton equations for H, i.e.

$$(2.2.1a) \quad \dot{x}^i = X^i = \frac{\partial H}{\partial y^i}$$

$$(2.2.1b) \quad \dot{y}^i = \tilde{X}^i = -\frac{\partial H}{\partial x^i}.$$

In other words X is *the gradient of* H *with respect to* ω that is

$$(2.2.2) \quad X^a = \omega^{ab}\frac{\partial H}{\partial x^a}$$

where $a = 1, \ldots, 2m$, $X^{m+i} = \tilde{X}^i$ and $x^{m+i} = y^i$.

The key result is

Proposition 2.2.2 *If* X *is Hamiltonian vector field then* X *satisfy* $(2.1.4a) - (2.1.4c)$. *The Hamilton equations* $(2.2.1a) - (2.2.1b)$ *can be write*

$$(2.2.3) \quad \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \omega \cdot \begin{pmatrix} \frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial y} \end{pmatrix}$$

with ω *given by* (2.1.2).

Corollary 2.2.3 *A Hamiltonian vector field is solenoidal.*

Applying remark 2.1.4(ii) we get

Proposition 2.2.4([7, p. 45]) *If* $n = 2$ *then* $X \in \mathcal{X}(M)$ *is Hamiltonian if and only if is solenoidal.*

One can apply the results of previously section and one obtain

Proposition 2.2.4 *Let* X *be a Hamiltonian vector field and* $Y \in \mathcal{X}(M)$ *a symmetry for* X. *If* $S \in \mathcal{X}(M)$ *is* Y*-pseudosymmetry for* X *then*

$$(2.2.4) \quad \phi = S^i\tilde{Y}^i - \tilde{S}^iY^i$$

is a conservation law for the Hamiltonian system (2.2.1) *or* (2.2.2). *Particularly, if* Y, S *are symmetries for* X *then* ϕ *is conservation law.*

Corollary 2.2.5(Jones) *If* X *is Hamiltonian vector field and* $S \in \mathcal{X}(M)$ *is pseudosymmetry for then*

$$(2.2.5) \quad \phi = -S^i\tilde{X}^i + \tilde{S}^iX^i = S^i\frac{\partial H}{\partial x^i} + \tilde{S}^i\frac{\partial H}{\partial y^i} = \mathcal{L}_S H$$

is a conservation law for the associate Hamiltonian system.

Remark 2.2.6

(i) G. L. Jones obtains the last result in [3, p. 108] using the properties of *canonical transformations* for Hamiltonian systems. Our approach does not require these properties.

(ii) If S is a symmetry for the Hamiltonian H, that is

$$(2.2.6) \quad \mathcal{L}_S H = 0$$

then ϕ given by (2.2..5) vanishes and the method is ineffectual. Then to get a non-zero conservation law one must find a pseudosymmetry of X which is not a symmetry for H or to apply proposition 2.2.4.

2.3 Applications to general Hamiltonian systems

Let (M, ω) be a symplectic manifold of dimension $n = 2m$.

Definition 2.3.1 $X \in \mathcal{X}(M)$ is said to be a *Hamiltonian vector field* if there exists $H \in C^\infty(M)$ such that

$$(2.3.1) \quad i_X \omega = -dH$$

i.e. X is the ω-dual of H, which is called a *Hamiltonian.*

The following version of proposition 2.2.2 is classical.

Proposition 2.3.2 *If X is Hamiltonian vector field on the symplectic manifold (M, ω) then ω is invariant 2-form for X.*

Proof

$$\mathcal{L}_X \omega = di_X \omega + i_X d\omega = di_X \omega = d(-dH) = 0. \qquad \square$$

Because ω is nondegenerate

$$(2.3.2) \quad \Omega := \omega^m$$

is a volume form on M. Then for every $X \in \mathcal{X}(M)$ its *divergence* is the function $div_\omega X \in C^\infty(M)$ uniquely determined by

$$(2.3.3) \quad \mathcal{L}_X \Omega = (div_\omega X)\, \Omega.$$

Applying proposition 2.3.2 one obtain the second part of proposition 2.2.2.

Proposition 2.3.3 *The divergence of a Hamiltonian vector field vanishes.*

One can apply the main result of section 1.

Proposition *Let X be a Hamiltonian vector field and $Y \in \mathcal{X}(M)$ a symmetry of X. If $S \in \mathcal{X}(M)$ is a Y-pseudosymmetry of X then*

$$(2.3.4) \qquad \phi = \omega(S, Y)$$

is a conservation law for the Hamiltonian flow defined by X.

3 Applications to Lagrangian systems

3.1 Applications to regular Lagrangian systems

The usual framework of Hamiltonian systems is the cotangent bundle T^*Q of a m-dimensional manifold Q. Using the Legendre transformation we obtain the Lagrangian point of view, which will be developed in this section.

Let Q be a m-dimensional manifold with TQ the tangent bundle and one consider a smooth function $L : TQ \to \mathbf{R}$ usually called *Lagrangian*.

On TQ an important structure is the $C^\infty(TQ)$-linear mapping([4, p. 108]) $J : \mathcal{X}(TQ) \to \mathcal{X}(TQ)$

$$(3.1.1) \qquad J\left(\frac{\partial}{\partial q^i}\right) = \frac{\partial}{\partial \dot{q}^i}, \quad J\left(\frac{\partial}{\partial \dot{q}^i}\right) = 0$$

or, equivalently

$$(3.1.2) \qquad J = \frac{\partial}{\partial \dot{q}^i} \otimes dq^i$$

where $(q^i) = q$ are the coordinates on Q and $(q^i, \dot{q}^i) = (q, \dot{q})$ the associated coordinates on TQ.

To the Lagrangian L we can associate two forms, usually called *Cartan forms*([1, p. 346])

$$(3.1.3) \qquad \theta_L = J^*(dL)$$

where J^* is the adjoint of structure J given by (3.1.1) and

$$(3.1.4) \qquad \omega_L = d\theta_L.$$

In local coordinates

$$(3.1.5) \qquad \theta_L = \frac{\partial L}{\partial \dot{q}^i} dq^i$$

$$(3.1.6) \qquad \omega_L = \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j} dq^j \wedge dq^i + \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} d\dot{q}^i \wedge dq^j.$$

Recall that the L-extremals i.e. the extremals of the action

$$(3.1.7) \qquad S = \int L\left(q\left(t\right), \dot{q}\left(t\right)\right) dt$$

are the solutions of *Euler-Lagrange equations*([4, p. 160])

$$(3.1.8) \qquad E_i\left(L\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i}, \quad i = 1, \ldots, m.$$

If in Euler-Lagrange equations we compute the derivative with respect to time then

$$(3.1.9) \qquad \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\ddot{q}^j + \frac{\partial^2 L}{\partial \dot{q}^i \partial q^k}\dot{q}^k - \frac{\partial L}{\partial q^i} = 0.$$

If we wish to write this equations in *the normal form,* that is the second derivatives appear explicitly, it must to consider

$$(3.1.10) \qquad g_{ij} = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}$$

which is called *the metric of* L and (3.1.9) becomes:

$$(3.1.11) \qquad g_{ij}\ddot{q}^j + \frac{\partial^2 L}{\partial \dot{q}^i \partial q^k}\dot{q}^k - \frac{\partial L}{\partial q^i} = 0.$$

Definition 3.1.1 The Lagrangian L is called *regular* or *nondegenerate* if the metric tensor (g_{ij}) is invertible that is

$$(3.1.12) \qquad rang\left(g_{ij}\right) = n$$

or equivalently

$$(3.1.13) \qquad \det\left(g_{ij}\right) \neq 0.$$

If L is regular denote by (g^{ij}) the inverse matrix of (g_{ij}). By multiplication of (3.1.11) with (g^{ij}) we get

Proposition 3.1.2 *If the Lagrangian L is regular then the L-extremals are solutions of equations*

$$(3.1.14) \qquad \ddot{q}^i + G^i = 0, \quad i = 1, \ldots, m$$

where

$$(3.1.15) \qquad G^i = g^{ij}\left(\frac{\partial^2 L}{\partial \dot{q}^j \partial q^k}\dot{q}^k - \frac{\partial L}{\partial q^j}\right).$$

But the systems (3.1.14) are exactly the flow system for the vector field $S \in \mathcal{X}(TQ)$ with

$$(3.1.16) \quad S = \dot{q}^i \frac{\partial}{\partial q^i} - G^i \frac{\partial}{\partial \dot{q}^i}$$

which is called *the canonical semispray of* L.

This semispray generates a *nonlinear connection* on Q that is a distribution N on Q with([4, p. 107])

$$(3.1.17) \quad TTQ = N \oplus V(TQ)$$

where $V(TQ)$ is *the vertical distribution* of Q. Recall that V has as basis the vector fields $\left(\frac{\partial}{\partial \dot{q}^i}\right)$ and that a basis for N is given by the vector fields $\left(\frac{\delta}{\delta q^i}\right)$ where([4, p. 108])

$$(3.1.18) \quad \frac{\delta}{\delta q^i} = \frac{\partial}{\partial q^i} - N_i^j \frac{\partial}{\partial \dot{q}^j}$$

with

$$(3.1.19) \quad N_i^j = \frac{1}{2} \frac{\partial G^j}{\partial \dot{q}^i}.$$

Let $(dq^i, \delta\dot{q}^i)$ be the dual basis of $\left(\frac{\delta}{\delta q^i}, \frac{\partial}{\partial \dot{q}^i}\right)$ where

$$(3.1.20) \quad \delta\dot{q}^i = d\dot{q}^i + N_j^i dq^j.$$

With respect to this basis one have

$$(3.1.21) \quad \omega_L = g_{ij} \delta\dot{q}^i \wedge dq^j$$

or, in matrix notation

$$(3.1.22) \quad \omega_L = \begin{pmatrix} 0 & -g_{ij} \\ g_{ij} & 0 \end{pmatrix}.$$

Returning to canonical semispray S one have another characterization, namely([1, p. 347])

$$(3.1.23) \quad i_S \omega_L = -dH$$

where i_S denotes the interior product with respect to S and H is *the energy* of L

$$(3.1.24) \quad H = \frac{\partial L}{\partial \dot{q}^i} \dot{q}^i - L.$$

This characterization yields a very important and well-known result in the theory of autonomous Lagrangian systems:

Theorem 3.1.3(Conservation of energy for autonomous Lagrangian systems) *If the Lagrangian L is time-independent then the energy H is a conservation law for L.*

Proof We have

$$\mathcal{L}_S H = dH(S) = -(i_S\omega_L)(S) = \omega_L(S,S) = 0. \qquad \Box$$

A straightforward computation give

$$\mathcal{L}_S\theta_L = di_S\theta_L + i_S d\theta_L = d < \theta_L, S > + i_S\omega_L = d\left(\frac{\partial L}{\partial \dot{q}^i}\dot{q}^i\right) - dH = d\left(\frac{\partial L}{\partial \dot{q}^i}\dot{q}^i - H\right) = dL$$

and

$$\mathcal{L}_S\omega_L = \mathcal{L}_S d\theta_L = d\mathcal{L}_S\theta_L = d(dL) = 0$$

that is

Proposition 3.1.4([1, p. 348, ex. 56]) *If the Lagrangian L is regular then the Cartan 2-form is invariant for the canonical semispray S.*

Applying the results of section 1 we obtain

Proposition 3.1.5 *Let L be a regular Lagrangian and $Y \in \mathcal{X}(TQ)$ be a symmetry of the canonical semispray S. If $Z \in \mathcal{X}(TQ)$ is a Y-pseudosymmetry for S then*

$$(3.1.25) \qquad \phi = \omega_L(Y, Z)$$

is a conservation law for the Lagrangian L that is a conservation law for Euler-Lagrange equations, equivalently for equations (3.1.14)*. Particularly, if Y and Z are symmetries for the canonical spray S then ϕ given by* (3.1.25) *is conservation law for L.*

If

$$Y = Y^i\frac{\delta}{\delta q^i} + \tilde{Y}^i\frac{\partial}{\partial \dot{q}^i}$$

$$Z = Z^i\frac{\delta}{\delta q^i} + \tilde{Z}^i\frac{\partial}{\partial \dot{q}^i}$$

then (3.1.25) becomes

$$(3.1.26) \qquad \phi = \left(Y^i, \tilde{Y}^i\right)\begin{pmatrix} 0 & -g_{ij} \\ g_{ij} & 0 \end{pmatrix}\begin{pmatrix} Z^j \\ \tilde{Z}^j \end{pmatrix} = g_{ij}\tilde{Y}^i Z^j - g_{ij}Y^i\tilde{Z}^j.$$

Corollary 3.1.6 *If the Lagrangian* L *is regular and* $Z \in \mathcal{X}(TQ)$ *is pseudosymmetry for the canonical semispray* S *then*

$$\phi = \omega_L(S,Z) = -\mathcal{L}_Z H \tag{3.1.27}$$

is a conservation law for the Lagrangian.

Remark that

(i) If in (3.1.27) we take $Z = S$ we obtain $\phi = 0$.

(ii)

$$S = \dot{q}^i \frac{\partial}{\partial q^i} - G^i \frac{\partial}{\partial \dot{q}^i} = \dot{q}^i \left(\frac{\delta}{\delta q^i} + N_i^k \frac{\partial}{\partial \dot{q}^k} \right) - G^k \frac{\partial}{\partial \dot{q}^k} = \dot{q}^i \frac{\delta}{\delta q^i} + \left(\dot{q}^i N_i^k - G^k \right) \frac{\partial}{\partial \dot{q}^k} \tag{3.1.28}$$

and then (3.1.26) and (3.1.27) give

$$\phi = g_{ij} \left(\dot{q}^k N_k^i - G^i \right) Z^j - g_{ij} \dot{q}^i \tilde{Z}^j. \tag{3.1.29}$$

Definition 3.1.7 The semispray S is called *spray* if the functions (G^i) are 2-positive homogeneous with respect to velocity that is

$$G^i(q, \lambda \dot{q}) = \lambda^2 G^i(q, \dot{q}), \quad i = 1, \ldots, m, \forall \lambda > 0. \tag{3.1.30}$$

Applying (3.1.19) and Euler Theorem on homogeneous functions we get

Proposition 3.1.8 *The semispray is spray if and only if*

$$G^k = \dot{q}^i N_i^k. \tag{3.1.31}$$

Applying (3.1.28) we obtain

Corollary 3.1.9 *If* S *is spray then*

$$S = \dot{q}^i \frac{\delta}{\delta q^i} \tag{3.1.32}$$

and ϕ *given by* (3.1.29) *is*

$$\phi = g_{ij} \dot{q}^i \tilde{Z}^j. \tag{3.1.33}$$

Another characterization of sprays is

Proposition 3.1.10([4, p. 112]) *A semispray* S *is spray if and only if*

$$\left[S, \dot{q}^i \frac{\partial}{\partial \dot{q}^i} \right] = S. \tag{3.1.34}$$

But relation (3.1.34) means that $\Upsilon = \dot{q}^i \frac{\partial}{\partial \dot{q}^i}$ is a pseudosymmetry for S and then relation (3.1.33) yields

Proposition 3.1.11 *If the Lagrangian L is regular and S is spray then*

$$(3.1.35) \quad \phi = g_{ij} \dot{q}^i \dot{q}^j$$

is a conservation law for L.

The expression $\mathcal{E}(g) = \frac{1}{2} g_{ij} \dot{q}^i \dot{q}^j$ is usually called *the kinetic energy of metric g_{ij}*. Then the proposition 3.1.11 give conservation of kinetic energy.

A class of Lagrangians which generates sprays is given by

Proposition 3.1.12 *If the Lagrangian L is regular and r-positively homogeneous with respect to velocity then G^i are 2-positively homogeneous i.e. S is spray for any r.*

The most important cases are([4])

(i) Finslerian case, $r = 2$

(ii) Riemannian case, $r = 2$.

But for a r-positively homogeneous Lagrangian L a straightforward computation give

$$(3.1.36) \quad H = (r-1) L$$

$$(3.1.37) \quad r(r-1) L = 2\mathcal{E}(g)$$

that is

$$(3.1.38) \quad rH = 2\mathcal{E}(g)$$

and then these relations via the theorem 3.1.3 give another proof of conservation of kinetic energy for r-positively homogeneous Lagrangians with $r \neq 0$.

3.2 Conservation of a kinetic energy for weak regular generalized Lagrangian metrics

Definition 3.2.1 A d-tensor field of type (r, s) on TM is a set of functions $\left(T^{i_1 \ldots i_r}_{j_1 \ldots j_r} (q, \dot{q}) \right)$ locally defined on TM such that with respect to a change of coordinates $(q^i) \to (\tilde{q}^i)$ on M we have

$$(3.2.1) \quad T^{i_1 \ldots i_r}_{j_1 \ldots j_s} (q, \dot{q}) = \frac{\partial q^{i_1}}{\partial \tilde{q}^{a_1}} \cdots \frac{\partial q^{i_r}}{\partial \tilde{q}^{a_r}} \frac{\partial \tilde{q}^{b_1}}{\partial q^{j_1}} \cdots \frac{\partial \tilde{q}^{b_s}}{\partial q^{j_s}} T^{a_1 \ldots a_r}_{b_1 \ldots b_s} \left(\tilde{q}, \dot{\tilde{q}} \right)$$

which is just the classical law of transformation of the local components of a tensor field of type (r, s) on the base manifold M.

Professor R. Miron give the following generalization of Riemannian metrics

Definition 3.2.2 A d-tensor field of type $(0,2)$ $g = (g_{ij}(q,\dot{q}))$ is called *generalized Lagrange metric,* shortly a *GL-metric* if ([4, p. 181]):

$$(3.2.2) \quad g_{ij} = g_{ji} \quad (symmetry)$$

$$(3.2.3) \quad \det(g_{ij}) \neq 0 \quad (non-degeneracy)$$

$$(3.2.4) \quad thequadraticformg_{ij}\xi^i\xi^j \ \ (\xi \in \mathbf{R}^n)\, hasconstantsignature.$$

Example If g does not depend on $\dot{q}$ then g is a Riemannian metric.

To a GL-metric we associate *the kinetic energy*

$$(3.2.5) \quad \mathcal{E}(g) = \frac{1}{2} g_{ij}\dot{q}^i\dot{q}^j.$$

As in Riemannian case we define

Definition 3.2.3 An $\mathcal{E}(g)$-extremal is called *geodesic of g*.

Remark 3.2.4 A Lagrangian L yields

$$L \to g = (g_{ij}) \to \mathcal{E}(g).$$

Then if g is induced by a Lagrangian it is possible that the L-extremals are different of $\mathcal{E}(g)$-extremals but there are GL-metrics which are not Lagrangian metrics([4])and then we can consider only $\mathcal{E}(g)$-extremals.

The equations of geodesics of g are

$$(3.2.6) \quad g^*_{ij}\ddot{q}^j + \left(\frac{\partial^2\mathcal{E}(g)}{\partial\dot{q}^i\partial\dot{q}^k}\dot{q}^k - \frac{\partial\mathcal{E}(g)}{\partial q^i}\right) = 0$$

where

$$(3.2.7) \qquad g^*_{ij}(q,\dot{q}) = \frac{\partial^2\mathcal{E}(g)}{\partial\dot{q}^i\partial\dot{q}^j} = g_{ij} + \frac{1}{2}\frac{\partial^2 g_{ab}}{\partial\dot{q}^i\partial\dot{q}^j}\dot{q}^a\dot{q}^b + \left(\frac{\partial g_{ia}}{\partial\dot{q}^j} + \frac{\partial g_{ja}}{\partial\dot{q}^i}\right)\dot{q}^a$$

These equations lead to:

Definition 3.2.5 ([4, p. 183]) A GL-metric $g(q,\dot{q})$ is said to be with *weakly regular metric* if $\mathcal{E}(g)$ is a regular Lagrangian.

Then $g^* = \left(g^*_{ij}\right)$ is regular matrix and the semispray equations for $\mathcal{E}(g)$ become:

$$(3.2.8) \qquad \ddot{q}^i + G^i(q,\dot{q}) = 0$$

where $G = (G^i)$ is called *the canonical spray of* (M, g):

$$(3.2.9) \qquad G^i = g^{*ij}\left(\frac{\partial^2 \mathcal{E}(g)}{\partial \dot{q}^j \partial q^k}\dot{q}^k - \frac{\partial \mathcal{E}(g)}{\partial q^j}\right).$$

An important particular case of weakly regular GL-metrics is

Definition 3.2.6(R. Miron) A weakly regular GL-metric is said to be *regular GL-metric* if

$$(3.2.10) \qquad \frac{\partial \mathcal{E}(g)}{\partial \dot{q}^i} = g_{ij}\dot{q}^j.$$

Proposition 3.2.7 *Let* L *be a Lagrangian and* $g = \left(g_{ij} = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\right)$ *the associated GL-metric. Suppose that* L *is* r*-positively homogeneous in velocity with* $r \notin \{0,1\}$*. Then:*
(i) The GL-metric is weakly regular metric .
(ii) g *is regular metric if and only if* $r = 2$*.*

Proof *(i)* $\frac{\partial L}{\partial \dot{q}^i}\dot{q}^i = rL \Longrightarrow g_{ij}\dot{q}^i = (r-1)\frac{\partial L}{\partial \dot{q}^j} \Longrightarrow \mathcal{E}(g) = \frac{1}{2}g_{ij}\dot{q}^i\dot{q}^j = \frac{1}{2}(r-1)\frac{\partial L}{\partial \dot{q}^i}\dot{q}^i = \frac{1}{2}r(r-1)L \Longrightarrow g^* = \frac{1}{2}r(r-1)g$
Because L is regular the d-tensor g is regular and then g^* is regular.
(ii) $\frac{\partial \mathcal{E}}{\partial \dot{q}^i} = \frac{1}{2}r(r-1)\frac{\partial L}{\partial \dot{q}^i} = \frac{1}{2}rg_{ij}\dot{q}^j$ and then (3.2.10) hold if and only if $r = 2$.
□

Example 3.2.8 A Riemannian(more generally Finsler) metric is a regular GL-metric.

Proposition 3.2.9 *If* g *is a regular GL-metric then* $\mathcal{E}(g)$ *is* 2*-positively homogeneous in velocity.*

Proof We have

$$\frac{\partial \mathcal{E}(g)}{\partial \dot{q}^i} = g_{ij}\dot{q}^j \Longrightarrow \frac{\partial \mathcal{E}(g)}{\partial \dot{q}^i}\dot{q}^i = g_{ij}\dot{q}^i\dot{q}^j = 2\mathcal{E}(g)$$

and Euler's theorem give the conclusion. □

The "total" energy of a GL-metric is the energy of the kinetic energy

$$(3.2.11) \qquad H = \frac{\partial \mathcal{E}(g)}{\partial \dot{q}^i}\dot{q}^i - \mathcal{E}(g).$$

A straightforward computation give

$$(3.2.12) \qquad H = \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial \dot{q}^a}\dot{q}^a + g_{ij}\right)\dot{q}^i\dot{q}^j$$

This relation becomes more interesting if the GL-metric g is r-positively homogeneous. Then

$$(3.2.13)\quad \frac{\partial \mathcal{E}(g)}{\partial \dot{q}^i}\dot{q}^i = \frac{1}{2}\frac{\partial g_{ab}}{\partial \dot{q}^i}\dot{q}^a\dot{q}^b\dot{q}^i + g_{ij}\dot{q}^i\dot{q}^j = \frac{r}{2}g_{ab}\dot{q}^a\dot{q}^b + g_{ij}\dot{q}^i\dot{q}^j = (r+2)\,\mathcal{E}(g)\,.$$

Proposition 3.2.10 *If the GL-metric g is r-positively homogeneous then*

$$(5.14)\quad \mathcal{E}(g)\, is r + 2 positively homogeneous$$

$$(5.15)\quad H = (r+1)\,\mathcal{E}(g)\,.$$

By (3.2.14) and proposition 3.2.9 it result

Corollary 3.2.11 *If the GL-metric g is r-positively homogeneous and regular then $r = 0$.*

Returning to a weakly regular metric $g = (g_{ij})$ and applying the results of previous section we get

Proposition 3.2.12 *If the kinetic energy of the weakly regular GL-metric g generates a spray then*

$$(3.2.16)\quad \mathcal{E}(g^*) = \frac{1}{2}g^*_{ij}\dot{q}^i\dot{q}^j = \mathcal{E}(g) + \frac{1}{4}\frac{\partial^2 g_{ab}}{\partial \dot{q}^i \partial \dot{q}^j}\dot{q}^a\dot{q}^b\dot{q}^i\dot{q}^j + \frac{1}{2}\left(\frac{\partial g_{ia}}{\partial \dot{q}^j} + \frac{\partial g_{ja}}{\partial \dot{q}^i}\right)\dot{q}^a\dot{q}^i\dot{q}^j$$

is constant along the geodesics of g.

By propositions 3.2.9 and 3.1.11 the kinetic energy of a regular GL-metric generates spray and in this case

$$(3.2.17)\quad g^*_{ij} = \frac{1}{2}\left(\frac{\partial g_{ia}}{\partial \dot{q}^j} + \frac{\partial g_{ja}}{\partial \dot{q}^i}\right)\dot{q}^a + g_{ij}$$

$$(3.2.18)\quad \mathcal{E}(g^*) = \mathcal{E}(g) + \frac{1}{4}\left(\frac{\partial g_{ia}}{\partial \dot{q}^j} + \frac{\partial g_{ja}}{\partial \dot{q}^i}\right)\dot{q}^a\dot{q}^i\dot{q}^j.$$

3.3 From conservation laws to exact Cartan symmetries. Classical Noether theorem

Returning to the framework of subsection 3.1 let for the Lagrangian system (M, L) the following type of symmetry:

Definition 3.3.1([1, p. 349]) $X \in \mathcal{X}(TM)$ is called *Cartan symmetry for* (M, L) if

$$(3.3.1)\quad \mathcal{L}_X \omega_L = 0$$

and

$$(3.3.2)\quad \mathcal{L}_X H = 0.$$

Proposition 3.3.2 *The canonical spray is Cartan symmetry.*

Proposition 3.3.3([1, p. 349]) *If (M, L) is a Lagrange space and $X \in \mathcal{X}(TM)$ is Cartan symmetry for L then X is symmetry for the canonical spray of L that is*

$$(3.3.3) \quad \mathcal{L}_X S = 0.$$

Let $X \in \mathcal{X}(TM)$ be a Cartan symmetry. By (3.3.1) we have

$$(3.3.4) \quad d\mathcal{L}_X \theta_L = 0$$

that is $L_X \theta_L$ is closed.

Definition 3.3.4([1, p. 349]) If $\mathcal{L}_X \theta_L$ is exact then X is called *exact Cartan symmetry.*

A key result in the theory of conservation laws is

Proposition 3.3.5([1, p. 349]; **Generalized Noether theorem**) *If $X \in \mathcal{X}(TM)$ is an exact Cartan symmetry with*

$$(3.3.5) \quad \mathcal{L}_X \theta_L = df$$

then

$$(3.3.6) \quad P_X = J(X) L - f$$

is a conservation law for L. Conversely if $F \in C^\infty(TM)$ is conservation law for L then $X \in \mathcal{X}(TM)$ uniquely defined by

$$(3.3.7) \quad i_X \omega_L = -dF$$

is exact Cartan symmetry.

This result say that there is a bijective correspondence between exact Cartan symmetries and conservation laws for L.

In this subsection we give the local expression of exact Cartan symmetry X given by (3.3.7) which we not found in literature. Suppose that $X = X^i \frac{\delta}{\delta q^i} + \tilde{X}^i \frac{\partial}{\partial \dot{q}^i}$. Because

$$(3.3.8) \quad dF = \frac{\delta F}{\delta q^i} dq^i + \frac{\partial F}{\partial \dot{q}^i} \delta \dot{q}^i$$

with

$$(3.3.9) \quad \frac{\delta F}{\delta q^i} = \frac{\partial F}{\partial q^i} - N_i^j \frac{\partial F}{\partial \dot{q}^j}$$

relation (3.3.7) becomes

$$\left(X^i, \tilde{X}^i\right) \begin{pmatrix} 0 & -g_{ij} \\ g_{ij} & 0 \end{pmatrix} = -\left(\frac{\delta F}{\delta q^j}, \frac{\partial F}{\partial \dot{q}^j}\right)$$

which give

Proposition 3.3.6 *If* $X \in \mathcal{X}(TM)$ *is exact Cartan symmetry with conservation law* $F \in C^\infty(TM)$ *then*

$$
\begin{aligned}
(3.3.10a) \quad X^i &= g^{ij}\frac{\partial F}{\partial \dot{q}^j} \\
(3.3.10b) \quad \tilde{X}^i &= -g^{ij}\frac{\delta F}{\delta q^j}.
\end{aligned}
$$

The original Noether theorem ([5]) covered the case in which X is the complete lift of a vector field on M. Let $X \in \mathcal{X}(M)$ and denote by $(\phi_t)_t$ the flow it generates. This flow lifts to a flow $(\psi_t)_t$ on TM given by :

$$
(3.3.11) \qquad \psi_t(q,\dot{q}) = \left(\phi_t(q), (\phi_t)_{*,q}(\dot{q})\right)
$$

Definition 3.3.7 The generator of the flow $(\psi_t)_t$, denoted by X^C, is called *the complete lift of* X.

Definition 3.3.8 $X \in \mathcal{X}(M)$ is *an invariant of* L(or L-*invariant*) if:

$$
(3.3.12) \quad L \circ \psi_t = L \qquad \forall t
$$

Because $(\psi_t)_t$ is generated by the complete lift of X we have

Proposition 3.3.9 $X \in \mathcal{X}(M)$ *is invariant of* L *if and only if*

$$
(3.3.13) \quad \mathcal{L}_{X^C} L = 0.
$$

Then:

Proposition 3.3.10(Characterization of invariant vector fields) *If* $X = X^i\frac{\partial}{\partial q^i}$ *then* X *is invariant of* L *if and only if*:

$$
(3.3.14) \qquad X^k\frac{\partial L}{\partial q^k} + \dot{q}^s\frac{\partial X^k}{\partial q^s}\frac{\partial L}{\partial \dot{q}^k} = 0.
$$

Proposition 3.3.11([1, p. 349]; **E. Noether)** *If* X *is invariant of* L *then* X^C *is an exact Cartan symmetry with*

$$
(3.3.15) \quad f = 0
$$

that is the quantity

$$
(3.3.16) \qquad P_X = X^i\frac{\partial L}{\partial \dot{q}^i} = X^i p_i
$$

is conservation law for L.

The conservation laws obtained with this last result will be called "classical".

3.4 A nonclassical conservation law generated by symmetries

Let the 2-dimensional isotropic harmonic oscillator

$$\begin{aligned} (3.4.1a) \quad \ddot{q}^1 + \omega^2 q^1 &= 0 \\ (3.4.1b) \quad \ddot{q}^2 + \omega^2 q^2 &= 0 \end{aligned}$$

a toy model for many methods to finding conservation laws.

The Lagrangian is

$$(3.4.2) \quad L = \frac{1}{2}\left[\left(\dot{q}^1\right)^2 + \left(\dot{q}^2\right)^2\right] - \frac{\omega^2}{2}\left[\left(q^1\right)^2 + \left(q^2\right)^2\right]$$

and then applying the conservation of energy(theorem 3.1.3) we have two conservation laws

$$\begin{aligned} (3.4.3a) \quad \phi_1 &= \left(\dot{q}^1\right)^2 + \omega^2\left(q^1\right)^2 \\ (3.4.3b) \quad \phi_2 &= \left(\dot{q}^2\right)^2 + \omega^2\left(q^2\right)^2. \end{aligned}$$

A straightforward computation give that the unique L-invariant vector field is

$$(3.4.4) \quad X = q^2\frac{\partial}{\partial q^1} - q^1\frac{\partial}{\partial q^2}$$

and then the associate classical Noetherian conservation law is

$$(3.4.5) \quad \phi_3 = P_X = q^2\dot{q}^1 - q^1\dot{q}^2.$$

But we can obtain a nonclassical conservation law with symmetries. The spray of (3.4.1) is

$$(3.4.6) \quad S = \dot{q}^1\frac{\partial}{\partial q^1} + \dot{q}^2\frac{\partial}{\partial q^2} - \omega^2 q^1\frac{\partial}{\partial \dot{q}^1} - \omega^2 q^2\frac{\partial}{\partial \dot{q}^2}$$

and another calculus give that

$$(3.4.7) \quad Y = \dot{q}^2\frac{\partial}{\partial q^1} + \dot{q}^1\frac{\partial}{\partial q^2} - \omega^2 q^2\frac{\partial}{\partial \dot{q}^1} - \omega^2 q^1\frac{\partial}{\partial \dot{q}^2}$$

is a symmetry for S. Also, because S is total 1-homogeneous, i.e. with respect to all variables $(q, \dot{q})$ it result that

$$(3.4.8) \quad Z = q^1\frac{\partial}{\partial q^1} + q^2\frac{\partial}{\partial q^2} + \dot{q}^1\frac{\partial}{\partial \dot{q}^1} + \dot{q}^2\frac{\partial}{\partial \dot{q}^2}$$

is symmetry for S. We have

$$(3.4.9) \quad \mathcal{L}_Y H = 0, \quad \mathcal{L}_Z H = 2H$$

and then

$$(3.4.10) \quad \phi = \omega_L(S, Y) = 0, \quad \phi = \omega_L(S, Z) = 2H$$

i.e. we not have new conservation law applying corollary 3.1.6. But

$$(3.4.11) \quad \phi_4 = \omega_L(Y, Z) = \dot{q}^1\dot{q}^2 + \omega^2 q^1 q^2$$

is a new conservation law given by proposition 3.1.5. We remark that
(i) ϕ_4 is nonclassical conservation law
(ii) ϕ_4 represent the energy of a new Lagrangian of (3.4.1), that is

$$(3.4.12) \quad \tilde{L} = \dot{q}^1\dot{q}^2 - \omega^2 q^1 q^2$$

a result very important from the point of view of Inverse Problem of Analytical Mechanics([6]). Our Lagrangian L^* appear in [6, p. 122].

References

[1] Crampin, M., Pirani, F.A.E., *Applicable differential geometry*, London Math. Society Lectures Notes Series., no. 59, Cambridge University Press, 1986.

[2] ten Eikelder, H.M.M., *Symmetries for dynamical and Hamiltonian systems*, CWI Tract, no. 17, Amsterdam, 1985.

[3] Jones, G.L., *Symmetry and conservation laws of differential equations*, Il Nuovo Cimento, 112(1997), no. 7, 1053-1059.

[4] Miron, R., Anastasiei, M., *The Geometry of Lagrange Spaces (Theory and Applications)*, Fundamental Theories of Physics, no. 59, Kluwer Academic Publishers, 1994.

[5] Noether, E., *Invariante Variationsprobleme,* in "Gesammelte Abhandlungen", Springer Verlag, 1983, 249-270.

[6] Santilli, R. M., *Foundations of Theoretical Mechanics I: The Inverse Problem in Newtonian Mechanics*, Texts and Monographs in Physics, Springer-Verlag, N.Y., 1978.

[7] Taylor, M.E., *Partial differential equations*(vol. I- "Basic Theory"), Texts in applied Mathematics, no. 23, Springer, 1996.

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