Hamilton spaces with figuratrices as constant mean curvature surfaces or minimal surfaces

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Abstract: Constant mean curvature, particularly minimal, surfaces given by figuratrices of Hamilton and generalized Hamilton spaces are studied.

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1. Introduction

The search of minimal surfaces in $I\!R^3$ is an old exciting problem (Lucas et al [7]) and several methods appear in the study of these surfaces: Lie groups methods (Bîlă [3]), theory of integrable systems via the Weierstrass representation (Hélein [5]).

Also, very interesting generalizations are fruitful: minimal surfaces in Riemannian manifolds (Ji et al [6]), constant mean curvature (CMC) surfaces (Hélein [5], Wente [10]).

In this paper we search CMC surfaces, particularly minimal surfaces, provided by figuratrices of Hamilton and generalized Hamilton manifolds of dimension three. Several equations of CMC and minimal figuratrices are obtained in the Hamilton and generalized Hamilton framework and the last section is devoted to examples.

Let in $I\!R^3$ a surface S given in implicit form $S : F(x, y, z) = 0$. Recall that the mean curvature of S is (Bloch [4, p. 290]):

$$H = \frac{F_x^2 (F_{yy} + F_{zz}) + F_y^2 (F_{zz} + F_{xx}) + F_z^2 (F_{xx} + F_{yy})}{2\left(F_x^2 + F_y^2 + F_z^2\right)^{\frac{3}{2}}} \tag{1.1}$$

where the low indices denote the derivatives with respect to corresponding variables. The equation H=constant is called *CMC surfaces equation* and in particular the equation $H = 0$ is called *minimal surfaces equation.*

So, we get the CMC surfaces equation:

$$2\left(F_{xy}F_xF_y + F_{yz}F_yF_z + F_{zx}F_zF_x\right) -$$

$$-\left[F_{xx}\left(F_y^2 + F_z^2\right) + F_{yy}\left(F_z^2 + F_x^2\right) + F_{zz}\left(F_x^2 + F_y^2\right)\right] = 2H\left(F_x^2 + F_y^2 + F_z^2\right)^{\frac{3}{2}} \tag{1.2}$$

and the minimal surfaces equation:

$$2(F_{xy}F_xF_y+F_{yz}F_yF_z+F_{zx}F_zF_x) = F_{xx}\left(F_y^2 + F_z^2\right)+F_{yy}\left(F_z^2 + F_x^2\right)+F_{zz}\left(F_x^2 + F_y^2\right). \tag{1.3}$$

2. CMC figuratrices in Hamilton geometry

Let us denote $T^*\mathbb{R}^3$ the cotangent bundle of $\mathbb{R}^3$ for which we use the coordinates $(x,p) = (x^i, p_i)_{1\leq i\leq 3}$ with $x = (x^i)$ the coordinates in $\mathbb{R}^3$ and $p = (p_i)$ the coordinates in the fiber $T_x^*\mathbb{R}^3$. A function $f \in C^\infty\left(T^*\mathbb{R}^3\right)$ which does not depends of x i.e. $f = f(p)$ is called *Minkowskian function.* A tensor field of (r,s)-type on $T^*\mathbb{R}^3$ with law of change, at a change of coordinates on $T\mathbb{R}^3$, exactly as a tensor field of (r,s)-type on $\mathbb{R}^3$ is called *d-tensor field* of (r,s)-type.

After Miron et al [9], a smooth Hamiltonian $H : T^*\mathbb{R}^3 \to \mathbb{R}$ is called *regular* if the matrix $g = (g^{ij})_{1\leq i,j\leq 3}, g^{ij} = \frac{1}{2}\dot{\partial}^i\dot{\partial}^j H$, is of rank 3 i.e. $\det g \neq 0$ where $\dot{\partial}^i = \frac{\partial}{\partial p_i}$. The pair $\left(\mathbb{R}^3, H\right)$ is called then *Hamilton space* and the d-tensor field $g = (g^{ij})$ is called *the Hamilton metric.* In the following H_x will be a real number related to a value of mean curvature.

For every $x \in \mathbb{R}^3$ we have *the figuratrix* of H, $F_x = \{p \in T_x^*\mathbb{R}^3; H(x,p) = 1\}$ which appears as a surface defined by $F_x(p) = H(x,p) - 1$, x being fixed. Using the last relations of previous section it results that F_x is CMC surface if:

$$2\left(F^{12}F^1F^2 + F^{23}F^2F^3 + F^{31}F^3F^1\right) -$$

$$-\left[F^{11}\left(\left(F^2\right)^2 + \left(F^3\right)^2\right) + F^{22}\left(\left(F^3\right)^2 + \left(F^1\right)^2\right) + F^{33}\left(\left(F^1\right)^2 + \left(F^2\right)^2\right)\right] =$$

$$= 2H_x\left(\left(F^1\right)^2 + \left(F^2\right)^2 + \left(F^3\right)^2\right)^{\frac{3}{2}} \tag{2.1}$$

and F_x is minimal surface if:

$$2\left(F^{12}F^1F^2 + F^{23}F^2F^3 + F^{31}F^3F^1\right) =$$

$$= F^{11}\left(\left(F^2\right)^2 + \left(F^3\right)^2\right) + F^{22}\left(\left(F^3\right)^2 + \left(F^1\right)^2\right) + F^{33}\left(\left(F^1\right)^2 + \left(F^2\right)^2\right) \tag{2.2}$$

where $F^i = \dot{\partial}^i H$ and $F^{ij} = \dot{\partial}^i \dot{\partial}^j H$ and we delete the subscript x of F. From $F^{ij} = 2g^{ij}$ it follows:

Proposition 2.1 (i) *CMC figuratrices are given by*:

$$2\left(g^{12}F^1F^2 + g^{23}F^2F^3 + g^{31}F^3F^1\right) -$$

$$-\left[g^{11}\left(\left(F^2\right)^2 + \left(F^3\right)^2\right) + g^{22}\left(\left(F^3\right)^2 + \left(F^1\right)^2\right) + g^{33}\left(\left(F^1\right)^2 + \left(F^2\right)^2\right)\right] =$$

$$= H_x\left(\left(F^1\right)^2 + \left(F^2\right)^2 + \left(F^3\right)^2\right)^{\frac{3}{2}} \tag{2.3}$$

(ii) *minimal figuratrices are given by*:

$$2\left(g^{12}F^1F^2 + g^{23}F^2F^3 + g^{31}F^3F^1\right) =$$

$$= g^{11}\left(\left(F^2\right)^2 + \left(F^3\right)^2\right) + g^{22}\left(\left(F^3\right)^2 + \left(F^1\right)^2\right) + g^{33}\left(\left(F^1\right)^2 + \left(F^2\right)^2\right). \tag{2.4}$$

Let us remark that for a Minkowski Hamiltonian if there exists a CMC (minimal) figuratrix then all figuratrices are CMC (minimal) surfaces.

A particular important case is that of a r-homogeneous Hamiltonian i.e. $H(x, \lambda p) = \lambda^r H(x, p)$ for every $\lambda \in I\!R$.

Proposition 2.2 *If H is r-homogeneous with $r \neq 1$ then*:

(i) *CMC figuratrices are given by*:

$$2\left(g^{12}g^{1a}g^{2b} + g^{23}g^{2a}g^{3b} + g^{31}g^{3a}g^{1b}\right)p_a p_b -$$

$$-\{g^{11}\left[\left(g^{2a}p_a\right)^2 + \left(g^{3a}p_a\right)^2\right] + g^{22}\left[\left(g^{3a}p_a\right)^2 + \left(g^{1a}p_a\right)^2\right] + g^{33}\left[\left(g^{1a}p_a\right)^2 + \left(g^{2a}p_a\right)^2\right]\} =$$

$$= \frac{2H_x}{r-1}\left[\left(g^{1a}p_a\right)^2 + \left(g^{2a}p_a\right)^2 + \left(g^{3a}p_a\right)^2\right]^{\frac{3}{2}} \tag{2.5}$$

(ii) *minimal figuratrices are given by*:

$$2\left(g^{12}g^{1a}g^{2b} + g^{23}g^{2a}g^{3b} + g^{31}g^{3a}g^{1b}\right)p_a p_b =$$

$$= g^{11}\left[\left(g^{2a}p_a\right)^2 + \left(g^{3a}p_a\right)^2\right] + g^{22}\left[\left(g^{3a}p_a\right)^2 + \left(g^{1a}p_a\right)^2\right] + g^{33}\left[\left(g^{1a}p_a\right)^2 + \left(g^{2a}p_a\right)^2\right]. \tag{2.6}$$

Proof. From Euler relation $\dot{\partial}^i Hp_i = rH$ applying $\dot{\partial}^j$ we have $2g^{ij}p_i + F^j = rF^j$ which means that:

$$F^j = \frac{2}{r-1}g^{ia}p_a \tag{2.7}$$

and substituting this relation in (2.3) and (2.4) we get (2.5) and (2.6). □

The most important case is $r = 2$ for:

(i) *Riemann spaces* when $g = (g^{ij}(x))$ is a contravariant Riemannian metric and H is the kinetic energy of g i.e. $H = g^{ij} p_i p_j$

(ii) *Cartan spaces* when $g = (g^{ij}(x, p))$ is a Cartan metric (Miron et al [9]) and $H = g^{ij} p_i p_j$.

Proposition 2.3 *For Cartan, particularly Riemann, spaces*:

(i) *the CMC figuratrices are given by*:

$$2\left(g^{12}g^{1a}g^{2b} + g^{23}g^{2a}g^{3b} + g^{31}g^{3a}g^{1b}\right)p_a p_b -$$

$$-\{g^{11}\left[\left(g^{2a}p_a\right)^2 + \left(g^{3a}p_a\right)^2\right] + g^{22}\left[\left(g^{3a}p_a\right)^2 + \left(g^{1a}p_a\right)^2\right] + g^{33}\left[\left(g^{1a}p_a\right)^2 + \left(g^{2a}p_a\right)^2\right]\} =$$

$$= 2H_x\left[\left(g^{1a}p_a\right)^2 + \left(g^{2a}p_a\right)^2 + \left(g^{3a}p_a\right)^2\right]^{\frac{3}{2}} \tag{2.8}$$

(ii) *the minimal figuratrices are given by* (2.6).

Returning to the general case of proposition 2.1 because the matrix $g = (g^{ij})$ is symmetric let us suppose that this matrix is diagonal: $g^{12} = g^{32} = g^{31} = 0$. Let us call *diagonal Hamilton space* this type of Hamilton spaces.

Proposition 2.4 A) *In a diagonal Hamilton space*:

(i) *the CMC figuratrices are given by*:

$$g^{11}\left(\left(F^2\right)^2 + \left(F^3\right)^2\right) + g^{22}\left(\left(F^3\right)^2 + \left(F^1\right)^2\right) + g^{33}\left(\left(F^1\right)^2 + \left(F^2\right)^2\right) =$$

$$= -H_x\left(\left(F^1\right)^2 + \left(F^2\right)^2 + \left(F^3\right)^2\right)^{\frac{3}{2}} \tag{2.9}$$

(ii) *the minimal figuratrices are given by*:

$$g^{11}\left(\left(F^2\right)^2 + \left(F^3\right)^2\right) + g^{22}\left(\left(F^3\right)^2 + \left(F^1\right)^2\right) + g^{33}\left(\left(F^1\right)^2 + \left(F^2\right)^2\right) = 0. \tag{2.10}$$

If the diagonal Hamilton metric is positive definite i.e. $g^{ii} > 0, 1 \leq i \leq 3$, *it results that there are no minimal figuratrices.*

B) *In a diagonal r-homogeneous Hamilton space*:

(i) *the CMC figuratrices are given by*:

$$g^{11}\left[\left(g^{22}p_2\right)^2 + \left(g^{33}p_3\right)^2\right] + g^{22}\left[\left(g^{33}p_3\right)^2 + \left(g^{11}p_1\right)^2\right] + g^{33}\left[\left(g^{11}p_1\right)^2 + \left(g^{22}p_2\right)^2\right] =$$

$$= \frac{2H_x}{1-r}\left[\left(g^{11}p_1\right)^2 + \left(g^{22}p_2\right)^2 + \left(g^{33}p_3\right)^2\right]^{\frac{3}{2}} \tag{2.11}$$

(ii) *the minimal figuratrices are given by*:

$$g^{11}\left[\left(g^{22}p_2\right)^2 + \left(g^{33}p_3\right)^2\right] + g^{22}\left[\left(g^{33}p_3\right)^2 + \left(g^{11}p_1\right)^2\right] + g^{33}\left[\left(g^{11}p_1\right)^2 + \left(g^{22}p_2\right)^2\right] = 0 \tag{2.12}$$

C) *In a diagonal Cartan, particularly Riemann, space*:

(i) *the CMC figuratrices are given by*:

$$g^{11}\left[\left(g^{22}p_2\right)^2+\left(g^{33}p_3\right)^2\right]+g^{22}\left[\left(g^{33}p_3\right)^2+\left(g^{11}p_1\right)^2\right]+g^{33}\left[\left(g^{11}p_1\right)^2+\left(g^{22}p_2\right)^2\right]=$$

$$=-2H_x\left[\left(g^{11}p_1\right)^2+\left(g^{22}p_2\right)^2+\left(g^{33}p_3\right)^2\right]^{\frac{3}{2}} \qquad (2.13)$$

(ii) *the minimal figuratrices are given by* (2.12).

Example 2.5 (**The Euclidean case**) Let $g^{ij}=\delta^{ij}$ be the usual contravariant Euclidean metric of $\mathbb{R}^3$ which is a diagonal Riemann metric. The relation (2.13) becomes:

$$2\left[(p_1)^2+(p_2)^2+(p_3)^2\right]=-2H_x\left[(p_1)^2+(p_2)^2+(p_3)^2\right]^{\frac{3}{2}}.$$

In this case $H=(p_1)^2+(p_2)^2+(p_3)^2$ and thus the equation of F_x is $(p_1)^2+(p_2)^2+(p_3)^2=1$ and then for every $x\in\mathbb{R}^3$:

(i) the only CMC figuratrix is the unit sphere S^2 with $H_x=-1$,

(ii) there are no minimal figuratrices.

3. CMC figuratrices in generalized Hamilton spaces

A d-tensor field of $(2,0)$-type on $T^*\mathbb{R}^3$, denoted $g=\left(g^{ij}(x,p)\right)$, is called *generalized Hamilton metric* (*GH-metric*, on short) if the following properties hold (Miron et al [9]):

(i) symmetry, $g^{ij}=g^{ji}$

(ii) nondegeneracy: $\det\left(g^{ij}\right)\neq 0$

(iii) the signature of quadratic form $g(\xi)=g^{ij}\xi_i\xi_j, \xi=(\xi_i)\in\mathbb{R}^3$, is constant.

The function $\mathcal{E}(g)=g^{ij}p_ip_j$ is called *the absolute energy* of the given GH-metric.

Definition 3.1 (Miron et al [9]) The GH-metric is called *weak regular* if $\mathcal{E}(g)$ is a regular Hamiltonian.

It follows that for a weak regular GH-metric the d-tensor field of $(2,0)$-type:

$$g^{*ij}=\frac{1}{2}\dot{\partial}^i\dot{\partial}^j\,\mathcal{E}(g) \qquad (3.1)$$

is a Hamilton metric and then we can associate the figuratrix:

$$F_x=\{p\in T_x^*\mathbb{R}^3;\mathcal{E}(g)(x,p)=1\}.$$

Applying proposition 2.1 we get:

Proposition 3.2 *For a weak regular GH-metric*:

(i) *the CMC figuratrices are given by*:

$$2\left[g^{*12}\,\dot{\partial}^1\,\mathcal{E}(g)\,\dot{\partial}^2\,\mathcal{E}(g)+g^{*23}\,\dot{\partial}^2\,\mathcal{E}(g)\,\dot{\partial}^3\,\mathcal{E}(g)+g^{*31}\,\dot{\partial}^3\,\mathcal{E}(g)\,\dot{\partial}^1\,\mathcal{E}(g)\right]-$$

$$-\{g^{*11}\left[\left(\dot{\partial}^2\,\mathcal{E}(g)\right)^2+\left(\dot{\partial}^3\,\mathcal{E}(g)\right)^2\right]+g^{*22}\left[\left(\dot{\partial}^3\,\mathcal{E}(g)\right)^2+\left(\dot{\partial}^1\,\mathcal{E}(g)\right)^2\right]+$$

$$+g^{*33}\left[\left(\dot{\partial}^1\,\mathcal{E}(g)\right)^2+\left(\dot{\partial}^2\,\mathcal{E}(g)\right)^2\right]\}=$$

$$=H_x\left[\left(\dot{\partial}^1\,\mathcal{E}(g)\right)^2+\left(\dot{\partial}^2\,\mathcal{E}(g)\right)^2+\left(\dot{\partial}^3\,\mathcal{E}(g)\right)^2\right]^{\frac{3}{2}} \tag{3.2}$$

(ii) *the minimal figuratrices are given by*:

$$2\left[g^{*12}\,\dot{\partial}^1\,\mathcal{E}(g)\,\dot{\partial}^2\,\mathcal{E}(g)+g^{*23}\,\dot{\partial}^2\,\mathcal{E}(g)\,\dot{\partial}^3\,\mathcal{E}(g)+g^{*31}\,\dot{\partial}^3\,\mathcal{E}(g)\,\dot{\partial}^1\,\mathcal{E}(g)\right]=$$

$$=g^{*11}\left[\left(\dot{\partial}^2\,\mathcal{E}(g)\right)^2+\left(\dot{\partial}^3\,\mathcal{E}(g)\right)^2\right]+g^{*22}\left[\left(\dot{\partial}^3\,\mathcal{E}(g)\right)^2+\left(\dot{\partial}^1\,\mathcal{E}(g)\right)^2\right]+$$

$$+g^{*33}\left[\left(\dot{\partial}^1\,\mathcal{E}(g)\right)^2+\left(\dot{\partial}^2\,\mathcal{E}(g)\right)^2\right]. \tag{3.3}$$

A straightforward computation gives:

$$\begin{cases} g^{*ij}=g^{ij}+\left(\dot{\partial}^i\dot{\partial}^j\;g^{ab}\right)p_ap_b+\left(\dot{\partial}^i\;g^{ja}+\dot{\partial}^j\;g^{ia}\right)p_a \\ \dot{\partial}^i\;\mathcal{E}(g)=\left(\dot{\partial}^i\;g^{ab}\right)p_ap_b+2g^{ia}p_a \end{cases}. \tag{3.4}$$

The above formulae become more simple in the following case:

Definition 3.3 (Miron et al [9]) A weak regular GH-metric is called *regular* if:

$$\dot{\partial}^i\;\mathcal{E}(g)=2g^{ij}p_j. \tag{3.5}$$

It results:

$$g^{*ij}=g^{ij}+\left(\dot{\partial}^j\;g^{ik}\right)p_k \tag{3.6}$$

and then:

Proposition 3.4 *For a regular GH-metric*:

(i) *the CMC figuratrices are given by*:

$$2\{\left[g^{12}+\left(\dot{\partial}^2\;g^{1k}\right)p_k\right]g^{1a}g^{2b}+\left[g^{23}+\left(\dot{\partial}^3\;g^{2k}\right)p_k\right]g^{2a}g^{3b}+\left[g^{31}+\left(\dot{\partial}^1\;g^{3k}\right)p_k\right]g^{3a}g^{1b}\}p_ap_b-$$

$$-\{\left[g^{11}+\left(\dot{\partial}^1\;g^{1k}\right)p_k\right]\left[\left(g^{2a}p_a\right)^2+\left(g^{3a}p_a\right)^2\right]+\left[g^{22}+\left(\dot{\partial}^2\;g^{2k}\right)p_k\right]\left[\left(g^{3a}p_a\right)^2+\left(g^{1a}p_a\right)^2\right]+$$

$$+\left[g^{33}+\left(\dot{\partial}^3\, g^{3k}\right)p_k\right]\left[\left(g^{1a}p_a\right)^2+\left(g^{2a}p_a\right)^2\right]\}=$$

$$2H_x\left[\left(g^{1a}p_a\right)^2+\left(g^{2a}p_a\right)^2+\left(g^{3a}p_a\right)^2\right]^{\frac{3}{2}} \tag{3.7}$$

(ii) *the minimal figuratrices are given by*:

$$2\{\left[g^{12}+\left(\dot{\partial}^2\, g^{1k}\right)p_k\right]g^{1a}g^{2b}+\left[g^{23}+\left(\dot{\partial}^3\, g^{2k}\right)p_k\right]g^{2a}g^{3b}+\left[g^{31}+\left(\dot{\partial}^1\, g^{3k}\right)p_k\right]g^{3a}g^{1b}\}p_ap_b=$$

$$=\left[g^{11}+\left(\dot{\partial}^1\, g^{1k}\right)p_k\right]\left[\left(g^{2a}p_a\right)^2+\left(g^{3a}p_a\right)^2\right]+\left[g^{22}+\left(\dot{\partial}^2\, g^{2k}\right)p_k\right]\left[\left(g^{3a}p_a\right)^2+\left(g^{1a}p_a\right)^2\right]+$$

$$+\left[g^{33}+\left(\dot{\partial}^3\, g^{3k}\right)p_k\right]\left[\left(g^{1a}p_a\right)^2+\left(g^{2a}p_a\right)^2\right]. \tag{3.8}$$

Another approach in the regular case is provided by homogeneity. By multiplication of (3.5) with p_i we have:

$$\dot{\partial}^i\,\mathcal{E}\left(g\right)p_i=2g^{ij}p_ip_j=2\mathcal{E}\left(g\right) \tag{3.9}$$

which means that $\mathcal{E}(g)$ is 2-homogeneous i.e. $\mathcal{E}(g)$ is a Cartan function. Then we apply proposition 2.3:

Proposition 3.5 *For a regular GH-metric*:

(i) *the CMC figuratrices are given by*:

$$2\left(g^{*12}g^{*1a}g^{*2b}+g^{*23}g^{*2a}g^{*3b}+g^{*31}g^{*3a}g^{*1b}\right)p_ap_b-$$

$$-\{g^{*11}\left[\left(g^{*2a}p_a\right)^2+\left(g^{*3a}p_a\right)^2\right]+g^{*22}\left[\left(g^{*3a}p_a\right)^2+\left(g^{*1a}p_a\right)^2\right]+g^{*33}\left[\left(g^{*1a}p_a\right)^2+\left(g^{*2a}p_a\right)^2\right]\}=$$

$$=2H_x\left[\left(g^{*1a}p_a\right)^2+\left(g^{*2a}p_a\right)^2+\left(g^{*3a}p_a\right)^2\right]^{\frac{3}{2}} \tag{3.10}$$

(ii) *the minimal figuratrices are given by*:

$$2\left(g^{*12}g^{*2a}g^{*3b}+g^{*23}g^{*2a}g^{*3b}+g^{*31}g^{*3a}g^{*1b}\right)p_ap_b=$$

$$=g^{*11}\left[\left(g^{*2a}p_a\right)^2+\left(g^{*3a}p_a\right)^2\right]+g^{*22}\left[\left(g^{*3a}p_a\right)^2+\left(g^{*1a}p_a\right)^2\right]+g^{*33}\left[\left(g^{*1a}p_a\right)^2+\left(g^{*2a}p_a\right)^2\right] \tag{3.11}$$

where for g^{*ij} *we use the relation* (3.6).

4. Hamilton-Beil type metrics as examples

Let $\widetilde{g} = \left(\widetilde{g}^{ij}(x,p)\right)$ be a Cartan metric and $B = B_i(x,p)\,\dot{\partial}^i$ a d-covector field for which we denote $B^i = \widetilde{g}^{ij}B_j$ and $B_0 = B^i p_i$. Let also $a, b \in C^\infty\left(T^*I\!R^3\right)$. Using an idea from Anastasiei et al [1], [2], the following GH-metric we can consider:

$$g^{ij} = a\widetilde{g}^{ij} + bB^iB^j. \tag{4.1}$$

These GH-metrics, which we call *Hamilton-Beil type metrics*, are not Hamilton metrics. From:

$$\mathcal{E}(g) = a\mathcal{E}(\widetilde{g}) + b(B_0)^2 \tag{4.2}$$

we get:

$$\dot{\partial}^i\,\mathcal{E}(g) = \left(\dot{\partial}^i\,a\right)\mathcal{E}(\widetilde{g}) + a\left(\dot{\partial}^i\,\mathcal{E}(\widetilde{g})\right) + \left(\dot{\partial}^i\,b\right)(B_0)^2 + 2bB_0\left(\dot{\partial}^i\,B_0\right) \tag{4.3}$$

$$\begin{aligned} 2g^{*ij} = 2a\widetilde{g}^{ij} + \dot{\partial}^i\dot{\partial}^j\,a\mathcal{E}(\widetilde{g}) + \dot{\partial}^i\,a\,\dot{\partial}^j\,\mathcal{E}(\widetilde{g}) + \dot{\partial}^j\,a\,\dot{\partial}^i\,\mathcal{E}(\widetilde{g}) + \dot{\partial}^i\dot{\partial}^j\,b(B_0)^2 + \\ +2B_0\left(\dot{\partial}^i\,b\,\dot{\partial}^j\,B_0 + \dot{\partial}^j\,b\,\dot{\partial}^i\,B_0 + b\,\dot{\partial}^i\dot{\partial}^j\,B_0\right) + 2b\,\dot{\partial}^i\,B_0\,\dot{\partial}^j\,B_0. \end{aligned} \tag{4.4}$$

I) On $T_0^*I\!R^3 = T^*I\!R^3 \backslash \{\text{null section}\}$ let:

$$a = \frac{1}{2}, b = \frac{1}{2\|p\|_C^2} \tag{4.5}$$

where $\|,\|_C$ is the norm induced by the Cartan metric $\widetilde{g}$ i.e. $\|p\|_C^2 = E(\widetilde{g}) = \widetilde{g}^{ij}p_ip_j$. Let $B = p_i\,\dot{\partial}^i$ be *the Liouville covector field*, it results $B^i = \widetilde{g}^{ij}p_j \overset{denoted}{=} \widetilde{p}^i$. The associated Hamilton-Beil type metric is:

$$g = \frac{1}{2\|p\|_C^2}\begin{pmatrix} \left(\widetilde{p}^1\right)^2 + \widetilde{g}^{11}\|p\|_C^2 & \widetilde{p}^1\widetilde{p}^2 + \widetilde{g}^{12}\|p\|_C^2 & \widetilde{p}^1\widetilde{p}^3 + \widetilde{g}^{13}\|p\|_C^2 \\ \widetilde{p}^1\widetilde{p}^2 + \widetilde{g}^{12}\|p\|_C^2 & \left(\widetilde{p}^2\right)^2 + \widetilde{g}^{22}\|p\|_C^2 & \widetilde{p}^2\widetilde{p}^3 + \widetilde{g}^{23}\|p\|_C^2 \\ \widetilde{p}^1\widetilde{p}^3 + \widetilde{g}^{13}\|p\|_C^2 & \widetilde{p}^2\widetilde{p}^3 + \widetilde{g}^{23}\|p\|_C^2 & \left(\widetilde{p}^3\right)^2 + \widetilde{g}^{33}\|p\|_C^2 \end{pmatrix}. \tag{4.6}$$

Thus:

$$\mathcal{E}(g) = \|p\|_C^2 = \mathcal{E}(\widetilde{g}) \tag{4.7}$$

which is 2-homogeneous and then a Cartan function. It results that the Hamilton-Beil type metric is regular GH-metric with $g^{*ij} = \widetilde{g}^{ij}$ and then the CMC (minimal) figuratrices of this Hamilton-Beil type metric are exactly the CMC (minimal) indicatrices of Cartan metric $\widetilde{g}$.

II) (**Hamilton-Miron-Tavakol metrics**) For $a = \exp(2\sigma), b = 0$ with $\sigma \in C^\infty\left(T^*I\!R^3\right)$ and $\widetilde{g} = \widetilde{g}(x)$ a Riemannian contravariant metric we have the so-called *Hamilton-Miron-Tavakol metrics*, in analogy with Lagrangian Miron-Tavakol metrics (Miron et al [8]):

$$g^{ij}(x,p) = e^{2\sigma(x,p)}\widetilde{g}^{ij}(x) \tag{4.8}$$

for which:

$$\dot{\partial}^{i}\ \mathcal{E}\left(g\right)=2\left(g^{ia}p_{a}+\left(\dot{\partial}^{i}\ \sigma\right)g^{ab}p_{a}p_{b}\right) \tag{4.9}$$

$$g^{*ij}=g^{ij}+\left(\dot{\partial}^{i}\dot{\partial}^{j}\ \sigma+2\ \dot{\partial}^{i}\ \sigma\ \dot{\partial}^{j}\ \sigma\right)g^{ab}p_{a}p_{b}+2\left(g^{ja}\ \dot{\partial}^{i}\ \sigma+g^{ia}\ \dot{\partial}^{j}\ \sigma\right). \tag{4.10}$$

Particular cases:

II1. (inspired by Miron et al [8]) $\sigma=\frac{1}{2}\mathcal{E}\left(\widetilde{g}\right)=\frac{1}{2}\widetilde{g}^{ij}p_{i}p_{j}$

$$\dot{\partial}^{i}\ \mathcal{E}\left(g\right)=2e^{\mathcal{E}\left(\widetilde{g}\right)}\left(1+\mathcal{E}\left(\widetilde{g}\right)\right)\widetilde{g}^{ia}p_{a} \tag{4.11}$$

$$g^{*ij}=g^{ij}+\left(\widetilde{g}^{ij}+2\widetilde{g}^{ia}\widetilde{g}^{jb}p_{a}p_{b}\right)g^{uv}p_{u}p_{v}+2\left(g^{ja}\widetilde{g}^{iu}p_{u}+g^{ia}\widetilde{g}^{ju}p_{u}\right)y^{a}. \tag{4.12}$$

II2. $\sigma=\gamma^{i}\left(x\right)p_{i}$ with $\gamma_{i}\in C^{\infty}\left(I\!R^{3}\right),1\leq i\leq 3$

$$\dot{\partial}^{i}\ \mathcal{E}\left(g\right)=2\left(g^{ia}p_{a}+\gamma^{i}\mathcal{E}\left(g\right)\right) \tag{4.13}$$

$$g^{*ij}=g^{ij}+2\gamma^{i}\gamma^{j}\mathcal{E}\left(g\right)+2\left(\gamma^{i}g^{ja}+\gamma^{j}g^{ia}\right)p_{a}. \tag{4.14}$$

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