The Gaussian curvature for the indicatrix of a generalized Lagrange space

Mircea Crâşmăreanu

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Abstract

Using the formulae for the Gaussian curvature of a indicatrix in a Lagrange space from [3] and [6] similar computations are obtained for generalized Lagrange spaces. Some examples are discussed.

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1 The Gaussian curvature for the indicatrix of a Lagrange space

In the n-dimensional Euclidean space $I\!R^n$ let us consider a hypersurface S defined by $f \in C^{\infty}(I\!R^n)$ as: $S = \{x \in I\!R^n \mid f(x) = 0, \nabla f(x) \neq 0\}$ where ∇f denotes the gradient of f namely $\nabla f = (f_i)$ where $f_i = \frac{\partial f}{\partial x^i}$. Suppose that the normal vector field of S is: $N = -\frac{\nabla f}{\|\nabla f\|}$. In [6, p. 37] and [3, p. 23] the following classical result is proved in a direct way:

Proposition 1.1 *The Gaussian curvature of pair* (S, N) *is*:

$$K = -\frac{\begin{vmatrix} f_{ij} & f_i \\ f_j & 0 \end{vmatrix}}{\| \nabla f \|^{n+1}}. \tag{1.1}$$

This formulae is used in the cited papers in order to obtain the Gaussian curvature for the indicatrix of a *Lagrange space*. More precisely, let $L : T\mathbb{R}^n \to \mathbb{R}$ be a regular Lagrangian that is the matrix $g_{ij} = \frac{1}{2} \dot{\partial}_i \dot{\partial}_j L$ is of rank n i. e. $\det(g_{ij}) \neq 0$ where $\dot{\partial}_i = \frac{\partial}{\partial y^i}$. Associated to this Lagrangian we have *the indicatrix of* L: for every $x \in \mathbb{R}^n$, $I_x = \{(x, y) \in T\mathbb{R}^n; L(x, y) = 1\}$ appears as a hypersurface defined by: $f(x, y) = L(x, y) - 1$.Then it is proved in [6, p. 39] that:

Proposition 1.2 *Let* $(\mathbb{R}^n, L)$ *be a Lagrange space. At each* $x \in \mathbb{R}^n$ *the Gaussian curvature* K_x *of the indicatrix* I_x *oriented in the direction* $N_x = -\frac{\dot{\nabla} L}{\|\dot{\nabla} L\|}$ *is*:

$$K_x = -\frac{\begin{vmatrix} \dot{\partial}_i \dot{\partial}_j L & \dot{\partial}_i L \\ \dot{\partial}_j L & 0 \end{vmatrix}}{\left(\sum_{i=1}^{n} \left(\dot{\partial}_i L \right)^2 \right)^{\frac{n+1}{2}}}. \tag{1.2}$$

In this proposition $\dot{\nabla} L$ denotes the gradient of L with respect to y i.e. $\dot{\nabla} L = \left(\dot{\partial}_i L \right)_{1 \leq i \leq n}$.

A function $f \in C^\infty(T\mathbb{R}^n)$ which does not depends on x i.e. $f = f(y)$ is called *Minkowskian function*. A tensor field of (r, s)-type on $T\mathbb{R}^n$ with law of change, at a change of coordinates on $T\mathbb{R}^n$, exactly as a tensor field of (r, s)-type on $\mathbb{R}^n$ is called *d-tensor field of* (r, s)*-type* on $T\mathbb{R}^n$. We denote $T_0\mathbb{R}^n$ the tangent bundle of $\mathbb{R}^n$ without the null section.

2 The Gaussian curvature for the indicatrix of a generalized Lagrange space

A d-tensor field of $(0, 2)$-type on $T\mathbb{R}^n$, denoted $(g_{ij}(x, y))$, is called *generalized Lagrange metric* (*GL-metric*, on short) if has the following properties ([5]):

(i) symmetry, $g_{ij} = g_{ji}$

(ii) nondegeneracy, $g := \det(g_{ij}) \neq 0$.

The function $\mathcal{E}(g) = g_{ij} y^i y^j$ is called *the absolute energy* of the given GL-metric.

Definition 2.1([5]) The GL-metric (g_{ij}) is called *weak regular* if $\mathcal{E}(g)$ is a regular Lagrangian.

It follows that for a weak regular GL-metric the d-tensor of $(0,2)$-type:

$$g^*_{ij} = \frac{1}{2} \dot{\partial}_i \dot{\partial}_j \, \mathcal{E}(g) \tag{2.1}$$

is a Lagrange metric and then we can associate the indicatrix

$$I_x = \{(x, y) ; \mathcal{E}(g)(x, y) = 1, \dot{\nabla} \, \mathcal{E}(g)(x, y) \neq 0\}.$$

Applying proposition 1.2 we get the main result of this paper:

Proposition 2.2 *For a weak regular Lagrange metric the Gaussian curvature of I_x is:*

$$K_x = -\frac{\begin{vmatrix} g^*_{ij} & \dot{\partial}_i \, \mathcal{E}(g) \\ \dot{\partial}_j \, \mathcal{E}(g) & 0 \end{vmatrix}}{\left(\sum\limits_{i=1}^{n} \left(\dot{\partial}_i \, \mathcal{E}(g)\right)^2\right)^{\frac{n+1}{2}}}. \tag{2.2}$$

A straightforward computation gives:

$$\begin{cases} g^*_{ij} = g_{ij} + \frac{1}{2}\left(\dot{\partial}_i \dot{\partial}_j \, g_{ab}\right) y^a y^b + \left(\dot{\partial}_i \, g_{ja} + \dot{\partial}_j \, g_{ia}\right) y^a \\ \dot{\partial}_i \, \mathcal{E}(g) = \frac{1}{2}\left(\dot{\partial}_i \, g_{ab}\right) y^a y^b + g_{ia} y^a \end{cases} . \tag{2.3}$$

The above formulae become more simple in the following case:

Definition 2.3([5]) A weak regular generalized Lagrange metric is called *regular* if:

$$\frac{1}{2} \dot{\partial}_i \, \mathcal{E}(g) = g_{ij} y^j. \tag{2.4}$$

It results([5]):

$$g^*_{ij} = g_{ij} + (\dot{\partial}_j \, g_{ik}) y^k \tag{2.5}$$

and then:

$$K_x = -\frac{\begin{vmatrix} g_{ij} + \dot{\partial}_j \, g_{ik} y^k & 2 g_{ik} y^k \\ 2 g_{jk} y^k & 0 \end{vmatrix}}{2^{n+1} \left[\sum\limits_{i=1}^{n} (g_{ik} y^k)^2\right]^{\frac{n+1}{2}}}. \tag{2.6}$$

Using the notation $y_i = g_{ik}y^k$ a simple form can be given:

$$K_x = -\frac{\begin{vmatrix} g_{ij}+ \dot{\partial}_j \ g_{ik}y^k & 2y_i \\ 2y_j & 0 \end{vmatrix}}{2^{n+1}\left[\sum\limits_{i=1}^{n} y_i^2\right]^{\frac{n+1}{2}}}. \tag{2.7}$$

For example $n = 2$ the denominator of (2.6) is:

$$D_x = \begin{vmatrix} g_{11}+ \dot{\partial}_1 \ g_{11}y^1+ \dot{\partial}_1 \ g_{12}y^2 & g_{12}+ \dot{\partial}_2 \ g_{11}y^1+ \dot{\partial}_2 \ g_{12}y^2 & 2y_1 \\ g_{12}+ \dot{\partial}_1 \ g_{12}y^1+ \dot{\partial}_1 \ g_{22}y^2 & g_{22}+ \dot{\partial}_2 \ g_{12}y^1+ \dot{\partial} \ g_{22}y^2 & 2y_2 \\ 2y_1 & 2y_2 & 0 \end{vmatrix}. \tag{2.8}$$

Let us suppose in addition that the GL-metric is diagonal i. e. $g_{12} = g_{21} = 0$. It follows:

$$D_x = \begin{vmatrix} g_{11}+ \dot{\partial}_1 \ g_{11}y^1 & \dot{\partial}_2 \ g_{11}y^1 & 2g_{11}y^1 \\ \dot{\partial}_1 \ g_{22}y^2 & g_{22}+ \dot{\partial}_2 \ g_{22}y^2 & 2g_{22}y^2 \\ 2g_{11}y^1 & 2g_{22}y^2 & 0 \end{vmatrix} =$$

$$= 2g_{11}y^1\left[2y^1y^2\left(g_{22} \ \dot{\partial}_2 \ g_{11} - g_{11} \ \dot{\partial}_2 \ g_{22}\right) - 2g_{11}g_{22}y^1\right] -$$
$$-2g_{22}y^2\left[2y^1y^2\left(g_{22} \ \dot{\partial}_1 \ g_{11} - g_{11} \ \dot{\partial}_1 \ g_{22}\right) + 2g_{11}g_{22}y^2\right]. \tag{2.9}$$

Another approach in the regular case is provided by homogeneity. By multiplication of eq. (2.4) with y^i it results:

$$\left(\dot{\partial}_i \ \mathcal{E}\left(g\right)\right) y^i = 2g_{ij}y^iy^j = 2\mathcal{E}\left(g\right) \tag{2.10}$$

which means that $\mathcal{E}\left(g\right)$ is 2-homogeneous i. e. $E\left(g\right)$ is a Finsler function([5]). But for a Finsler function in [6, p. 40] it is proved:

Proposition 2.4 *Let $(I\!R^n, F)$ be a Finsler space and $g = \frac{1}{2}\det\left(\dot{\partial}_i\dot{\partial}_j \ F^2\right)$. At each point $x \in I\!R^n$ the Gaussian curvature K_x of the indicatrix I_x oriented in the direction opposite to $\dot{\nabla} \ L$ is:*

$$K_x = \frac{g}{\left(\sum\limits_{i=1}^{n} l_i^2\right)^{\frac{n+1}{2}}} \tag{2.11}$$

where $l_i = \dot{\partial}_i \ F$.

It follows in our framework:

Proposition 2.5 *If $g=(g_{ij}(x,y))$ is a regular generalized Lagrange metric then the Gaussian curvature K_x of the indicatrix I_x is:*

$$K_x=\frac{g^*}{\left(\sum_{i=1}^{n} y_i^2\right)^{\frac{n+1}{2}}} \tag{2.12}$$

where $g^*=\det\left(g^*_{ij}\right)$.

3 Beil metrics as examples

Let $\widetilde{g}=(\widetilde{g}_{ij})$ be a Finsler metric and $B=B^i(x,y)\,\dot{\partial}_i$ a d-vector field for which we denote $B_i=\widetilde{g}_{ij}B^j$ and $B_0=B_iy^i$. Let also $a,b\in C^\infty(TM)$. In [1] and [2] the following GL-metric is studied:

$$g_{ij}=a\widetilde{g}_{ij}+bB_iB_j. \tag{3.1}$$

This GL-metrics, called *Beil metrics*, are not Lagrange metrics. Because:

$$\mathcal{E}(g)=a\mathcal{E}(\widetilde{g})+b(B_0)^2 \tag{3.2}$$

we get:

$$\dot{\partial}_i\,\mathcal{E}(g)=\dot{\partial}_i\,a\mathcal{E}(\widetilde{g})+a\,\dot{\partial}_i\,\mathcal{E}(\widetilde{g})+\dot{\partial}_i\,b(B_0)^2+2bB_0\,\dot{\partial}_i\,B_0 \tag{3.3}$$

$$2g^*_{ij}=2\widetilde{g}^*_{ij}+\dot{\partial}_i\dot{\partial}_j\,aE(\widetilde{g})+\dot{\partial}_i\,a\,\dot{\partial}_j\,E(\widetilde{g})+\dot{\partial}_j\,a\,\dot{\partial}_i\,E(\widetilde{g})+\dot{\partial}_i\dot{\partial}_j\,b(B_0)^2+$$
$$+2B_0\left(\dot{\partial}_i\,b\,\dot{\partial}_j\,B_0+\dot{\partial}_j\,b\,\dot{\partial}_i\,B_0+b\,\dot{\partial}_i\dot{\partial}_j\,B_0\right)+2b\,\dot{\partial}_i\,B_0\,\dot{\partial}_j\,B_0. \tag{3.4}$$

I (**A Beil metric with hypersphere as indicatrix**) On $T_0\mathbb{R}^n$ let:

$$a=\frac{1}{2},b=\frac{1}{2\|y\|_F^2},B_i=y_i=\widetilde{g}_{ik}y^k \tag{3.5}$$

where $\|,\|_F$ is the norm induced by the Finsler metric $\widetilde{g}$ i.e. $\|y\|_F^2=\mathcal{E}(\widetilde{g})=\widetilde{g}_{ij}y^iy^j$. Let us remark that for this choice $B=y^i\,\dot{\partial}_i$is exactly *the Liouville vector field* on TM. The considered Beil metric is:

$$g_{ij}=\frac{1}{2\|y\|_F^2}\begin{pmatrix} (y^1)^2+\|y\|_F^2 & y^1y^2 & \dots & y^1y^n \\ y^1y^2 & (y^2)^2+\|y\|_F^2 & \dots & y^2y^n \\ \dots & \dots & \dots & \dots \\ y^1y^n & y^2y^n & \dots & (y^n)^2+\|y\|_F^2 \end{pmatrix}. \tag{3.6}$$

A straightforward computation gives:

$$\mathcal{E}(g) = \|y\|_F^2 \tag{3.7}$$

which is 2-homogeneous and then a Finsler function. It results that (g_{ij}) is a Minkowskian regular generalized metric with:

$$g^*_{ij} = \delta_{ij}, \quad y_i = y^i \tag{3.8}$$

and then applying (2.12):

$$K_x = \frac{1}{\|y\|_F^{n+1}} = 1 \tag{3.9}$$

because we are on the indicatrix $I_x : \|y\|_F^2 = 1$.

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Faculty of Mathematics
University "Al. I. Cuza"
Iaşi, 6600
România
e-mail: mcrasm@uaic.ro