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NONLINEAR CONNECTIONS AND SEMISPRAYS ON TANGENT MANIFOLDS

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Abstract. The well–known notions from tangent bundle geometry, like nonlinear connections and semisprays, are extended to bundle–type tangent manifolds. Also, new objects interesting from a dynamical point of view, like symmetries of nonlinear connections, are introduced.

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1. Introduction

Almost tangent structures, introduced by Clark and Bruckheimer, [3], and Eliopoulos, [6], around 1960, have been investigated by several authors, see [1], [4], [5], [11]. As is well–known, the tangent bundle of a manifold carries a canonical integrable almost tangent structure, hence the name. This almost tangent structure plays an important role in the Lagrangian description of analytical mechanics ([5], [7], [8]).

The aim of the present paper is to extend two natural objects, namely nonlinear connections and semisprays, from the tangent bundles to tangent manifolds geometry. The former geometrical object is studied by means of vertical projectors and the latter implies the existence of a global vector field of the Liouville type.

The paper is structured as follows. In the second section nonlinear connections are introduced and interpreted as kernels of vertical projectors and the equivalence with other two types of vector 1–forms is proved. In the third section the notion of second order differential system (semispray in short) is defined and the relationship between semisprays and nonlinear connections is discussed in detail. As a particular case, the notion of spray corresponds to a homogeneity condition. In the last section, a completely new notion (to the best of our knowledge!), namely symmetry of a vertical projector(=nonlinear connection), is considered and studied. The paper ends with types of curves associated in a natural manner to nonlinear connections and semisprays.

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As methods, both global and local expressions are used in order to obtain several characterizations. Since vector 1–forms are used throughout the paper the Frölicher-Nijenhuis formalism has a main role in the global descriptions.

2. Nonlinear connections on tangent manifolds

Let M be a smooth, m-dimensional real manifold for which we denote: $C^{\infty}(M)$ – the real algebra of smooth real functions on M, $\mathcal{X}(M)$ – the Lie algebra of vector fields on M, $T^r_s(M)$ – the $C^{\infty}(M)$–module of tensor fields of (r,s)–type on M. An element of $T^1_1(M)$ is usually called *vector 1–form*, [9, p. 176].

The framework of our paper is fixed by:

Definition 2.1. *$J \in T^1_1(M)$ is called* almost tangent structure *on M if*

$$imJ = \ker J. \tag{2.1}$$

The pair (M,J) is an almost tangent manifold.

The name is motivated by the fact that (2.1) implies the nilpotence $J^2 = 0$ exactly as the natural tangent structure of tangent bundles, [8].

Denoting $rankJ = n$ it results $m = 2n$. In addition, we suppose that J is integrable, i.e.

$$N_J(X,Y) := [JX,JY] - J[JX,Y] - J[X,JY] + J^2[X,Y] = 0 \tag{2.2}$$

and in this case J is called *tangent structure* and (M,J) is called *tangent manifold*.

In the following we shall work only on tangent manifolds. From [10, p. 6–7] we get:

(i) the distribution $imJ(=\ker J)$ defines a foliation denoted $V(M)$ and called *the vertical distribution*

Example 2.1. *$M = \mathbb{R}^2, J(x,y) = (0,x)$ is a tangent structure with $\ker J$ the y–axis, hence the name.*

(ii) there exists an atlas on M with local coordinates $(x,y) = \left(x^i,y^i\right)_{1\leq i\leq n}$ such that $J = \frac{\partial}{\partial y^i} \otimes dx^i$, i.e.

$$J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}, \quad J\left(\frac{\partial}{\partial y^i}\right) = 0. \tag{2.3}$$

We call *canonical coordinates* the above (x,y) and the change of canonical coordinates $(x,y) \to (\widetilde{x},\widetilde{y})$ is given by, [10, p. 7],

$$\begin{cases} \widetilde{x}^i = \widetilde{x}^i(x) \\ \widetilde{y}^i = \dfrac{\partial \widetilde{x}^i}{\partial x^a} y^a + B^i(x) \end{cases}. \tag{2.4}$$

It results in an alternative description in terms of G–structures. Namely, a tangent structure is a G–structure with

$$G = \{C = \begin{pmatrix} A & O_n \\ B & A \end{pmatrix} \in GL(2n, \mathbb{R}); A \in GL(n, \mathbb{R})\}$$

and G is the invariance group of the matrix $J = \begin{pmatrix} O_n & O_n \\ I_n & O_n \end{pmatrix}$, i.e. $C \in G$ if and only if $C \cdot J = J \cdot C$.

Inspired by Definition 1.1 of [2, p. 71] we give a first main notion:

Definition 2.2. *A vector 1–form* $v : \mathcal{X}(M) \to \mathcal{X}(M)$ *satisfying*

$$\begin{cases} J \circ v = 0 \\ v \circ J = J \end{cases} \tag{2.5}$$

is called vertical projector.

From (2.5_1) $imv \subseteq \ker J = V(M)$ and from (2.5_2) $v|_{imJ} = 1_{V(M)}$. In conclusion $imv = V(M)$ and $v^2 = v$; these facts explain the name of v.

Another well–known notion in tangent bundles geometry extends to:

Definition 2.3. ([10, p. 19]) *A supplementary distribution N to the vertical distribution $V(M)$:*

$$\mathcal{X}(M) = N \oplus V(M) \tag{2.6}$$

is called normalization *or* horizontal distribution *or* nonlinear connection. *A vector field belonging to N is called* horizontal *and one belonging to $V(M)$ is called* vertical.

Because a vertical projector v is $C^\infty(M)$–linear with $imv = V(M)$ we have a first important result:

Proposition 2.1. *A vertical projector v yields a nonlinear connection denoted $N(v)$ through relation $N(v) = \ker v$.*

This relation is a generalization of remarks from [2, p. 71] where the tangent bundles case is treated. An important remark is that last result admits a converse. Namely, if N is a nonlinear connection let h_N, v_N the horizontal and vertical projection with respect to the decomposition (2.6).

Proposition 2.2. *v_N is a vertical projector with $N(v_N) = N$.*

Proof. From $imv_N = V(M) = \ker J$ it follows (2.5_1). v_N being projector satisfy $v_N(V(M)) = V(M) = imJ$ and then we have (2.5_2). The second fact comes immediately from the definition of $N(v_N)$. □

With respect to the identification nonlinear connection=vertical projector let us point other two equivalent choices:

I) Following [7] we get:

Definition 2.4. *A vector 1–form* Γ *is called* nonlinear connection of almost product type *if*

$$\begin{cases} \Gamma \circ J = -J \\ J \circ \Gamma = J \end{cases} . \tag{2.7}$$

Proposition 2.3. *If* Γ *is a nonlinear connection of almost product type then*

(i) $v_\Gamma = \frac{1}{2}\left(1_{\mathcal{X}(M)} - \Gamma\right)$ *is a vertical projector,*

(ii) $V(M)$ *is the* (-1)*–eigenspace of* Γ*,*

(iii) $N(v_\Gamma)$ *is the* $(+1)$*–eigenspace of* Γ*.*

It comes out that every vertical projector v *yields a nonlinear connection of almost product type:* $\Gamma = 1_{\mathcal{X}(M)} - 2v$*. From this last relation it results that* $\Gamma^2 = 1_{X(M)}$*, i.e.* Γ *is an almost product structure on* M *(hence the name).*

Proof.

(*i*) $J \circ v_\Gamma = \frac{1}{2}(J - J \circ \Gamma) \overset{(2.7_2)}{=} \frac{1}{2}(J - J) = 0$ and $v_\Gamma \circ J = \frac{1}{2}(J - \Gamma \circ J) \overset{(2.7_1)}{=} \frac{1}{2}(J + J) = J$.

(*ii*) $V(M) = im v_\Gamma = \{X \in \mathcal{X}(M) ; \Gamma(X) = -X\}$.

(*iii*) $N(v_\Gamma) = \ker v_\Gamma = \{X \in \mathcal{X}(M) ; \Gamma(X) = X\}$. □

II) Inspired by [9, p. 180] we define:

Definition 2.5. *A vector 1–form* h *is called* horizontal projector *if*

$$\begin{cases} h^2 = h \\ \ker h = V(M) \end{cases} . \tag{2.8}$$

Proposition 2.4. *If* h *is a horizontal projector then*

(i) $v_h = 1_{\mathcal{X}(M)} - h$ *is a vertical projector,*

(ii) $N(v_h)$ *is the* $(+1)$*–eigenspace of* h*.*

It follows that every vertical projector v *yields a horizontal projector* $h = 1_{\mathcal{X}(M)} - v$*.*

Proof. (*i*) From $h\left(1_{\mathcal{X}(M)} - h\right) = 0$ we have $im\left(1_{\mathcal{X}(M)} - h\right) \subseteq \ker h = V(M) = \ker J$, then $J \circ v_h = 0$. Also, $imJ = V(M) = \ker h$ imply $v_h \circ J = J - h \circ J = J$.

(*ii*) $N(v_h) = \ker v_h = \{X \in \mathcal{X}(M) ; h(X) = X\}$. □

In canonical coordinates a vertical projector reads

$$v = N^i_j \frac{\partial}{\partial y^i} \otimes dx^j + \frac{\partial}{\partial y^i} \otimes dy^i = \frac{\partial}{\partial y^i} \otimes \left(N^i_j dx^j + dy^i\right) \tag{2.9}$$

and the functions $\left(N^i_j(x,y)\right)_{1 \leq i,j \leq n}$ are called *the coefficients* of v, respectively $N(v)$. A basis of $\mathcal{X}(M)$ adapted to the decomposition (1.6) is

$$\left\{ \frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^i} \right\}_{1 \leq i \leq n}$$

and it is called *the Berwald basis.* Then

$$v = \frac{\partial}{\partial y^i} \otimes \delta y^i, \quad h = \frac{\delta}{\delta x^i} \otimes dx^i,$$

where $\{dx^i, \delta y^i = dy^i + N^i_j dx^j\}$ is the dual of the Berwald basis.

3. Semisprays on bundle-type tangent manifolds

In the following we suppose that $V(M)$ admits a global section $E = y^i \frac{\partial}{\partial y^i}$ called *Euler vector field* after [10, p. 4] (on tangent bundles E is called *Liouville vector field* , cf. [2, p. 70]). Again after [10, p. 7] the triple (M, J, E) will be called *bundle–type tangent manifold* and in this case $\left(B^i\right)$ from (2.4_2) are zero, cf. [10, p. 7]. For examples of bundle–type tangent manifolds see [10].

As in the tangent bundle case, [2, p. 70], we give a second main notion:

Definition 3.1. *If (M, J, E) is a bundle–type tangent manifold then $S \in \mathcal{X}(M)$ is called* semispray *or* second order differential equation *(*sode *in short) if*

$$J(S) = E. \tag{3.1}$$

In canonical coordinates

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y) \frac{\partial}{\partial y^i} \tag{3.2}$$

and the functions $\left(G^i(x,y)\right)$ are *the coefficients* of S.

Another important result is:

Proposition 3.1. *A vertical projector v yields an unique horizontal semispray denoted $S(v)$.*

Proof. This proposition is a generalization of a similar result (without proof) from [2, p. 71]. The formula

$$G^i = \frac{1}{2} N^i_j y^j \tag{3.3}$$

gives the conclusion. □

In other words

$$S(v) = y^i \frac{\delta}{\delta x^i}. \tag{3.3'}$$

The converse of last result is:

Proposition 3.2. *If S is a semispray, then $v_S : \mathcal{X}(M) \to \mathcal{X}(M)$ given by*

$$v_S(X) = \frac{1}{2}\left(X + [S, JX] + J[X, S]\right) \tag{3.4}$$

is a vertical projector.

Proof. Because

$$J \circ v_S(X) = \frac{1}{2}\left(JX - J[JX, S]\right), \quad v_S \circ J(X) = \frac{1}{2}\left(JX + J[JX, S]\right)$$

it must prove that

$$J[JX, S] = JX \tag{3.5}$$

for every $X \in \mathcal{X}(M)$.

But from (2.2) with $Y = S$ we have

$$[JX, E] - J[JX, S] - J[X, E] = 0 \tag{3.6}$$

and then (3.5) is equivalent with

$$[JX, E] = J\left([X, E] + X\right). \tag{3.7}$$

Case 1) $X = \frac{\partial}{\partial x^i} \Rightarrow \left[\frac{\partial}{\partial y^i}, y^a \frac{\partial}{\partial y^a}\right] = \frac{\partial}{\partial y^i} = J\left(\frac{\partial}{\partial x^i}\right)$, i.e. (3.7) is true for this case.

Case 2) $X = \frac{\partial}{\partial y^i} \Rightarrow [0, E] = 0 = J\left(\frac{\partial}{\partial y^i} + \frac{\partial}{\partial y^i}\right)$, i.e. (3.7) is true for this case. □

If S is given by (3.2) then the coefficients of v_S are

$$N^i_j = \frac{\partial G^i}{\partial y^j}. \tag{3.8}$$

A first natural question is: given the vertical projector $v\left(= N^i_j\right)$, does a semispray S such that $v = v_S$ exist? Looking at (3.8) it results that $\left(N^i_j\right)$ must be a gradient with respect to $\left(y^i\right)$. Then if we define: $t^k_{ij} = \frac{\partial N^k_i}{\partial y^j} - \frac{\partial N^k_j}{\partial y^i}$ it results in:

Corollary 3.1. *There exists a semispray S such that $v = v_S$ if and only if $t^k_{ij} = 0, 1 \le i, j, k \le n$.*

A second natural question is related to the sequence

$$S \to v_S \to S(v_S)$$
$$G^i \overset{(3.8)}{\to} \frac{\partial G^i}{\partial y^j} \overset{(3.3)}{\to} \frac{1}{2}\frac{\partial G^i}{\partial y^j} y^j ;$$

when $S = S(v_S)$?

Corollary 3.2. *Let S be a semispray and v_S the associated vertical projector. Then S is exactly $S(v_S)$ given by Proposition 3.1 if and only if*

$$[E, S] = S. \tag{3.9}$$

Proof. $v_S(S) = 0 \overset{(3.4)}{\Leftrightarrow} S + [S, E] = 0.$ □

Definition 3.2. *A semispray satisfying* (3.9) *is called* spray.

Locally (3.9) means

$$2G^i = y^j \frac{\partial G^i}{\partial y^j} \tag{3.10}$$

i.e. the functions $\left(G^i\right)$ are homogeneous of degree 2 with respect to variables $\left(y^i\right)$. In terms of associated vertical projector $v_S = \left(N^i_j\right)$, using (3.8), it results that $\left(N^i_j\right)$ are homogeneous of degree 1 with respect to $\left(y^i\right)$

$$N^i_j = y^a \frac{\partial N^i_j}{\partial y^a}. \tag{3.11}$$

The above formulae can be put in a compact form using the Frölicher–Nijenhuis formalism. Recall that for a vector 1–form K and $Z \in \mathcal{X}(M)$ we have the bracket $[K, Z]_{FN} : \mathcal{X}(M) \to \mathcal{X}(M)$ given by, [9, p. 177],

$$[K, Z]_{FN}(X) = [K(X), Z] - K[X, Z] \tag{3.12}$$

where in the R.H.S. we have the usual Lie bracket of vector fields. Then (3.4) becomes

$$v_S = \frac{1}{2}\left(1_{X(M)} - [J, S]_{FN}\right) \tag{3.4$'$}$$

and looking to Proposition 2.3 it comes out that $[J, S]_{FN}$ is exactly the nonlinear connection of almost product type Γ associated to v_S.

Corollary 3.3. *A semispray S is a spray if and only if*

$$[v_S, E]_{FN} = 0. \tag{3.9$'$}$$

Proof. Let $X \in \mathcal{X}(M)$. The above relation means $[v_S(X), E] = v_S([X, E])$.

I) If $X = \frac{\partial}{\partial y^i}$, then
$$\left[\frac{\partial}{\partial y^i}, E\right] = v_S\left(\left[\frac{\partial}{\partial y^i}, E\right]\right),$$
which is true because $\left[\frac{\partial}{\partial y^i}, E\right] = E \in V(M)$,

II) If $X = \frac{\delta}{\delta x^i}$, then
$$0 = v_S\left(\left[\frac{\delta}{\delta x^i}, E\right]\right) = v_S\left(\left(y^a \frac{\partial N_i^j}{\partial y^a} - N_i^j\right)\frac{\partial}{\partial y^j}\right) = \left(y^a \frac{\partial N_i^j}{\partial y^a} - N_i^j\right)\frac{\partial}{\partial y^j}$$
which is equivalent with characterization (3.11). □

A third natural question is related to the sequence
$$v \to S(v) \to v_{S(v)}$$
$$N_j^i \overset{(3.3)}{\to} G^i = \frac{1}{2} N_k^i y^k \overset{(3.8)}{\to} \frac{\partial G^i}{\partial y^j};$$
when $v = v_{S(v)}$? We must have $N_j^i = \frac{1}{2}\frac{\partial}{\partial y^j}\left(N_k^i y^k\right) = \frac{1}{2}N_j^i + \frac{1}{2}y^k \frac{\partial N_k^i}{\partial y^j}$ and then:

Corollary 3.4. *Let $v\left(= N_j^i\right)$ be a vertical projector and $S(v)$ the associated semispray. Then v is exactly $v_{S(v)}$ given by Proposition 3.2 if and only if*
$$N_j^i = y^k \frac{\partial N_k^i}{\partial y^j}. \tag{3.13}$$
If $v = v_{S(v)}$, then $S(v)$ is a spray, $t_{ij}^k = 0$ and (2.11) holds.

A last question is: given the semispray $S\left(= G^i\right)$ does a vertical projector v such that $S = S(v)$ exist? So, we have to solve the system $G^i = N_j^i y^j$ in the unknowns $\left(N_j^i\right)$. We do not know the general answer but is obvious that if S is spray then the answer is positive with $v = v_S$.

4. Symmetries and paths of nonlinear connections

Let N be a nonlinear connection with associated vertical projector $v = \left(N_j^i\right)_{1\le i,j\le n}$. With respect to the Berwald basis $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}_{1\le i\le n}$ we have
$$\begin{cases} \left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right] = R_{ij}^a \frac{\partial}{\partial y^a} \\ \left[\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\right] = \frac{\partial N_i^a}{\partial y^j}\frac{\partial}{\partial y^a} \\ \left[\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right] = 0 \end{cases} \tag{4.1}$$

where

$$R^a_{ij} = \frac{\delta N^a_i}{\delta x^j} - \frac{\delta N^a_j}{\delta x^i}. \tag{4.2}$$

Then the horizontal distribution N is integrable if and only if $R^k_{ij} = 0, 1 \leq i, j, k \leq n$.

A notion which does not appear in tangent bundle geometry (to our knowledge!) but is inspired by (3.9′) and in general is important from a dynamical point of view is:

Definition 4.1. *$X \in \mathcal{X}(M)$ is a* symmetry of v *(or N) if* $[v, X]_{FN} = 0$.

A characterization of symmetries is given by:

Proposition 4.1. *The vector field $X = X^a \frac{\delta}{\delta x^a} + X^{n+a} \frac{\partial}{\partial y^a}$ is a symmetry of v if and only if*

$$(4.3) \qquad \begin{cases} \dfrac{\partial X^a}{\partial y^i} = 0 \quad \textit{(then } X^a \textit{ depends only of } x!) \\ \dfrac{\delta X^{n+a}}{\delta x^i} + R^a_{ij} X^j + X^{n+j} \dfrac{\partial N^a_i}{\partial y^j} = 0 \end{cases} .$$

Proof. I)

$$\begin{aligned} 0 &= [v, X]_{FN} \left(\frac{\partial}{\partial y^i} \right) = \left[v \left(\frac{\partial}{\partial y^i} \right), X \right] - v \left[\frac{\partial}{\partial y^i}, X \right] = h \left[\frac{\partial}{\partial y^i}, X \right] \\ &= h \left(\frac{\partial X^a}{\partial y^i} \frac{\delta}{\delta x^a} + \left(\frac{\partial X^{n+a}}{\partial y^i} - X^j \frac{\partial N^a_j}{\partial y^i} \right) \frac{\partial}{\partial y^a} \right) = \frac{\partial X^a}{\partial y^i} \frac{\delta}{\delta x^a} \end{aligned}$$

which gives (4.3_1);

II)

$$\begin{aligned} 0 &= [v, X]_{FN} \left(\frac{\delta}{\delta x^i} \right) \\ &= -v \left(\frac{\delta X^a}{\delta x^i} \frac{\delta}{\delta x^a} + \left(\frac{\delta X^{n+a}}{\delta x^i} + R^a_{ij} X^j + X^{n+j} \frac{\partial N^a_i}{\partial y^j} \right) \frac{\partial}{\partial y^a} \right) \\ &= - \left(\frac{\delta X^{n+a}}{\delta x^i} + R^a_{ij} X^j + X^{n+j} \frac{\partial N^a_i}{\partial y^j} \right) \frac{\partial}{\partial y^a} \end{aligned}$$

which gives (4.3_2). □

Corollary 4.1. *(i) An horizontal vector field $X = X^a \frac{\delta}{\delta x^a}$ is a symmetry of v if and only if the coefficients (X^a) depend only of x and $R^a_{ij} X^j = 0$, $1 \leq a$,*

$i \leq n$. In particular, if the nonlinear connection N is integrable, then X is the symmetry of N (or v) if and only if $X = X^a(x)\frac{\delta}{\delta x^a}$.

(ii) A vertical vector field $X = X^{n+a}\frac{\partial}{\partial y^a}$ is a symmetry of v if and only if

$$\frac{\delta X^{n+a}}{\delta x^i} + X^{n+j}\frac{\partial N_i^a}{\partial y^j} = 0, \quad 1 \leq a, \quad i \leq n.$$

Let us suppose that $v = v_S$ for the semispray $S\left(= G^i\right)$. From (3.4′) we get that X is a symmetry for v_S if and only if: $\left[1_{\mathcal{X}(M)} - [J,S]_{FN}, X\right]_{FN} = 0$; but $\left[1_{\mathcal{X}(M)}, X\right]_{FN} = 0$ for every X and then X is a symmetry for v_S if and only if

$$[[J,S]_{FN}, X]_{FN} = 0. \tag{4.4}$$

Looking at local expressions let us note that R_{ij}^a for v_S is

$$R_{ij}^a = \frac{\delta}{\delta x^j}\left(\frac{\partial G^a}{\partial y^i}\right) - \frac{\delta}{\delta x^i}\left(\frac{\partial G^a}{\partial y^j}\right) \tag{4.5}$$

and Proposition 4.1 yields:

Proposition 4.2. *The vector field $X = X^a\frac{\delta}{\delta x^a} + X^{n+a}\frac{\partial}{\partial y^a}$ is a symmetry for v_S if and only if* (4.3_1) *and*

$$\frac{\partial X^{n+a}}{\partial x^i} - \frac{\partial G^j}{\partial y^i}\frac{\partial X^{n+a}}{\partial y^j} + R_{ij}^a X^j + X^{n+j}\frac{\partial^2 G^a}{\partial y^i \partial y^j} = 0$$

holds, where R_{ij}^a is given by (4.5).

It comes out that S can not be symmetry for v_S because (4.3_1) does not hold.

Also, Corollary 4.1 yields:

Corollary 4.2. *(i) The horizontal vector field $X = X^a\frac{\delta}{\delta x^a}$ is symmetry for v_S if and only if the coefficients (X^a) do not depend on $\left(y^i\right)$ and $R_{ij}^a X^j = 0$, $1 \leq a$, $i \leq n$, where R_{ij}^a is given by* (4.5). *In particular, if the nonlinear connection N_S is integrable, i.e.*

$$\frac{\delta}{\delta x^j}\left(\frac{\partial G^a}{\partial y^i}\right) = \frac{\delta}{\delta x^i}\left(\frac{\partial G^a}{\partial y^j}\right),$$

with

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \frac{\partial G^k}{\partial y^i}\frac{\partial}{\partial y^k},$$

then X is a symmetry for v_S if and only if $X = X^a(x)\frac{\delta}{\delta x^a}$.

(ii) The vertical vector field $X = X^{n+a}\frac{\partial}{\partial y^a}$ is a symmetry for v_S if and only if

$$\frac{\delta X^{n+a}}{\delta x^i} + X^{n+j}\frac{\partial^2 G^a}{\partial y^i \partial y^j} = 0, \quad 1 \leq a, \quad i \leq n.$$

In a more particular case S=spray from $(3.9')$ it comes out that E is symmetry for v_S.

Because we are interested in dynamics let us study the curves on bundle–type tangent manifolds. Let $c = c(t)$ be a curve on M with local expression $c(t) = (x(t), y(t)) = \left(x^i(t), y^i(t)\right)$. Three cases are of importance:

I) c is an *integral curve* of the semispray S. It results from (3.2) that the differential system

$$\begin{cases} \dfrac{dx^i}{dt}(t) = y^i(t) \\ \dfrac{dy^i}{dt}(t) + 2G^i(x(t), y(t)) = 0 \end{cases} \tag{4.6}$$

which explains the name *sode* for S.

II) the tangent field of c is horizontal with respect to the vertical projector v. From (2.7)

$$v\left(\frac{dc}{dt}\right) = v\left(\frac{dx^i}{dt}\frac{\partial}{\partial x^i} + \frac{dy^i}{dt}\frac{\partial}{\partial y^i}\right) = \left(N^i_j\frac{dx^j}{dt} + \frac{dy^i}{dt}\right)\frac{\partial}{\partial y^i}. \tag{4.7}$$

Such a curve is called *h–path* of v and it is a solution of the differential system

$$\frac{dy^i}{dt}(t) + N^i_j(x(t), y(t))\frac{dx^j}{dt}(t) = 0. \tag{4.8}$$

III) an h–path of v satisfying in addition $\frac{dx^i}{dt} = y^i$ will be called *h–integral curve* of v and it is a solution to

$$\begin{cases} \dfrac{dx^i}{dt}(t) = y^i(t) \\ \dfrac{dy^i}{dt}(t) + N^i_j\left(x(t), \dfrac{dx}{dt}\right)\dfrac{dx^j}{dt}(t) = 0 \end{cases}. \tag{4.9}$$

With respect to Proposition 3.1 comparing (4.6) and (4.9) it results via (3.3):

Proposition 4.3. *An h–integral curve of v is an integral curve of $S(v)$.*

With respect to Proposition 3.2 there is no relation between integral curves of S and v_S in the general case. But in the homogeneous case (3.9) – (3.10) we get:

Proposition 4.4. *If S is a spray then an integral curve of S is an h–integral curve of v_S.*

References

[1] Brickell, F., Clark, R.S., Integrable almost tangent structures. J. Diff. Geom. 9 (1974), 557–563.

[2] Bucătaru, I., The Jacobi fields for a spray on the tangent bundle. Novi Sad J. Math. 29(3) (1999), 69–78.

[3] Clark, R.S., Bruckheimer, M., Sur les structures presque tangents. C.R.A.S. Paris 251 (1960), 627–629.

[4] Clark, R.S., Goel, D.S., On the geometry of an almost tangent manifold. Tensor 24 (1972), 243–252.

[5] Crampin, M., Thompson, G., Affine bundles and integrable almost tangent structures. Math. Proc. Camb. Phil. Soc. 98 (1985), 61–71.

[6] Eliopoulos, H.A., Structures presque tangents sur les variétés différentiables. C.R.A.S. Paris 255 (1962), 1563–1565.

[7] Grifone, J., Structure presque–tangente et connexions, I, II. Ann. Inst. Fourier (Grenoble) 22(13) (1972), 287–334, 291–338.

[8] Miron, R., Anastasiei, M., The geometry of Lagrange spaces: theory and applications. Kluwer Academic Publishers, FTPH no. 59, 1994.

[9] Szilasi, J., Muznay, Z., Nonlinear connections and the problem of metrizability. Publ. Math. Debrecen 42(1–2) (1993), 175–192.

[10] Vaisman, I., Lagrange geometry on tangent manifolds. math.DG/0212080.

[11] Yano, K., Davies, E.T., Differential geometry on almost tangent manifolds. Ann. Mat. Pura Appl. 103(4) (1975), 131–160.

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