Quadratic first integrals for natural Lagrangian systems

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Scientific Annals of University of Agricultural Sciences and Veterinary Medicine "Ion Ionescu de la Brad", Iaşi, 2003, v. 2, 125-128.

Abstract

Quadratic in moments first integrals for Lagrangian systems of natural type are discussed and the two-dimensional case is detailed.

2000 Math. Subject Classification: 70H33; 70H03.
Key words: natural Lagrangian system, first integral.

Several dynamical systems modelling physical or biological processes are of Lagrangian type, namely are described by a Lagrangian $L(x, \dot{x})$ where x is the dependent variables and the dot means the derivative with respect to the independent variable, usually considered as proper time, denoted t. If the given system has n degrees of freedom and the Lagrangian has the form $L = \frac{1}{2}\delta_{ij}\ \dot{x}^i\dot{x}^j\ - V(x)$ where $x = (x^i)_{1\leq i\leq n}, \dot{x}= \left(\dot{x}^i\right)_{1\leq i\leq n}$ the we call it *natural Lagrangian system* and V is *the potential.* The evolution paths of the system are solutions of Euler-Lagrange equations:

$$E_i(L) : \frac{d^2x}{dt^2}(t) + \frac{\partial V}{\partial x^i}(x(t)) = 0, 1 \leq i \leq n \tag{1}$$

which is a system of ordinary differential equations with special features for the "natural " case.

The present paper is devoted to a main notion in the study of (1), with great importance from a dynamical point of view, namely *first integrals.*

These are functions $\mathcal{F}(x,\dot{x})$ such that for every solution $x(t)$ of (1) we have: $\mathcal{F}\left(x(t), \frac{dx}{dt}(t)\right)$ =constant (does not depends of t).

More precisely, we search for quadratic in moments first integrals, namely:

$$\mathcal{F} = A_{ab}(x)\,\dot{x}^a\dot{x}^b + \Phi(x). \tag{2}$$

There is a strong motivation for this search; it's well-known that (1) admits the first integral:

$$H = \frac{1}{2}\delta_{ab}\,\dot{x}^a\dot{x}^b + V \tag{3}$$

called *Hamiltonian* of (1) which is quadratic.

In [1, p. 193] there are given the equations in order to obtain $\mathcal{F}$ from (2). We present on short the arguments of cited paper. Namely, set $p_i = \frac{\partial L}{\partial \dot{x}^i} = \delta_{ij}\,\dot{x}^j = \dot{x}^i$, then $F = A^{ab}(x)\,p_a p_b + \Phi(x)$ where $A^{ab} = \delta^{ai}\delta^{bj}A_{ij}$ is the contravariant version of the covariant tensor A. The Hamiltonian becomes: $H = \frac{1}{2}\delta^{ab}p_a p_b + V$ and then $\mathcal{F}$ is a first integral if and only if the usual Poisson bracket $\{\mathcal{F}, H\}$ vanishes. In [1, p. 193] is obtained:

$$\{\mathcal{F}, H\} = \delta^{ab}p_b\left(A^{cd}{}_{,a}\,p_c p_d + \Phi_{,a}\right) - 2A^{ab}p_a V_{,b} \tag{4}$$

where $T^{\cdots}_{\cdots}{}_{,a}$ means the derivative with respect to x^a i.e. $T^{\cdots}_{\cdots}{}_{,a} = \frac{\partial T^{\cdots}_{\cdots}}{\partial x^a}$. It follows:

Proposition([1]) $\mathcal{F}$ *is first integral if and only if:*

$$\begin{cases} A_{(ab,c)} = 0 \\ 2A^b_a V_{,b} = \Phi_{,a} \end{cases}. \tag{5}$$

Our main remark is that (5) can be obtained exactly in the Lagrangian framework and the Hamiltonian setting (moments, Poisson bracket) is not necessary. The point of start for our approach is:

Lemma *The function $f(q)$ vanishes on the solutions of the system $\phi_\alpha(q) = 0, 1 \leq \alpha \leq n$ if there exist functions $\mu^\alpha(q), 1 \leq \alpha \leq n$ such that $f = \mu^\alpha \phi_\alpha$.*

Therefore $\mathcal{F}$ is first integral of (1) if there exist functions $\mu^i(x,\dot{x}), 1 \leq i \leq n$ such that $\frac{d\mathcal{F}}{dt} = \mu^i E_i(L)$ and this relation yields exactly (5). We give here a proof only for the case $n = 2$.

For this case we have:

$$\begin{cases} E_1(L) = \ddot{x}^1 + V_{,1} \\ E_2(L) = \ddot{x}^2 + V_{,2} \\ \mathcal{F} = A\left(\dot{x}^1\right)^2 + 2B\,\dot{x}^1\dot{x}^2 + C\left(\dot{x}^2\right)^2 + \Phi \end{cases} \tag{6}$$

or denoting $A^{(2)} = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$ it results: $\mathcal{F} = \left(\dot{x}^1, \dot{x}^2\right) \cdot A^{(2)} \cdot \begin{pmatrix} \dot{x}^1 \\ \dot{x}^2 \end{pmatrix} + \Phi$ which yields:

$$\frac{d\mathcal{F}}{dt} = \left(\ddot{x}^1, \ddot{x}^2\right) A^{(2)} \begin{pmatrix} \dot{x}^1 \\ \dot{x}^2 \end{pmatrix} + \left(\dot{x}^1, \dot{x}^2\right) \frac{d}{dt} A^{(2)} \begin{pmatrix} \dot{x}^1 \\ \dot{x}^2 \end{pmatrix} + \left(\dot{x}^1, \dot{x}^2\right) A^{(2)} \begin{pmatrix} \ddot{x}^1 \\ \ddot{x}^2 \end{pmatrix} + \frac{d\Phi}{dt} =$$

$$= \mu^1\left(\ddot{x}^1 + V_{,1}\right) + \mu^2\left(\ddot{x}^2 + V_{,2}\right). \tag{7}$$

Then:

$$\begin{cases} \mu^1 = 2\left(A\,\dot{x}^1 + B\,\dot{x}^2\right) \\ \mu^2 = 2\left(B\,\dot{x}^1 + C\,\dot{x}^2\right) \end{cases} \; or \; \begin{pmatrix} \mu^1 \\ \mu^2 \end{pmatrix} = 2A^{(2)} \cdot \begin{pmatrix} \dot{x}^1 \\ \dot{x}^2 \end{pmatrix}. \tag{8}$$

Returning to (7) it follows:

$$\begin{cases} A_{,1} = 0 \\ A_{,2} + 2B_{,1} = 0 \\ 2B_{,2} + C_{,1} = 0 \\ C_{,2} = 0 \end{cases}$$

which is exactly (5_1) and:

$$\begin{cases} \Phi_{,1} = 2\left(AV_{,1} + BV_{,2}\right) \\ \Phi_{,2} = 2\left(BV_{,1} + CV_{,2}\right) \end{cases}$$

which is exactly (5_2). The last relation can be writed:

$$\begin{pmatrix} \frac{\partial}{\partial x^1} \\ \frac{\partial}{\partial x^2} \end{pmatrix} \Phi = 2A^{(2)} \cdot \begin{pmatrix} \frac{\partial}{\partial x^1} \\ \frac{\partial}{\partial x^2} \end{pmatrix} V. \tag{9}$$

An important remark is that (5_1) does not depend of potential V i.e. does not depend of system. In [1, p. 195] is given the general solution of (5_1):

$$A^{(2)} = aM + bL_1 + cL_2 + eE_1 + dE_2 + gE_3 \tag{10}$$

with a, b, c, d, e, g arbitrary real numbers and:

$$M = \frac{1}{2}\begin{pmatrix} y^2 & -xy \\ -xy & x^2 \end{pmatrix}, L_1 = \frac{1}{2}\begin{pmatrix} 0 & -y \\ -y & 2x \end{pmatrix}, L_2 = \frac{1}{2}\begin{pmatrix} 2y & -x \\ -x & 0 \end{pmatrix} \tag{11}$$

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, E_3 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{12}$$

References

[1] Boccaletti, D., Pucacco, G., *Killing equations in classical mechanics,* Il Nuovo Cimento, 112B(1997), N. 2-3, 181-212.

[2] Crâşmăreanu, M., *Conservation laws generated by pseudosymmetries with applications to Hamiltonian systems*, Ital. J. Pure Appl. Math., 8(2000), 91-98.

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