Noether symmetries for 1D spinning particle

Mircea Crâşmăreanu

Dedicated to Academician Radu Miron on the occasion of 75th birthday
Libertas Mathematica, 24(2004), 29-32.

Abstract

In this paper we obtain five Noether symmetries and corresponding Noetherian first integrals for one-dimensional spinning particle. Only the first from these Noetherian conservation quantities was previously known.

Subject Classification: 70H50, 70H33, 37J15, 34A26.
Key words and phrases: Noether symmetry, Killing equation, Noetherian first integral.

The aim of present paper is to obtain some Noether symmetries for classical spinning particle. To this purpose, let us recall the theory of Noether symmetries for higher-order Lagrangians.

In our study we treat only one dependent variable. This is for purposes of clarity because the case of several dependent variables is obtained by the inclusion of appropriately positioned indices. The general results will simple be stated. Let us point that an interesting discussion appear for the 2D case; see [2].

Let $L = L(t, q^{(0)}, q^{(1)}, ..., q^{(k)})$ be a Lagrangian of order k with usual action integral $A = \int Ldt$ where $q^{(0)} = q, q^{(1)} = \frac{dq}{dt}, ..., q^{(k)} = \frac{d^kq}{dt^k}$. Let $S = \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial q}$

be a given vector field which yields the infinitesimal transformation (flow):

$$\begin{cases} \widetilde{t} = t + \varepsilon\tau \\ \widetilde{q} = q + \varepsilon\eta \end{cases} . \tag{1}$$

The new Lagrangian will be: $\widetilde{L} = L(\widetilde{t}, \widetilde{q}, \frac{d\widetilde{q}}{d\widetilde{t}}, ..., \frac{d^k\widetilde{q}}{d\widetilde{t}^k})$ and the corresponding action $\widetilde{A} = \int \widetilde{L} d\widetilde{t}$.

In the following, we adopt the approach of [8](see also [5]); let us point that another way to obtain a Noether-type theorem in higher-order mechanics, based on higher-order energies, appears in [7] for the autonomous case (i.e. independent of time Lagrangians) and in [1] for the time-dependent case; also, for a non-Noetherian way to obtain first integrals for higher-order differential equations see [3]. Namely, S is called *Noether symmetry* if the action is invariant by the flow (1) of S i.e. $\widetilde{A} = A$. This is equivalent with the existence of a function f, usually called *gauge*(or *boundary term* in Noether's original work), such that the following *Killing equation* hold ([8, p. 224-225]):

$$\frac{df}{dt} = \frac{d\tau}{dt}L + \tau\frac{\partial L}{\partial t} + \sum_{i=0}^{k} \xi_i \frac{\partial L}{\partial q^{(i)}} \tag{2}$$

where:

$$\begin{cases} \xi_0 = \eta \\ \xi_1 = \frac{d\xi_0}{dt} - q^{(1)}\frac{d\tau}{dt} \\ \dots \\ \xi_i = \frac{d\xi_{i-1}}{dt} - q^{(i)}\frac{d\tau}{dt} \end{cases} . \tag{3}$$

In this case the function:

$$\mathcal{F} = f - [\tau L + \sum_{j=0}^{k-1}\sum_{i=j+1}^{k} (-1)^j(\xi_{i-j-1} - \tau q^{(i-j)})\frac{d^j}{dt^j}(\frac{\partial L}{\partial q^{(i)}})] \tag{4}$$

is a Noetherian first integral.

Let us treat in detail the particular case $k = 2$. The Killing equation becomes:

$$\frac{df}{dt} = \frac{d\tau}{dt}L + \tau\frac{\partial L}{\partial t} + \eta\frac{\partial L}{\partial q} + \left(\frac{d\eta}{dt} - q^{(1)}\frac{d\tau}{dt}\right)\frac{\partial L}{\partial q^{(1)}} +$$

$$+ \left(\frac{d^2\eta}{dt^2} - q^{(1)}\frac{d^2\tau}{dt^2} - 2q^{(2)}\frac{d\tau}{dt}\right)\frac{\partial L}{\partial q^{(2)}} \tag{5}$$

and the Noetherian first integral is:

$$\mathcal{F} = f - \tau L - - \left(\eta - q^{(1)}\tau\right)\left[\frac{\partial L}{\partial q^{(1)}} - \frac{d}{dt}\left(\frac{\partial L}{\partial q^{(2)}}\right)\right] - \left[\frac{d\eta}{dt} - q^{(1)}\frac{d\tau}{dt} - q^{(2)}\tau\right]\frac{\partial L}{\partial q^{(2)}}. \tag{6}$$

Example: the spinning particle

After [6, p. 4147], the Lagrangian of classical spinning particle is of second order, namely:

$$L = \frac{1}{2}\left(q^{(1)}\right)^2 - \frac{1}{2}\left(q^{(2)}\right)^2 \tag{7}$$

with the Euler-Lagrange equation:

$$q^{(4)} + q^{(2)} = 0. \tag{8}$$

(For a detailed discussion with historical arguments see [13]). Let us point that the two-dimensional case is also very interesting from the point of view of Noetherian symmetries, cf. [2]. The Killing equation (5) is:

$$\frac{df}{dt} = \frac{d\tau}{dt}L + q^{(1)}\left(\frac{d\eta}{dt} - q^{(1)}\frac{d\tau}{dt}\right) - q^{(2)}\left(\frac{d^2\eta}{dt^2} - q^{(1)}\frac{d^2\tau}{dt^2} - 2q^{(2)}\frac{d\tau}{dt}\right) \tag{9}$$

and the Noetherian first integral (6) is:

$$\mathcal{F} = f - \tau L - \left(\eta - q^{(1)}\tau\right)\left(q + q^{(3)}\right) + \left[\frac{d\eta}{dt} - q^{(1)}\frac{d\tau}{dt} - q^{(2)}\tau\right]q^{(2)}. \tag{10}$$

Looking at some calculations for harmonic oscillator from [4] we obtain the following Noether symmetries:

(i)

$$S_1 = \frac{\partial}{\partial t} \tag{11}$$

with the gauge $G_1 = 0$ and the Noetherian first integral:

$$\mathcal{F}_1 = \frac{1}{2}\left(q^{(1)}\right)^2 - \frac{1}{2}\left(q^{(2)}\right)^2 + q^{(1)}q^{(3)} \tag{12}$$

which is exactly the energy (or Hamiltonian) of the Lagrangian (7), cf. [6, p. 4148].

(ii)

$$S_2 = \cos t \frac{\partial}{\partial q} \tag{13}$$

with the gauge $G_2 = \cos t q^{(1)}$ and the Noetherian first integral:

$$\mathcal{F}_2 = q^{(2)} \sin t + q^{(3)} \cos t. \tag{14}$$

(iii)

$$S_3 = \sin t \frac{\partial}{\partial q} \tag{15}$$

with the gauge $G_3 = \sin t q^{(1)}$ and the Noetherian first integral:

$$\mathcal{F}_3 = q^{(2)} \cos t - q^{(3)} \sin t. \tag{16}$$

(iv)

$$S_4 = \sin 2t \frac{\partial}{\partial t} + q^{(1)} \cos 2t \frac{\partial}{\partial q} \tag{17}$$

with the gauge:

$$\begin{aligned} G_4 = \sin 2t \left(\left(q^{(2)}\right)^2 - \frac{1}{2}\left(q^{(1)}\right)^2 - \frac{1}{2}\left(q^{(3)}\right)^2 + 2q^{(1)}q^{(2)} - q^{(1)}q^{(3)} \right) + \\ + \cos 2t \left(\left(q^{(1)}\right)^2 - \left(q^{(2)}\right)^2 + 2q^{(1)}q^{(2)} + q^{(1)}q^{(3)} + q^{(2)}q^{3} \right) \end{aligned} \tag{18}$$

and the Noetherian first integral:

$$\mathcal{F}_4 = \frac{1}{2}\left(\left(q^{(2)}\right)^2 - \left(q^{(3)}\right)^2 \right) \sin 2t + q^{(2)}q^{(3)} \cos 2t \tag{19}$$

(v)

$$S_5 = \cos 2t \frac{\partial}{\partial t} - q^{(1)} \sin 2t \frac{\partial}{\partial q} \tag{20}$$

with the gauge:

$$\begin{aligned} G_5 = \cos 2t \left(\left(q^{(2)}\right)^2 - \frac{1}{2}\left(q^{(1)}\right)^2 - \frac{1}{2}\left(q^{(3)}\right)^2 + 2q^{(1)}q^{(2)} - q^{(1)}q^{(3)} \right) + \\ + \sin 2t \left(- \left(q^{(1)}\right)^2 + \left(q^{(2)}\right)^2 - 2q^{(1)}q^{(2)} - q^{(1)}q^{(3)} - q^{(2)}q^{(3)} \right) \end{aligned} \tag{21}$$

and the Noetherian first integral:

$$\mathcal{F}_5 = \frac{1}{2}\left(\left(q^{(2)}\right)^2 - \left(q^{(3)}\right)^2\right)\cos 2t - q^{(2)}q^{(3)}\sin 2t. \tag{22}$$

Let us point a common expression for last two cases. Namely, setting:

$$\begin{cases} A = A\left(q^{(1)}, q^{(2)}, q^{(3)}\right) = \left(q^{(1)}\right)^2 - \left(q^{(2)}\right)^2 + 2q^{(1)}q^{(2)} + q^{(1)}q^{(3)} + q^{(2)}q^{(3)} \\ B = B\left(q^{(1)}, q^{(2)}, q^{(3)}\right) = \left(q^{(2)}\right)^2 - \frac{1}{2}\left(q^{(1)}\right)^2 - \frac{1}{2}\left(q^{(3)}\right)^2 + 2q^{(1)}q^{(3)} \end{cases} \tag{23}$$

it results:

$$\begin{cases} G_4 = A\cos 2t + B\sin 2t \\ G_5 = B\cos 2t - A\sin 2t \end{cases} \tag{24}$$

and denoting:

$$\begin{cases} C = C\left(q^{(2)}, q^{(3)}\right) = q^{(2)}q^{(3)} \\ D = D\left(q^{(2)}, q^{(3)}\right) = \frac{1}{2}\left(\left(q^{(2)}\right)^2 - \left(q^{(3)}\right)^2\right) \end{cases} \tag{25}$$

we get:

$$\begin{cases} \mathcal{F}_4 = C\cos 2t + D\sin 2t \\ \mathcal{F}_5 = D\cos 2t - C\sin 2t \end{cases}. \tag{26}$$

Also, the relations (14) – (16), (24), (26) have the matrix form:

$$\begin{pmatrix} \mathcal{F}_2 \\ \mathcal{F}_3 \end{pmatrix} = \left(q^{(3)}, q^{(2)}\right) \cdot \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \tag{14'-16'}$$

$$\begin{pmatrix} G_4 \\ G_5 \end{pmatrix} = (A, B) \cdot \begin{pmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{pmatrix} \tag{24'}$$

$$\begin{pmatrix} \mathcal{F}_4 \\ \mathcal{F}_5 \end{pmatrix} = (C, D) \cdot \begin{pmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{pmatrix} \tag{26'}$$

and it's amazing the "birth" of Lie group $SO(2) \simeq S^1$ through the trigonometric matrix from the RHS of these equations!

Acknowledgment: Special thanks are offered to Academician Radu Miron for introduction in higher-order Lagrange mechanics!

References

[1] Crâşmăreanu, M., *Noether theorem for time-dependent higher-order Lagrangians*, Proceedings of Romanian Academy-Series A, 1(2000), no. 2, 77-79.

[2] Crâşmăreanu, M., *A Noetherian symmetry for 2D spinning particle*, Internat. J. Non-Linear Mech., 35(2000), no. 5, 947-951. MR **2001g:**70024

[3] Crâşmăreanu, M., Adjoint variables: a common way to first integrals and inverse problems, Mem. Differential Equations Math. Phys., 26(2002), 55-64. MR **2003j:**34015

[4] Govinder, K. S., Heil, T. G., Uzer, T., *Approximate Noether symmetries*, Physics Letters A, 240(1998), 127-131. MR **99a:**70027, Zbl 0945.70520

[5] Leach, P. G. L., Moyo, S., Cotsakis, S., Lemmer, R. L., *Symmetry, Singularities and Integrability in complex Dynamics III: Approximate Symmetries and Invariants*, J. Nonlinear Math. Phys., 8(2001), no. 1, 139-156. MR **2001m:**34008, Zbl 0992.34028

[6] de Leon, M., de Diego, D. M., *Symmetries and constants of the motion for higher-order Lagrangian systems*, J. Math. Phys., 36(1995), no. 8, 4138-4161. MR **96d:**58047, Zbl 0845.70012

[7] Miron, R., *Noether theorem in higher-order Lagrangian mechanics*, Internat. J. Theoret. Phys., 34(1995), no. 7, 1123-1146. MR **96h:**58067, Zbl 0833.58015

[8] Moyo, S., Leach, P. G. L., *Noether's theorem in classical mechanics*, Sci. Bull. Univ. Politeh. Bucureşti, Series A, Vol. 60(1998), Nr. 3-4, 221-234. MR **2000c:**70019

[9] Muslih, S. I., *Canonical formulation treatment of a free relativistic spinning particle*, arXiv:math-ph/0009034.

[10] Nesterenko, V. V., Feoli, A., Scarpetta, G., *Dynamics of relativistic particles with Lagrangians dependent on acceleration*, J. Math. Phys., 36(1995), no. 1, 5552-5564. MR **96g:**58061

[11] Nesterenko, V. V., *The Ostrogradski formalism for singular Lagrangians with higher-order derivatives*, Ukrainean Math. J., 53(2001), no. 8, 12201236. MR **2003b:**700029

[12] Rivas, M., *Classical relativistic spinning particle*, J. Math. Phys., 30(1989), no. 2, 318-329. MR **90d:**70037

[13] Rivas, M., *Generalized Lagrangians and spinning particles*, Ukrainean Math. J. 53(2001), no. 8, 1326-1339. MR **2003f:**70027, Zbl 1016.70012

Faculty of Mathematics
University "Al. I. Cuza"
Iaşi, 6600, Romania
E-mail: mcrasm@uaic.ro