Tzitzeica indicatrices in Lagrange geometry

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Abstract

Tzitzeica hypersurfaces provided by indicatrices of Lagrange and generalized Lagrange spaces are studied.

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Introduction

Gheorghe Tîțeica (1873-1938), writing in French under the name of Georges Tzitzeica, was a student of Gaston Darboux and thus a member of the second generation of classical differential geometers, after Gauss and Riemann.

Tzitzeica has introduced a class of surfaces, nowadays called Tzitzeica surfaces, in 1907 ([7]) and a class of curves, called Tzitzeica curves, in 1911. The relation between these objects is the following: for a Tzitzeica surface with negative Gaussian curvature, the asymptotic lines are Tzitzeica curves. Since their appearance these notions are a permanent subject of research, suitable for fruitful generalizations.

For example, in [3] the notion of ”Tzitzeica surface” is generalized to hypersurfaces as follows. In the $n$-dimensional Euclidean space $\mathbb{R}^n$ let us consider a hypersurface $S$ for which we denote by $K$ the Gaussian curvature and by $d$ the distance from the origin of $\mathbb{R}^n$ to the tangent space in an arbitrary point of $S$. Then, considering the function:

$$Tzitzeica(S) := \frac{K}{d^{n+1}}$$
the hypersurface $S$ is called Tzitzeica if this function is a constant. Let us note that the function $Tzitzeica(S)$ is a centroaffine invariant of $S$ ([9]).

In this paper we are interested in Tzitzeica hypersurfaces provided by indicatrices of Lagrange and generalized Lagrange spaces, a notion introduced by R. Miron ([5]) as generalizations of Riemann and Finsler manifolds. Our results are generalizations of similar results from [8] where the Finsler spaces are studied. As examples of generalized Lagrange metrics not reducible to Lagrange metrics the Beil metrics are considered.

1 Tzitzeica indicatrices in Lagrange geometry

Let us suppose that the hypersurface $S$ is defined implicitly by $f \in C^\infty(\mathbb{R}^n)$ as $S = \{ x \in \mathbb{R}^n; f(x) = 0, \nabla f(x) \neq 0 \}$ where $\nabla f$ denotes the gradient of $f$, namely $\nabla f = (f_i)$ with $f_i = \frac{\partial f}{\partial x^i}$. Suppose that the normal field of $S$ is: $N = -\frac{\nabla f}{\|\nabla f\|}$. In [6, p. 37] and [4, p. 23] the following classical result is proved in a direct way:

**Proposition 1.1** The Gaussian curvature of the pair $(S, N)$ is:

$$K = -\frac{\begin{vmatrix} f_{ij} & f_i \\ f_j & 0 \end{vmatrix}}{\|\nabla f\|^{n+1}}. \quad (1.1)$$

This formulae is used in the cited papers in order to obtain the Gaussian curvature for the indicatrices of a Lagrange space. More precisely, denoting $T\mathbb{R}^n = (x', y')$ the tangent bundle of $\mathbb{R}^n = (x^i)$, let $L : T\mathbb{R}^n \to \mathbb{R}$ be a regular Lagrangian, that is, the matrix with entries $g_{ij} = \frac{1}{2} \frac{\partial^2 L}{\partial x^i \partial y^j}$ is of rank $n$, i.e. $g = \det (g_{ij}) \neq 0$, where $\frac{\partial}{\partial y^i}$. Associated with this Lagrangian and $x \in \mathbb{R}^n$ we have the indicatrix of $L$ in $x$: $I_x = \{ y \in T_x \mathbb{R}^n; L(x, y) = 1 \}$ which appears as a hypersurface defined by $f_x(y) = L(x, y) - 1$. Then, it is proved in [6, p. 39] that:

**Proposition 1.2** Let $(\mathbb{R}^n, L)$ be a Lagrange space and $x \in \mathbb{R}^n$. The Gaussian curvature $K_x$ of the indicatrix $I_x$ oriented in the direction $N_x = \frac{\nabla f_x}{\|\nabla f_x\|}$ is: $K_x = -\frac{\begin{vmatrix} f_{ij} & f_i \\ f_j & 0 \end{vmatrix}}{\|\nabla f_x\|^{n+1}}$. \quad (1.1)
\[-\frac{\nabla L}{\|\nabla L\|}\] is:

\[
K_x = -\frac{\left| \frac{\partial_i \partial_j L}{\partial_j L} \right|}{\left( \sum_{i=1}^{n} \left( \frac{\partial_i L}{\partial_j L} \right)^2 \right)^{\frac{n+1}{2}}}. \tag{1.2}
\]

In this proposition \(\nabla L\) denotes the gradient of \(L\) with respect to \((y^i)_{1 \leq i \leq n}\).

Because the tangent hyperplane \(T_y I_x\) in an arbitrary point \(y = (y^i) \in I_x\) has the equation:

\[
\partial_i L \left( Y^i - y^i \right) = 0 \tag{1.3}
\]

it follows:

\[
d_x = \frac{|\partial_i Ly^i|}{\left( \sum_{i=1}^{n} \left( \frac{\partial_i L}{\partial_j L} \right)^2 \right)^{\frac{1}{2}}} \tag{1.4}
\]

and then, from (1.2) and (1.4) we get:

**Proposition 1.3** If \((\mathbb{R}^n, L)\) is a Lagrange space and \(x \in \mathbb{R}^n\) then the indicatrix \(I_x\) is a Tzitzeica hypersurface if and only if there exists a real number \(C(x)\) such that:

\[
\left| \frac{2g_{ij}}{\partial_j L} \dot{\partial}_i L \right| = -C(x) |\dot{\partial}_i Ly^i|^{n+1}. \tag{1.5}
\]

Let us analyze the last equality. Because all derivatives are with respect to \((y^i)\) it follows that for Lagrangians of type \(L(x, y) = A(x, y) + B(x)\) the Tzitzeica indicatrices depend only of \(A(x, y)\). Then, for *natural Lagrangians*:

\[
L = \text{kinetic energy} - \text{potential energy} = \text{kinetic energy} - V(x)
\]

the Tzitzeica indicatrices depend only of kinetic energy. Therefore, considering:

\[
\text{kinetic energy} = \sum_{i=1}^{n} (y^i)^2
\]

we get that the hyperspheres are Tzitzeica indicatrices.

Looking at the RHS of (1.5) it appears naturally to consider homogeneous Lagrangians with respect to velocity \((y^i)\). So, for a \(r\)-homogeneous
Lagrangian, i.e. \( L(x, \lambda y) = \lambda^r L(x, y) \) for every \((x, y) \in T\mathbb{R}^n\) and \(\lambda \in \mathbb{R}\), the Euler characterization gives:

\[
\dot{\partial}_i L y^i = rL
\]

and we have:

**Proposition 1.4** Let \((\mathbb{R}^n, L)\) be a \(r\)-homogeneous Lagrange space with \(r \notin \{0, 1\}\) and \(x \in \mathbb{R}^n\). Then \(I_x\) is a Tzitzeica indicatrix if and only if there exists a real number \(C(x)\) such that:

\[
g = C(x).
\]

**Proof** From (1.5) and (1.6) it results on \(I_x\):

\[
\begin{vmatrix}
2g_{ij} & \dot{\partial}_i L \\
\partial_j L & 0
\end{vmatrix} = -C(x) |r|^{n+1}.
\]

Applying \(\partial_j\) to (1.6) it follows:

\[
2g_{ij} y^i = (r - 1) \partial_j L.
\]

Multiplying the last relation with \(y^j\) and using (1.6) we get:

\[
2g_{ij} y^i y^j = (r - 1) \partial_j Ly^j = r (r - 1) L
\]

which yields on \(I_x\):

\[
g_{ij} y^i y^j = \frac{r(r - 1)}{2}.
\]

Using (1.9) the LHS of (8) reads:

\[
\begin{vmatrix}
2g_{ij} & \frac{2}{r-1} g_{ja} y^a \\
\frac{2}{r-1} g_{ja} y^a & 0
\end{vmatrix} = \frac{2^{n+1}}{(r-1)^2} \begin{vmatrix}
g_{ij} & g_{ja} y^a \\
g_{ja} y^a & 0
\end{vmatrix} = \frac{2^{n+1}}{(r-1)^2} (-g) g_{ij} y^i y^j =
\]

\[
\left.\partial_0 I_x\right| \frac{2^{n+1}}{r(r-1)} g \cdot \frac{r(r-1)}{2} = \frac{2^n r}{r-1} g.
\]

From (1.8) and (1.11) it follows:

\[
-\frac{2^n r}{r-1} g = -C(x) |r|^{n+1}
\]
which imply the conclusion because \( n \) and \( r \) are constants. \( \square \)

**Particular cases.** If \( F \) is a Finsler, particularly Riemann, fundamental function ([5]) then \( L = F^2 \) is a 2-homogeneous Lagrangian. Tzitzeica indicatrices in the Finslerian framework are studied in [8] and our result 1.4 is generalization of Theorem 2 from [8].

Let us introduce other notions. A function \( f \in C^\infty (TIR) \) which does not depends on \( x \) i.e. \( f = f (y) \), is called *Minkowskian function*. It results that for a Minkowskian Lagrangian if there exists a Tzitzeica indicatrix then all indicatrices are Tzitzeica hypersurfaces.

A tensor field of \( (r, s) \)-type on \( TIR^n \) with law of change, at a change of coordinates on \( TIR^n \), exactly as a tensor field of \( (r, s) \)-type on \( IR^n \) is called *d-tensor field of \( (r, s) \)-type on \( TIR^n \).* We denote \( T_0IR^n \) the tangent bundle of \( IR^n \) without the null section.

### 2 Tzitzeica indicatrices in generalized Lagrange geometry

A d-tensor field of \( (0, 2) \)-type on \( TIR^n \), denoted \( g = (g_{ij} (x, y)) \), is called *generalized Lagrange metric* (GL-metric, on short) if the following properties hold ([5]):

(i) symmetry, \( g_{ij} = g_{ji} \)

(ii) nondegeneracy: \( \det (g_{ij}) \neq 0 \)

(iii) the signature of quadratic form \( g (\xi) = g_{ij} \xi^i \xi^j, \xi = (\xi^i) \in IR^n \), is constant.

The function \( E (g) = g_{ij} y^i y^j \) is called the *absolute energy* of the given GL-metric.

**Definition 2.1** ([5]) The GL-metric is called *weak regular* if \( E (g) \) is a regular Lagrangian.

It follows that for a weak Lagrange metric the d-tensor field of \( (0, 2) \)-type:

\[
\dot{g}_{ij} = \frac{1}{2} \left. \partial_i \partial_j E (g) \right|
\]

is a Lagrange metric and then we can associate the indicatrix:

\[
I_x = \{(x, y) \in TIR^n; E (g) (x, y) = 1, \nabla E (g) (x, y) \neq 0 \}.
\]
Applying proposition 1.3 we get:

**Proposition 2.2** Let $\mathbb{R}^n$, $g$ be a weak regular GL-space and $x \in \mathbb{R}^n$. Then $I_x$ is a Tzitzeica indicatrix if and only if there exists a real number $C(x)$ such that:

$$
\begin{vmatrix}
2g^*_{ij} & \dot{\partial}_i \mathcal{E}(g) \\
\dot{\partial}_j \mathcal{E}(g) & 0
\end{vmatrix} = -C(x) \left| \dot{\partial}_i \mathcal{E}(g) y^i \right|^{n+1}.
$$

(2.2)

A straightforward computation gives:

$$
\begin{align*}
\dot{g}^*_{ij} &= g_{ij} + \left( \dot{\partial}_j g_{ik} \right) y^k \\
\dot{\partial}_i \mathcal{E}(g) &= \left( \dot{\partial}_i g_{ab} \right) y^a y^b + 2g_{ia} y^a
\end{align*}
$$

(2.3)

The above formulae become more simple in the following case:

**Definition 2.3** ([5]) A weak regular GL-metric is called regular if:

$$
\dot{\partial}_i \mathcal{E}(g) = 2g_{ij} y^j.
$$

(2.4)

It results ([5]):

$$
\dot{g}^*_{ij} = g_{ij} + \left( \dot{\partial}_j g_{ik} \right) y^k
$$

(2.5)

but the formulae is still complicated. Another approach in the regular case is provided by homogeneity. By multiplication of (2.4) with $y^i$ we have:

$$
\dot{\partial}_i \mathcal{E}(g) y^i = 2g_{ij} y^i y^j = 2\mathcal{E}(g)
$$

(2.6)

which means that $\mathcal{E}(g)$ is 2-homogeneous i.e. $\mathcal{E}(g)$ is a Finslerian function. Then we apply proposition 1.4:

**Proposition 2.4** Let $\mathbb{R}^n$, $g$ be a regular GL-space and $x \in \mathbb{R}^n$. Then $I_x$ is a Tzitzeica indicatrix if and only if there exists a real number $C(x)$ such that:

$$
g^* = C(x). 
$$

(2.7)

where $g^* = \det \left( g^*_{ij} \right)$.

### 3 Beil metrics as examples

Let $\tilde{g} = (\tilde{g}_{ij}(x,y))$ be a Finsler metric and $B = B^i(x,y) \partial_i$ a d-vector field for which we denote $B_i = \tilde{g}_{ij} B^j$ and $B_0 = B_i y^i$. Let also $a, b \in C^\infty(T\mathbb{R}^n)$. In [1] and [2] the following GL-metric is studied:

$$
g_{ij} = a\tilde{g}_{ij} + bB_i B_j.
$$

(3.1)
These GL-metrics, called Beil metrics, are not Lagrange metrics. From:
\[
\mathcal{E}(g) = a\mathcal{E}(\tilde{g}) + b (B_0)^2
\]
we get:
\[
\dot{\partial}_i \mathcal{E}(g) = \left(\dot{\partial}_i a\right) \mathcal{E}(\tilde{g}) + a \left(\dot{\partial}_i \mathcal{E}(\tilde{g})\right) + \left(\dot{\partial}_i b\right) (B_0)^2 + 2bB_0 \left(\dot{\partial}_i \dot{B}_0\right)
\]
\[
2\tilde{g}^*_{ij} = 2a\tilde{g}_{ij} + \dot{\partial}_i \dot{\partial}_j a\mathcal{E}(\tilde{g}) + \dot{\partial}_i a \dot{\partial}_j \mathcal{E}(\tilde{g}) + \dot{\partial}_i \mathcal{E}(\tilde{g}) + \dot{\partial}_i \dot{\partial}_j b (B_0)^2 + 2B_0 \left(\dot{\partial}_i b \dot{\partial}_j B_0 + \dot{\partial}_j b \dot{\partial}_i B_0 + b \dot{\partial}_i \dot{\partial}_j B_0\right) + 2b \dot{\partial}_i B_0 \dot{\partial}_j B_0.
\]

**Example** On $T_0\mathbb{R}^n$ let:
\[
a = \frac{1}{2}, \quad b = \frac{1}{2\|y\|_F^2},
\]
where $\|\cdot\|_F$ is the norm induced by the Finsler metric $\tilde{g}$ i.e. $\|y\|_F^2 = \mathcal{E}(\tilde{g}) = \tilde{g}_{ij}y^iy^j$. Let $B = y^i \dot{\partial}_i$ be the Liouville vector field, it results $B_i = \tilde{g}^{ij}y_j$ denoted $\tilde{y}_i$. The associated Beil metric is:
\[
g_{ij} = \frac{1}{2}\tilde{g}_{ij} + \frac{1}{2\|y\|_F^2} \tilde{y}_i \tilde{y}_j.
\]
Thus:
\[
\mathcal{E}(g) = \|y\|_F^2 = \mathcal{E}(\tilde{g})
\]
which is 2-homogeneous and then a Finsler function. It results that the Beil metric is regular GL-metric with $g^*_{ij} = \tilde{g}_{ij}$ and then the Tzitzeica indicatrices of this Beil metric are exactly the Tzitzeica indicatrices of Finsler metric $\tilde{g}$. Recall that remarkable examples of Tzitzeica indicatrices in the Finslerian approach are given in [8].

**References**


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