Tzitzeica indicatrices in Lagrange geometry

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Abstract

Tzitzeica hypersurfaces provided by indicatrices of Lagrange and generalized Lagrange spaces are studied.

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Introduction

Gheorghe Ţiţeica (1873-1938), writing in French under the name of Georges Tzitzeica, was a student of Gaston Darboux and thus a member of the second generation of classical differential geometers, after Gauss and Riemann.

Tzitzeica has introduced a class of surfaces, nowadays called *Tzitzeica surfaces*, in 1907 ([7]) and a class of curves, called *Tzitzeica curves*, in 1911. The relation between these objects is the following: for a Tzitzeica surface with negative Gaussian curvature, the asymptotic lines are Tzitzeica curves. Since their appearance these notions are a permanent subject of research, suitable for fruitful generalizations.

For example, in [3] the notion of "Tzitzeica surface" is generalized to hypersurfaces as follows. In the n-dimensional Euclidean space $I\!R^n$ let us consider a hypersurface S for which we denote by K the Gaussian curvature and by d the distance from the origin of $I\!R^n$ to the tangent space in an arbitrary point of S. Then, considering the function:

$$Tzitzeica\,(S) := \frac{K}{d^{n+1}}$$

the hypersurface S is called *Tzitzeica* if this function is a constant. Let us note that the function $Tzitzeica\,(S)$ is a centroaffine invariant of S ([9]).

In this paper we are interested in Tzitzeica hypersurfaces provided by indicatrices of Lagrange and generalized Lagrange spaces, a notion introduced by R. Miron ([5]) as generalizations of Riemann and Finsler manifolds. Our results are generalizations of similar results from [8] where the Finsler spaces are studied. As examples of generalized Lagrange metrics not reducible to Lagrange metrics the Beil metrics are considered.

1 Tzitzeica indicatrices in Lagrange geometry

Let us suppose that the hypersurface S is defined implicitly by $f \in C^{\infty}(I\!R^n)$ as $S = \{x \in I\!R^n; f(x) = 0, \nabla f(x) \neq 0\}$ where ∇f denotes the gradient of f, namely $\nabla f = (f_i)$ with $f_i = \frac{\partial f}{\partial x^i}$. Suppose that the normal field of S is: $N = -\frac{\nabla f}{\|\nabla f\|}$. In [6, p. 37] and [4, p. 23] the following classical result is proved in a direct way:

Proposition 1.1 *The Gaussian curvature of the pair* (S, N) *is*:

$$K = -\frac{\begin{vmatrix} f_{ij} & f_i \\ f_j & 0 \end{vmatrix}}{\|\nabla f\|^{n+1}}. \tag{1.1}$$

This formulae is used in the cited papers in order to obtain the Gaussian curvature for the indicatrices of a *Lagrange space*. More precisely, denoting $TI\!R^n = (x^i, y^i)$ the tangent bundle of $I\!R^n = (x^i)$, let $L : TI\!R^n \to I\!R$ be a *regular Lagrangian*, that is, the matrix with entries $g_{ij} = \frac{1}{2}\dot{\partial}_i \dot{\partial}_j\, L$ is of rank n, i.e. $g = \det(g_{ij}) \neq 0$, where $\dot{\partial}_i = \frac{\partial}{\partial y^i}$. Associated with this Lagrangian and $x \in I\!R^n$ we have *the indicatrix of* L *in* x: $I_x = \{y \in T_x I\!R^n; L(x, y) = 1\}$ which appears as a hypersurface defined by $f_x(y) = L(x, y) - 1$. Then, it is proved in [6, p. 39] that:

Proposition 1.2 *Let* $(I\!R^n, L)$ *be a Lagrange space and* $x \in I\!R^n$. *The Gaussian curvature* K_x *of the indicatrix* I_x *oriented in the direction* $N_x =$

$-\frac{\dot{\nabla} L}{\|\dot{\nabla} L\|}$ *is*:

$$K_x = -\frac{\begin{vmatrix} \dot{\partial}_i \dot{\partial}_j \; L & \dot{\partial}_i \; L \\ \dot{\partial}_j \; L & 0 \end{vmatrix}}{\left(\sum\limits_{i=1}^{n} \left(\dot{\partial}_i \; L \right)^2 \right)^{\frac{n+1}{2}}}. \tag{1.2}$$

In this proposition $\dot{\nabla} \; L$ denotes the gradient of L with respect to $\left(y^i\right)_{1 \leq i \leq n}$.

Because the tangent hyperplane $T_y I_x$ in an arbitrary point $y = \left(y^i\right) \in I_x$ has the equation:

$$\dot{\partial}_i \; L \left(Y^i - y^i \right) = 0 \tag{1.3}$$

it follows:

$$d_x = \frac{\left| \dot{\partial}_i \; L y^i \right|}{\left(\sum\limits_{i=1}^{n} \left(\dot{\partial}_i \; L \right)^2 \right)^{\frac{1}{2}}} \tag{1.4}$$

and then, from (1.2) and (1.4) we get:

Proposition 1.3 *If* $(I\!R^n, L)$ *is a Lagrange space and* $x \in I\!R^n$ *then the indicatrix* I_x *is a Tzitzeica hypersurface if and only if there exists a real number* $C(x)$ *such that*:

$$\begin{vmatrix} 2g_{ij} & \dot{\partial}_i \; L \\ \dot{\partial}_j \; L & 0 \end{vmatrix} = -C(x) \left| \dot{\partial}_i \; L y^i \right|^{n+1}. \tag{1.5}$$

Let us analyze the last equality. Because all derivatives are with respect to $\left(y^i\right)$ it follows that for Lagrangians of type $L(x, y) = A(x, y) + B(x)$ the Tzitzeica indicatrices depend only of $A(x, y)$. Then, for *natural Lagrangians*:

$$L = \textit{kinetic energy} - \textit{potential energy} = \textit{kinetic energy} - V(x)$$

the Tzitzeica indicatrices depend only of kinetic energy. Therefore, considering:

$$\textit{kinetic energy} = \sum_{i=1}^{n} \left(y^i \right)^2$$

we get that the hyperspheres are Tzitzeica indicatrices.

Looking at the RHS of (1.5) it appears naturally to consider homogeneous Lagrangians with respect to velocity $\left(y^i\right)$. So, for a r-homogeneous

Lagrangian, i.e. $L(x, \lambda y) = \lambda^r L(x, y)$ for every $(x, y) \in T\mathbb{R}^n$ and $\lambda \in \mathbb{R}$, the Euler characterization gives:

$$\dot{\partial}_i \, L y^i = rL \tag{1.6}$$

and we have:

Proposition 1.4 *Let $(\mathbb{R}^n, L)$ be a r-homogeneous Lagrange space with $r \notin \{0, 1\}$ and $x \in \mathbb{R}^n$. Then I_x is a Tzitzeica indicatrix if and only if there exists a real number $C(x)$ such that*:

$$g = C(x). \tag{1.7}$$

Proof From (1.5) and (1.6) it results on I_x:

$$\begin{vmatrix} 2g_{ij} & \dot{\partial}_i \, L \\ \dot{\partial}_j \, L & 0 \end{vmatrix} = -C(x) |r|^{n+1}. \tag{1.8}$$

Applying $\dot{\partial}_j$ to (1.6) it follows:

$$2g_{ij} y^i = (r-1) \, \dot{\partial}_j \, L. \tag{1.9}$$

Multiplying the last relation with y^j and using (1.6) we get:

$$2g_{ij} y^i y^j = (r-1) \, \dot{\partial}_j \, L y^j = r(r-1) L$$

which yields on I_x:

$$g_{ij} y^i y^j = \frac{r(r-1)}{2}. \tag{1.10}$$

Using (1.9) the LHS of (1.8) reads:

$$\begin{vmatrix} 2g_{ij} & \frac{2}{r-1} g_{ia} y^a \\ \frac{2}{r-1} g_{ja} y^a & 0 \end{vmatrix} = \frac{2^{n+1}}{(r-1)^2} \begin{vmatrix} g_{ij} & g_{ia} y^a \\ g_{ja} y^a & 0 \end{vmatrix} = \frac{2^{n+1}}{(r-1)^2} (-g) \, g_{ij} y^i y^j =$$

$$\stackrel{on I_x}{=} -\frac{2^{n+1}}{(r-1)^2} g \cdot \frac{r(r-1)}{2} = -\frac{2^n r}{r-1} g. \tag{1.11}$$

From (1.8) and (1.11) it follows:

$$-\frac{2^n r}{r-1} g = -C(x) |r|^{n+1}$$

which imply the conclusion because n and r are constants. $\square$

Particular cases. If F is a Finsler, particularly Riemann, fundamental function ([5]) then $L = F^2$ is a 2-homogeneous Lagrangian. Tzitzeica indicatrices in the Finslerian framework are studied in [8] and our result 1.4 is generalization of Theorem 2 from [8].

Let us introduce other notions. A function $f \in C^\infty(T\mathbb{R})$ which does not depends on x i. e. $f = f(y)$, is called *Minkowskian function.* It results that for a Minkowskian Lagrangian if there exists a Tzitzeica indicatrix then all indicatrices are Tzitzeica hypersurfaces.

A tensor field of (r, s)-type on $T\mathbb{R}^n$ with law of change, at a change of coordinates on $T\mathbb{R}^n$, exactly as a tensor field of (r, s)-type on $\mathbb{R}^n$ is called *d-tensor field of* (r, s)*-type* on $T\mathbb{R}^n$. We denote $T_0\mathbb{R}^n$ the tangent bundle of $\mathbb{R}^n$ without the null section.

2 Tzitzeica indicatrices in generalized Lagrange geometry

A d-tensor field of $(0, 2)$-type on $T\mathbb{R}^n$, denoted $g = (g_{ij}(x, y))$, is called *generalized Lagrange metric* (*GL-metric*, on short) if the following properties hold ([5]):

(i) symmetry, $g_{ij} = g_{ji}$

(ii) nondegeneracy: $\det(g_{ij}) \neq 0$

(iii) the signature of quadratic form $g(\xi) = g_{ij}\xi^i\xi^j, \xi = (\xi^i) \in \mathbb{R}^n$, is constant.

The function $\mathcal{E}(g) = g_{ij}y^iy^j$ is called *the absolute energy* of the given GL-metric.

Definition 2.1([5]) The GL-metric is called *weak regular* if $\mathcal{E}(g)$ is a regular Lagrangian.

It follows that for a weak Lagrange metric the d-tensor field of $(0, 2)$-type:

$$g^*_{ij} = \frac{1}{2}\,\dot{\partial}_i\dot{\partial}_j\,\mathcal{E}(g) \tag{2.1}$$

is a Lagrange metric and then we can associate the indicatrix:

$$I_x = \{(x, y) \in T\mathbb{R}^n; \mathcal{E}(g)(x, y) = 1, \dot{\nabla}\,\mathcal{E}(g)(x, y) \neq 0\}.$$

Applying proposition 1.3 we get:

Proposition 2.2 *Let* $(I\!R^n, g)$ *be a weak regular GL-space and* $x \in I\!R^n$*. Then* I_x *is a Tzitzeica indicatrix if and only if there exists a real number* $C(x)$ *such that*:

$$\begin{vmatrix} 2g^*_{ij} & \dot{\partial}_i \,\mathcal{E}(g) \\ \dot{\partial}_j \,\mathcal{E}(g) & 0 \end{vmatrix} = -C(x) \left| \dot{\partial}_i \,\mathcal{E}(g)\, y^i \right|^{n+1}. \tag{2.2}$$

A straightforward computation gives:

$$\begin{cases} g^*_{ij} = g_{ij} + \left(\dot{\partial}_i \dot{\partial}_j \, g_{ab}\right) y^a y^b + \left(\dot{\partial}_i \, g_{ja} + \dot{\partial}_j \, g_{ia}\right) y^a \\ \dot{\partial}_i \,\mathcal{E}(g) = \left(\dot{\partial}_i \, g_{ab}\right) y^a y^b + 2g_{ia} y^a \end{cases}. \tag{2.3}$$

The above formulae become more simple in the following case:

Definition 2.3([5]) A weak regular GL-metric is called *regular* if:

$$\dot{\partial}_i \,\mathcal{E}(g) = 2g_{ij} y^j. \tag{2.4}$$

It results([5]):

$$g^*_{ij} = g_{ij} + \left(\dot{\partial}_j \, g_{ik}\right) y^k \tag{2.5}$$

but the formulae is still complicated. Another approach in the regular case is provided by homogeneity. By multiplication of (2.4) with y^i we have:

$$\dot{\partial}_i \,\mathcal{E}(g)\, y^i = 2g_{ij} y^i y^j = 2\mathcal{E}(g) \tag{2.6}$$

which means that $\mathcal{E}(g)$ is 2-homogeneous i.e. $\mathcal{E}(g)$ is a Finslerian function. Then we apply proposition 1.4:

Proposition 2.4 *Let* $(I\!R^n, g)$ *be a regular GL-space and* $x \in I\!R^n$*. Then* I_x *is a Tzitzeica indicatrix if and only if there exists a real number* $C(x)$ *such that*:

$$g^* = C(x). \tag{2.7}$$

where $g^* = \det\left(g^*_{ij}\right)$.

3 Beil metrics as examples

Let $\widetilde{g} = (\widetilde{g}_{ij}(x, y))$ be a Finsler metric and $B = B^i(x, y)\, \dot{\partial}_i$ a d-vector field for which we denote $B_i = \widetilde{g}_{ij} B^j$ and $B_0 = B_i y^i$. Let also $a, b \in C^\infty(TI\!R^n)$. In [1] and [2] the following GL-metric is studied:

$$g_{ij} = a\widetilde{g}_{ij} + bB_i B_j. \tag{3.1}$$

These GL-metrics, called *Beil metrics*, are not Lagrange metrics. From:

$$\mathcal{E}(g) = a\mathcal{E}(\widetilde{g}) + b(B_0)^2 \tag{3.2}$$

we get:

$$\dot{\partial}_i\, \mathcal{E}(g) = \left(\dot{\partial}_i\, a\right)\mathcal{E}(\widetilde{g}) + a\left(\dot{\partial}_i\, \mathcal{E}(\widetilde{g})\right) + \left(\dot{\partial}_i\, b\right)(B_0)^2 + 2bB_0\left(\dot{\partial}_i\, B_0\right) \tag{3.3}$$

$$\begin{aligned} 2g^*_{ij} = {} & 2a\widetilde{g}_{ij} + \dot{\partial}_i\dot{\partial}_j\, a\mathcal{E}(\widetilde{g}) + \dot{\partial}_i\, a\, \dot{\partial}_j\, \mathcal{E}(\widetilde{g}) + \dot{\partial}_j\, a\, \dot{\partial}_i\, \mathcal{E}(\widetilde{g}) + \dot{\partial}_i\dot{\partial}_j\, b(B_0)^2 + \\ & 2B_0\left(\dot{\partial}_i\, b\, \dot{\partial}_j\, B_0 + \dot{\partial}_j\, b\, \dot{\partial}_i\, B_0 + b\, \dot{\partial}_i\dot{\partial}_j\, B_0\right) + 2b\, \dot{\partial}_i\, B_0\, \dot{\partial}_j\, B_0. \end{aligned} \tag{3.4}$$

Example On $T_0 I\!R^n$ let:

$$a = \frac{1}{2}, b = \frac{1}{2\|y\|^2_F} \tag{3.5}$$

where $\|, \|_F$ is the norm induced by the Finsler metric $\widetilde{g}$ i.e. $\|y\|^2_F = \mathcal{E}(\widetilde{g}) = \widetilde{g}_{ij}y^iy^j$. Let $B = y^i\, \dot{\partial}_i$ be *the Liouville vector field*, it results $B_i = \widetilde{g}_{ij}y^j \overset{denoted}{=} \widetilde{y}_i$. The associated Beil metric is:

$$g_{ij} = \frac{1}{2}\widetilde{g}_{ij} + \frac{1}{2\|y\|^2_F}\widetilde{y}_i\widetilde{y}_j. \tag{3.6}$$

Thus:

$$\mathcal{E}(g) = \|y\|^2_F = \mathcal{E}(\widetilde{g}) \tag{3.7}$$

which is 2-homogeneous and then a Finsler function. It results that the Beil metric is regular GL-metric with $g^*_{ij} = \widetilde{g}_{ij}$ and then the Tzitzeica indicatrices of this Beil metric are exactly the Tzitzeica indicatrices of Finsler metric $\widetilde{g}$. Recall that remarkable examples of Tzitzeica indicatrices in the Finslerian approach are given in [8].

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