Approximate Noether symmetries for higher-order Lagrangians with applications to spinning particle

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Abstract

In this paper we extend the theory of approximate Noether symmetries and corresponding Noetherian first integrals to higher-order Lagrangians. Applications to spinning particle are given in order to illustrate this approach. In this way, five first integrals are obtained, only the first being previously known.

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In [3] a theory of approximate Noether symmetries for usual Lagrangians, i.e. of first order, is proposed and an approximate Noetherian first integral, based on this approach, is obtained. The aim of present paper is to extend this theory to higher-order Lagrangians i.e. Lagrangians depending of higher-order velocities.

In our study we treat only one dependent variable. This is for purposes of clarity because the case of several dependent variables is obtained by the inclusion of appropriately positioned indices. The general results will simple be stated.

Let $L = L(t, q^{(0)}, q^{(1)}, ..., q^{(k)})$ be a Lagrangian of order k with usual action integral $A = \int L dt$ where $q^{(0)} = q, q^{(1)} = \frac{dq}{dt}, ..., q^{(k)} = \frac{d^k q}{dt^k}$. Let $S = \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial q}$ be a given vector field which yields the infinitesimal transformation (flow):

$$\begin{cases} \widetilde{t} = t + \varepsilon\tau \\ \widetilde{q} = q + \varepsilon\eta \end{cases}. \tag{1}$$

The new Lagrangian will be: $\widetilde{L} = L(\widetilde{t}, \widetilde{q}, \frac{d\widetilde{q}}{d\widetilde{t}}, ..., \frac{d^k\widetilde{q}}{d\widetilde{t}^k})$ and the corresponding action $\widetilde{A} = \int \widetilde{L} d\widetilde{t}$.

In the following, we adopt the approach of [9](see also [4]); let us point that another way to obtain a Noether-type theorem in higher-order mechanics, based on higher-order energies, appears in [6](see also [7] or [8]) for the autonomous case (i.e. independent of time Lagrangians) and in [1] for the time-dependent case. Namely, S is called *Noether symmetry* if the action is invariant by the flow (1) of S i.e. $\widetilde{A} = A$. This is equivalent with the existence of a function f, usually called *gauge*, such that the following *Killing equation* holds ([9, p. 224-225]):

$$\frac{df}{dt} = \frac{d\tau}{dt}L + \tau\frac{\partial L}{\partial t} + \sum_{i=0}^{k} \xi_i \frac{\partial L}{\partial q^{(i)}} \tag{2}$$

where:

$$\begin{cases} \xi_0 = \eta \\ \xi_1 = \frac{d\xi_0}{dt} - q^{(1)}\frac{d\tau}{dt} \\ \quad ... \\ \xi_i = \frac{d\xi_{i-1}}{dt} - q^{(i)}\frac{d\tau}{dt} \end{cases}. \tag{3}$$

In this case the function:

$$\mathcal{F} = f - [\tau L + \sum_{j=0}^{k-1} \sum_{i=j+1}^{k} (-1)^j (\xi_{i-j-1} - \tau q^{(i-j)}) \frac{d^j}{dt^j}(\frac{\partial L}{\partial q^{(i)}})] \tag{4}$$

is a Noetherian first integral.

Now, we search for approximate Noether symmetries, i.e. the components of S are:

$$\begin{cases} \tau = \tau^0 + \varepsilon\tau^1 \\ \eta = \eta^0 + \varepsilon\eta^1 \end{cases} \tag{5}$$

and the gauge function is: $f = f^0 + \varepsilon f^1$. With previous computations it results:

Proposition *The vector field* $S = (\tau^0 + \varepsilon\tau^1)\frac{\partial}{\partial t} + (\eta^0 + \varepsilon\eta^1)\frac{\partial}{\partial q}$ *is an approximate Noether symmetry for the Lagrangian* L *with the gauge function* $f = f^0 + \varepsilon f^1$ *if and only if the following Killing equation holds:*

$$\frac{df^0}{dt} + \varepsilon\frac{df^1}{dt} = \left(\frac{d\tau^0}{dt} + \varepsilon\frac{d\tau^1}{dt}\right)L + \left(\tau^0 + \varepsilon\tau^1\right)\frac{\partial L}{\partial t} + \sum_{i=0}^{k}\xi_i\frac{\partial L}{\partial q^{(i)}} \tag{6}$$

where ξ_i *are given by eq.* (3) *with* η *replaced by* $\eta^0 + \varepsilon\eta^1$ *and* τ *replaced by* $\tau^0 + \varepsilon\tau^1$*. Then, the function*:

$$\mathcal{F} = \left(f^0 + \varepsilon f^1\right) - \left(\tau^0 + \varepsilon\tau^1\right)L - \sum_{j=0}^{k-1}\sum_{i=j+1}^{k}(-1)^j\left(\xi_{i-j-1} - q^{(i-j)}\left(\tau^0 + \varepsilon\tau^1\right)\right)\frac{d^j}{dt^j}\left(\frac{\partial L}{\partial q^{(i)}}\right) \tag{7}$$

is an approximate Noetherian first integral.

Particular cases:

1) $k = 1$.

Our formula (6) is exactly the relation (7) from [3, p. 128] and our relation (7) is exactly the equation (8) from the same citation.

2) $k = 2$.

The Killing equation becomes, with the index $a = 0, 1$:

$$\frac{df^a}{dt} = \frac{d\tau^a}{dt}L + \tau^a\frac{\partial L}{\partial t} + \eta^a\frac{\partial L}{\partial q} + \left(\frac{d\eta^a}{dt} - q^{(1)}\frac{d\tau^a}{dt}\right)\frac{\partial L}{\partial q^{(1)}} + \left(\frac{d^2\eta^a}{dt^2} - q^{(1)}\frac{d^2\tau^a}{dt^2} - 2q^{(2)}\frac{d\tau^a}{dt}\right)\frac{\partial L}{\partial q^{(2)}} \tag{8}$$

and the approximate Noetherian first integral is:

$$\begin{aligned}\mathcal{F} = {} & \left(f^0 + \varepsilon f^1\right) - \left(\tau^0 + \varepsilon\tau^1\right)L - \\ & - \left[\left(\eta^0 + \varepsilon\eta^1\right) - q^{(1)}\left(\tau^0 + \varepsilon\tau^1\right)\right]\left[\frac{\partial L}{\partial q^{(1)}} - \frac{d}{dt}\left(\frac{\partial L}{\partial q^{(2)}}\right)\right] - \\ & - \left[\left(\frac{d\eta^0}{dt} + \varepsilon\frac{d\eta^1}{dt}\right) - q^{(1)}\left(\frac{d\tau^0}{dt} + \varepsilon\frac{d\tau^1}{dt}\right) - q^{(2)}\left(\tau^0 + \varepsilon\tau^1\right)\right]\frac{\partial L}{\partial q^{(2)}}.\end{aligned} \tag{9}$$

Example: the spinning particle

After [5, p. 4147], the Lagrangian of classical spinning particle is of second order, namely:

$$L = \frac{1}{2}\left(q^{(1)}\right)^2 - \frac{1}{2}\left(q^{(2)}\right)^2 \tag{10}$$

with the Euler-Lagrange equation:

$$q^{(4)} + q^{(2)} = 0. \tag{11}$$

Let us point that the two-dimensional case is also very interesting from the point of view of Noetherian symmetries, cf. [2]. The Killing equation (8) is:

$$\frac{df^a}{dt} = \frac{d\tau^a}{dt}L + q^{(1)}\left(\frac{d\eta^a}{dt} - q^{(1)}\frac{d\tau^a}{dt}\right) - q^{(2)}\left(\frac{d^2\eta^a}{dt^2} - q^{(1)}\frac{d^2\tau^a}{dt^2} - 2q^{(2)}\frac{d\tau^a}{dt}\right) \tag{11}$$

and the Noetherian first integral (9) is:

$$\mathcal{F} = f^0 + \varepsilon f^1 - \left(\tau^0 + \varepsilon\tau^1\right)L - \left[\eta^0 + \varepsilon\eta^1 - q^{(1)}\left(\tau^0 + \varepsilon\tau^1\right)\right]\left(q^{(1)} + q^{(3)}\right) +$$

$$+\left[\frac{d\eta^0}{dt} + \varepsilon\frac{d\eta^1}{dt} - q^{(1)}\left(\frac{d\tau^0}{dt} + \varepsilon\frac{d\tau^1}{dt}\right) - q^{(2)}\left(\tau^0 + \varepsilon\tau^1\right)\right]q^{(2)}. \tag{12}$$

We obtain the following approximate Noether symmetries:

(i)

$$S_1 = \frac{\partial}{\partial t}, \quad S_1^\varepsilon = \varepsilon\frac{\partial}{\partial t} \tag{13}$$

with the gauge $G_1 = 0$ and the approximate Noetherian first integral:

$$\mathcal{F}_1^\varepsilon = \varepsilon\mathcal{F}_1 = \varepsilon\left(\frac{1}{2}\left(q^{(1)}\right)^2 - \frac{1}{2}\left(q^{(2)}\right)^2 + q^{(1)}q^{(3)}\right). \tag{14}$$

Let us note that $\mathcal{F}_1$ is exactly the energy (or Hamiltonian) of the Lagrangian (10), cf. [5, p. 4148].

(ii)

$$S_2^\varepsilon = \varepsilon\cos t\frac{\partial}{\partial q} \tag{15}$$

with the gauge $G_2^\varepsilon = \varepsilon\cos t q^{(1)}$ and the approximate Noetherian first integral:

$$\mathcal{F}_2^\varepsilon = \varepsilon\left(q^{(2)}\sin t + q^{(3)}\cos t\right). \tag{16}$$

(iii)

$$S_3^\varepsilon = \varepsilon \sin t \frac{\partial}{\partial q} \tag{17}$$

with the gauge $G_3^\varepsilon = \varepsilon \sin t q^{(1)}$ and the approximate Noetherian first integral:

$$\mathcal{F}_3^\varepsilon = \varepsilon \left(q^{(2)} \cos t - q^{(3)} \sin t \right). \tag{18}$$

(iv)

$$S_4^\varepsilon = \varepsilon \left(\sin 2t \frac{\partial}{\partial t} + q^{(1)} \cos 2t \frac{\partial}{\partial q} \right) \tag{19}$$

with the gauge:

$$G_4^\varepsilon = \varepsilon \sin 2t \left(\left(q^{(2)}\right)^2 - \frac{1}{2}\left(q^{(1)}\right)^2 - \frac{1}{2}\left(q^{(3)}\right)^2 + 2q^{(1)}q^{(2)} - q^{(1)}q^{(3)} \right) +$$

$$+\varepsilon \cos 2t \left(\left(q^{(1)}\right)^2 - \left(q^{(2)}\right)^2 + 2q^{(1)}q^{(2)} + q^{(1)}q^{(3)} + q^{(2)}q^{3} \right) \tag{20}$$

and the approximate Noetherian first integral:

$$\mathcal{F}_4^\varepsilon = \varepsilon \left[\frac{1}{2} \left(\left(q^{(2)}\right)^2 - \left(q^{(3)}\right)^2 \right) \sin 2t + q^{(2)}q^{(3)} \cos 2t \right] \tag{21}$$

(v)

$$S_5^\varepsilon = \varepsilon \left(\cos 2t \frac{\partial}{\partial t} - q^{(1)} \sin 2t \frac{\partial}{\partial q} \right) \tag{22}$$

with the gauge:

$$G_5^\varepsilon = \varepsilon \cos 2t \left(\left(q^{(2)}\right)^2 - \frac{1}{2}\left(q^{(1)}\right)^2 - \frac{1}{2}\left(q^{(3)}\right)^2 + 2q^{(1)}q^{(2)} - q^{(1)}q^{(3)} \right) +$$

$$+\varepsilon \sin 2t \left(- \left(q^{(1)}\right)^2 + \left(q^{(2)}\right)^2 - 2q^{(1)}q^{(2)} - q^{(1)}q^{(3)} - q^{(2)}q^{(3)} \right) \tag{23}$$

and the approximate Noetherian first integrals:

$$\mathcal{F}_5^\varepsilon = \varepsilon \left[\frac{1}{2} \left(\left(q^{(2)}\right)^2 - \left(q^{(3)}\right)^2 \right) \cos 2t - q^{(2)}q^{(3)} \sin 2t \right]. \tag{24}$$

Let us point a common expression for last two cases. Namely, setting:

$$\begin{cases} A = A\left(q^{(1)}, q^{(2)}, q^{(3)}\right) = \left(q^{(1)}\right)^2 - \left(q^{(2)}\right)^2 + 2q^{(1)}q^{(2)} + q^{(1)}q^{(3)} + q^{(2)}q^{(3)} \\ B = B\left(q^{(1)}, q^{(2)}, q^{(3)}\right) = \left(q^{(2)}\right)^2 - \frac{1}{2}\left(q^{(1)}\right)^2 - \frac{1}{2}\left(q^{(3)}\right)^2 + 2q^{(1)}q^{(3)} \end{cases} \tag{25}$$

it results:

$$\begin{cases} G_4^\varepsilon = \varepsilon\left(A\cos 2t + B\sin 2t\right) \\ G_5^\varepsilon = \varepsilon\left(B\cos 2t - A\sin 2t\right) \end{cases} \tag{26}$$

and denoting:

$$\begin{cases} C = C\left(q^{(2)}, q^{(3)}\right) = q^{(2)}q^{(3)} \\ D = D\left(q^{(2)}, q^{(3)}\right) = \frac{1}{2}\left(\left(q^{(2)}\right)^2 - \left(q^{(3)}\right)^2\right) \end{cases} \tag{27}$$

we get:

$$\begin{cases} \mathcal{F}_4^\varepsilon = \varepsilon\left(C\cos 2t + D\sin 2t\right) \\ \mathcal{F}_5^\varepsilon = \varepsilon\left(D\cos 2t - C\sin 2t\right) \end{cases} . \tag{28}$$

Also, the relations (16) – (18), (26), (28) have the matrix form:

$$\begin{pmatrix} \mathcal{F}_2^\varepsilon \\ \mathcal{F}_3^\varepsilon \end{pmatrix} = \varepsilon\left(q^{(3)}, q^{(2)}\right) \cdot \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \tag{16′) – (18′}$$

$$\begin{pmatrix} G_4^\varepsilon \\ G_5^\varepsilon \end{pmatrix} = \varepsilon\left(A, B\right) \cdot \begin{pmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{pmatrix} \tag{26′}$$

$$\begin{pmatrix} \mathcal{F}_4^\varepsilon \\ \mathcal{F}_5^\varepsilon \end{pmatrix} = \varepsilon\left(C, D\right) \cdot \begin{pmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{pmatrix} \tag{28′}$$

and it's amazing the "birth" of Lie group $SO(2) \simeq S^1$ through the trigonometric matrix from the RHS of these equations!

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