Last multipliers on manifolds

Mircea Crâşmăreanu*

Tensor, 66(2005), no. 1, 18-25.

Abstract

Using the characterizations of last multipliers from [15] new results in this approach of ODE are given. For example, two methods for obtaining last multipliers generated by pseudosymmetries are pointed in the second section. The vector fields with constant (particularly vanishing) divergence appears as a special case and in first and third section some associated constructions are obtained. Applications to Hamiltonian vector fields on symplectic and Poisson manifolds and gradient vector fields on Riemannian manifolds are given and two examples are completely investigated.

2000 Math. Subject Classification: 58A15; 58A30; 34A26; 34C40.
Key words: last multiplier, pseudosymmetry, Lie derivative, divergence.

Introduction

The notion of *last multiplier* was introduced by Karl Gustav Jacob Jacobi in "Vorlesugen über Dynamik", edited by R.F.A. Clebsch at Berlin in 1866. So, sometimes was used under the name of "Jacobi multiplier". Since then, this approach to ODE is intensively studied by mathematicians in the usual Euclidean space $I\!R^n$, cf. [1], [3], [10], [11], [16], [17]. For many, very interesting historical aspects, an excellent survey can be found in [2].

*Partially supported by CNCSIS Grant 2004

The aim of the present paper is to point out some results in this useful theory on a generalized framework, namely differential manifolds. Our study is inspired by characterizations from [15] based on calculus with Lie derivative, the manifold corresponding generalization of usual derivative calculus. Then, using the notion of *pseudosymmetry* introduced by author in [5] and [6] a generalization of the method from [15] is proposed. Also, the Lie derivative is used in order to obtain a class of last multipliers via the gradient operator. Because the Riemannian geometry is the most used framework, we add to our approach a Riemannian metric and a situation which yields last multipliers is characterized in terms of harmonic maps. Also, the invariance of last multiplier to a change of metric is studied. The global description of all geometrical objects studied is preferred.

The content of paper is as follows. The first section is a review of definitions and previous characterizations. As general example the solenoidal vector fields are treated and two concrete differential systems, in 2D and 3D, are completely examined. The second section is devoted to the generalizations of results from [15] via pseudosymmetries. In the next section, the class of multipliers of divergence type is studied and applications to Hamiltonian systems are pointed. The last sections treats the Riemannian and Poisson frameworks.

Acknowledgments Thanks are offered to Zbigniew Oziewicz, Radu Miron and Mihai Anastasiei for several useful remarks!

1 Characterizations for last multipliers. The case of solenoidal vector fields

Let M be a real, smooth, n-dimensional manifold, $C^{\infty}(M)$ the ring of smooth real functions on M, $\mathcal{X}(M)$ the Lie algebra of vector fields and $\Lambda^{k}(M)$ the $C^{\infty}(M)$-module of k-differential forms, $0 \leq k \leq n$. Suppose that M is orientable with the fixed volume form $V \in \Lambda^{n}(M)$.

Let an ODE system on M

$$\dot{x}^{i}(t) = A^{i}\left(x^{1}(t), \ldots, x^{n}(t)\right), 1 \leq i \leq n$$

generated by the vector field $A \in \mathcal{X}(M), A = (A^{i})_{1 \leq i \leq n}$ and let us consider the $n-1$-form $\Omega = i_{A}V \in \Lambda^{n-1}(M)$.

Definition 1.1([7, p. 107], [15, p. 428]) The function $m \in C^{\infty}(M)$ is called *last multiplier* for the ODE system generated by A, (*last multiplier for* A, on short) if:

$$d(m\Omega) := (dm) \wedge \Omega + m d\Omega = 0. \tag{1.1}$$

For example, in dimension 2 the notions of last multiplier and integrating factor are identical and for this dimension Sophus Lie gives a method to associate a last multiplier to every symmetry vector field of A. The Lie method is extended to any dimension in [15].

Firstly, let us recall:

Definition 1.2 The vector field $S \in \mathcal{X}(M)$ is called *symmetry* of A if there exists $\lambda \in C^{\infty}(M)$ such that:

$$L_A S := [A, S] = \lambda A \tag{1.2}$$

where L_A means the Lie derivative with respect to A.

The following characterization of last multipliers will be useful:

Lemma 1.3([15, p. 428]) (i) *$m \in C^{\infty}(M)$ is last multiplier for A if and only if*:

$$A(m) + m \cdot div_V A = 0. \tag{1.1'}$$

(ii) *Let $0 \neq h \in C^{\infty}(M)$ such that:*

$$L_A h := A(h) = (div_V A) \cdot h \tag{1.3}$$

where $div_V A$ is the divergence of A with respect to volume form V. Then $m = h^{-1}$ is a last multiplier for A.

Remarks (i) In the terminology of [2, p. 89] a function h satisfying (1.3) is called *inverse multiplier*.

(ii) A first important result given by (1.1′) is the characterization of last multipliers for solenoidal i.e. divergence-free vector fields: *$m \in C^{\infty}(M)$ is last multiplier for the solenoidal vector field A if and only if m is first integral of A.*

(iii) Recall the formulae from [4, p. 140]:

$$div_V(fX) = X(f) + f div_V X.$$

It follows that m is last multiplier for A if and only if the vector field mA is solenoidal i.e. $div_V(mA) = 0$.

The main result of [15] is:

Proposition 1.4([15, p. 429-430]) *Let $\{S_1, \ldots, S_{n-1}\}$ be a set of $n-1$ linear independent symmetry vector fields for A and:*

$$h = i_{S_{n-1}} \ldots i_{S_1} \Omega. \tag{1.4}$$

Then $m = h^{-1}$ is a last multiplier for A.

Examples

1. Let us consider a 2-dimensional ODE system:

$$\begin{cases} \frac{dx}{dt} = xy^2 \\ \frac{dy}{dt} = x^2 y \end{cases} \tag{1.5}$$

who has the associated vector field:

$$A = xy^2 \frac{\partial}{\partial x} + x^2 y \frac{\partial}{\partial y} \tag{1.6}$$

with $div_V A = x^2 + y^2$ with respect to the canonical volume form of $\mathbb{R}^2$:

$$V = dx \wedge dy. \tag{1.7}$$

The eq. $(1.1')$ reads:

$$xy^2 \frac{\partial m}{\partial x} + x^2 y \frac{\partial m}{\partial y} = -m \left(x^2 + y^2\right) \tag{1.8}$$

and the associated characteristic system:

$$\frac{dx}{xy^2} = \frac{dy}{x^2 y} = \frac{dm}{-m\left(x^2 + y^2\right)} \tag{1.9}$$

with first integrals:

$$\begin{cases} H_1 = x^2 - y^2 \\ H_2 = xym \end{cases} . \tag{1.10}$$

Therefore the general solution of $(1.1')$ is:

$$m = \frac{\Phi\left(x^2 - y^2\right)}{xy} \tag{1.11}$$

with $\Phi \in C^\infty(\mathbb{R})$. Obviously, this result is not acceptable in an equilibrium point of A i.e. a point where A vanishes.

2. **The Nahm system of** $SU(2)$**-monopoles**([6]):

$$\begin{cases} \frac{dx^1}{dt} = x^2x^3 \\ \frac{dx^2}{dt} = x^3x^1 \\ \frac{dx^3}{dt} = x^1x^2 \end{cases} \tag{1.12}$$

has the associate vector field:

$$A = x^2x^3\frac{\partial}{\partial x^1} + x^3x^1\frac{\partial}{\partial x^2} + x^1x^2\frac{\partial}{\partial x^3} \tag{1.13}$$

with $div_V X = 0$ with respect to the canonical volume form of $\mathbb{R}^3$:

$$V = dx^1 \wedge dx^2 \wedge dx^3. \tag{1.14}$$

The associated characteristic system:

$$\frac{dx^1}{x^2x^3} = \frac{dx^2}{x^3x^1} = \frac{dx^3}{x^1x^2} \tag{1.15}$$

has the first integrals([6]):

$$\begin{cases} H_1 = \left(x^1\right)^2 - \left(x^2\right)^2 \\ H_2 = \left(x^1\right)^2 - \left(x^3\right)^2 \end{cases} \tag{1.16}$$

and then the general solution of $(1.1')$ is:

$$m = \Phi\left(\left(x^1\right)^2 - \left(x^2\right)^2, \left(x^1\right)^2 - \left(x^3\right)^2\right) \tag{1.17}$$

with $\Phi \in C^\infty\left(\mathbb{R}^2\right)$.

2 Last multipliers generated by pseudosymmetries

Definition 2.1([5], [6]) Let $X, Y \in \mathcal{X}(M)$. Then Y is called X-*pseudosymmetry for* A if there exists $\lambda \in C^\infty(M)$ such that:

$$L_A Y = \lambda X. \tag{2.1}$$

Obviously, an A-pseudosymmetry is exactly a symmetry of A. Our first generalization is based on remark that the function h from (1.4) can be written:

$$h = V(A, S_1, \ldots, S_{n-1})$$

because $\Omega = i_A V$. Then we get:

Proposition 2.2 *Let $S \in \mathcal{X}(M)$ be a symmetry for A and let $\{Y_1, \ldots, Y_{n-1}\}$ be a set of $n-1$ linear independent S-pseudosymmetries for A such that:*

$$\Omega(Y_1, \ldots, Y_{n-1}) = 0. \tag{2.2}$$

Then for:

$$\widetilde{h} = V(S, Y_1, \ldots, Y_{n-1}) \tag{2.3}$$

we have that $\widetilde{m} = \widetilde{h}^{-1}$ is a last multiplier for A.

Proof Using the Lie derivative's properties it results:

$$L_A\widetilde{h} = (L_AV)(S, Y_1, \ldots, Y_{n-1}) + V(L_AS, Y_1, \ldots, Y_{n-1}) + \sum_{i=1}^{n-1} V(S, \ldots L_AS_i \ldots) =$$

$$= (div_V A) V(S, Y_1, \ldots, Y_{n-1}) + \lambda V(A, Y_1, \ldots, Y_{n-1}) + \sum_{i=1}^{n-1} \lambda_i V(S, \ldots S \ldots).$$

Every term from the above sum vanishes because V is skew-symmetric. Also, the middle term is zero from (2.2). In conclusion $L_A\widetilde{h} = (div_V A) \cdot \widetilde{h}$ and then using lemma we have the result. $\square$

Let us compare proposition 1.4 and 2.2. In the first proposition there are not a constraint of (2.4)-type because the middle term of Lie derivative is $V(L_AA, S_1, \ldots, S_{n-1}) = 0$. So, the price for our generalization, namely pseudosymmetries instead of symmetries, is given by eq. (2.2). In order to eliminate this condition we need a strong version of symmetry:

Definition 2.3 $S \in \mathcal{X}(M)$ is called *strong symmetry of A* or *commuting symmetry of A* if:

$$L_AS = 0. \tag{2.4}$$

The following version of last proposition holds:

Proposition 2.4 *Let $S \in \mathcal{X}(M)$ be a strong symmetry of A and $\{Y_1, \ldots, Y_{n-1}\}$ a set of $n-1$ linear independent S-pseudosymmetries of A. Then, for:*

$$\widehat{h} = V(S, Y_1, \ldots, Y_{n-1}) \tag{2.5}$$

it results that $\widehat{m} = \widehat{h}^{-1}$ *is a last multiplier for* A.

Proof In this case $V(L_A S, Y_1, \ldots, Y_{n-1}) = 0$. $\square$

Let us consider a local chart $(x^i)_{1 \le i \le n}$ on M such that:

$$V = V_{i_1 \ldots i_n} dx^{i_1} \wedge \ldots \wedge dx^{i_n}$$

$$S = S^i \frac{\partial}{\partial x^i}, \quad Y_k = Y_k^{i_k} \frac{\partial}{\partial x_k^{i_k}}, 1 \le k \le n-1.$$

Then $\widetilde{h}$ and $\widehat{h}$ has the form:

$$\widetilde{h} = V_{j i_1 \ldots i_{n-1}} S^j Y_1^{i_1} \ldots Y_{n-1}^{i_{n-1}}. \tag{2.6}$$

3 Last multipliers of divergence type

In this section we search for a last multiplier of divergence type i.e. $m = div_V S$ for some $S \in \mathcal{X}(M)$. Using (1.1′) it follows:

$$A(div_V S) + div_V S \cdot div_V A = 0. \tag{3.1}$$

Let us multiply this relation with V:

$$\mathcal{L}_A (div_V S) \cdot V + div_V S \cdot \mathcal{L}_A V = 0$$

which means:

$$\mathcal{L}_A (div_V S \cdot V) = \mathcal{L}_A \mathcal{L}_S V = 0$$

Let us recall that:

$$\mathcal{L}_A \mathcal{L}_S = \mathcal{L}_{[A,S]} + \mathcal{L}_S \mathcal{L}_A$$

which yields:

Proposition 3.1 *If* $S \in \mathcal{X}(M)$ *satisfy*:

$$\left(\mathcal{L}_{[A,S]} + \mathcal{L}_S \mathcal{L}_A\right)(V) = 0 \tag{3.2}$$

then $m = div_V S$ *is last multiplier for* A.

This result is again useful for the solenoidal case of section 1.

Proposition 3.2 *If* A *and* $S \in \mathcal{X}(M)$ *satisfy:*

(i) A *has constant divergence with respect to* V, *particularly* A *is solenoidal*

(ii) $[A, S]$ *is solenoidal, particularly* S *is strong symmetry of* A

then $m = div_V S$ *is last multiplier, in particular first integral of* A.

This result, from the first integrals point of view, is known in the particular cases of divergence-free systems with a symmetry and Hamiltonian systems. For divergenceless systems with an associate symmetry the result appears in [8]. Recall that on a symplectic manifold there exists a canonical volume form induced by the symplectic form and any Hamiltonian vector field is solenoidal with respect to this volume form. For Hamiltonian systems with a symmetry the result appears in [12] cf. [13]. Also, the case of systems with constant divergence and a symmetry is treated in [9].

4 Last multipliers on Riemannian manifolds

4.1 Last multipliers for gradients vector fields

Let us suppose that on M there is given a Riemannian metric $g = <,>$ and the considered vector field A is the gradient of a function $a \in C^\infty(M)$ with respect to g i.e. $A = \nabla a$. Then there exists a canonical volume form V_g generated by g and $div_V A$ is the Laplacean of a, $div_{V_g} A = \Delta a$. The condition $(1.1')$ becomes:

$$< \nabla a, \nabla m > + m \cdot \Delta a = 0 \tag{4.1}$$

and using the well-known formula:

$$< \nabla a, \nabla m > = \frac{1}{2} \left(\Delta (am) - a \cdot \Delta m - m \cdot \Delta a \right) \tag{4.2}$$

it results:

$$\Delta (am) + m \cdot \Delta a = a \cdot \Delta m. \tag{4.3}$$

The last condition yields:

Proposition 4.1 *Let* $a, m \in C^\infty(M)$ *such that* a *is last multiplier for* ∇m *and* m *is last multiplier for* ∇a. *Then* $a \cdot m$ *is a harmonic function on* M.

Proof Adding to (4.3) a similar relation with a replaced by m gives the conclusion. $\Box$

Proposition 4.2 $a \in C^\infty(M)$ *is last multiplier for exactly* $A = \nabla a$ *if and only if* a^2 *is a harmonic function on* M.

Proof Equation (4.3) is $\Delta(a^2) + a \cdot \Delta a = a \cdot \Delta a$. $\Box$

If M is an orientable compact manifold then from last proposition it results that a^2 is a constant which gives that a is a constant but then $A = \nabla a = 0$. Therefore on a orientable compact manifold a function can not be considered as a last multiplier for its associated gradient vector field.

Also, recall the formulae from [4, p. 142]:

$$\Delta (fh) = f\Delta h + 2g (\nabla f, \nabla h) + h\Delta f.$$

It results that a^2 is a harmonic function if and only if:

$$a\Delta a + g (\nabla a, \nabla a) = 0.$$

4.2 Last multipliers subject to conformal changes

Let $\widetilde{g} = e^{\varphi} g$ be a conformal transformation of metrics on M with $\varphi \in C^{\infty}(M)$. Recall that:

$$div_{V_{\widetilde{g}}} A = div_{V_g} A + \frac{n}{2} A (\varphi). \tag{4.4}$$

Then, from (4.4) and (1.1′) it results:

Proposition 4.3 *The last multipliers for A with respect to g are the same with last multipliers for A with respect to $\widetilde{g}$ if and only if φ is first integral of A.*

5 Last multipliers on Poisson manifolds

Let us suppose that M is endowed with a Poisson bracket $\{\}$. Let $f \in C^{\infty}(M)$ and $A_f \in \mathcal{X}(M)$ the associated *Hamiltonian vector field* generated by the *Hamiltonian* f. Recall that given the volume form V there exists a unique vector field X_V, called the *modular vector field*, such that:

$$div_V A_f = X_V (f).$$

From (1.1′) it results:

$$0 = A_f (m) + mX_V (f) = \{f, m\} + mX_V (f) = -\{m, f\} + mX_V (f) = -A_m (f) + mX_V (f)$$

which means:

Proposition 5.1 *m is last multiplier for A_f if and only if f is first integral for the vector field $mX_V - A_m$.*

Because f is a first integral of A_f we get:

Corollary 5.2 *f is last multiplier for A_f if and only if f is first integral of the vector field fX_V.*

References

[1] Babariko, N. N.; Gorbuzov, V. N., *On the question of the construction of the first integral or the last multiplier of a nonlinear system of differential equations* (Russian), Dokl. Akad. Nauk. BSSR, 30(1986), no. 9, 791-792. MR **87k:**34004

[2] Berrone, L. R.; Giacomini, H., *Inverse Jacobi multipliers*, Rend. Circ. Mat. Palermo (2) 52(2003), no. 1, 77-130. MR **2004b:**34067

[3] Buslyuk, D. V., *Construction of the last multiplier of a system of partial differential equations on admissible operators* (Russian), Vestn. Beloruss. Gos. Univ. Ser. 1 Fiz. Mat. Inform., 1998, no. 3, 20-23. MR **99m:**35009

[4] Chavel, I., *Riemannian geometry: a modern introduction*, Cambridge Tracts in Mathematics, no. 108, Cambridge University Press, 1993. MR **95j:**53001.

[5] Crâşmăreanu, M., *First integrals generated by pseudosymetries in Nambu-Poisson dynamics*, J. of Nonlinear Math. Phys., 7(2000) no. 2, 126-135. MR **2001g:**37089

[6] Crâşmăreanu, M., *Conservation laws generated by pseudosymmetries with applications to Hamiltonian systems*, Ital. J. Pure Appl. Math., 8(2000), 91-98. MR **2001j:**37098

[7] Flanders, H., *Differential forms with applications to the physical sciences*, Academic Press, 1963. MR **28** #5397

[8] Gonzalez-Gascon, F., *On a new first integral of certain dynamical system*, Phys. Lett. A, 61(1977), no. 6, 375-376. MR **57** 14035a

[9] Gonzalez-Gascon, F., *A global first integral for certain dynamical systems and related remarks*, Lett. Nuovo Cimento(2), 20(1977), no. 2, 54-56. MR **57** 14035b

[10] Gorbuzov, V. N., *On integrals and last multipliers of multidimensional differential systems with symmetries* (Russian), Vests. Akad. Navuk. Belarus. Ser. Fiz. Mat., 1996, no. 1, 20-24. MR **97f:**35149

[11] Gorbuzov, V. N., *Construction of first integrals and last multipliers of polynomial autonomous multidimensional differential systems*, Differential Equations, 34(1998), no. 4, 564-566. MR **2000e**:34010

[12] Komar A., Phys. Rev. D, 8(1973), 1028.

[13] Lutzky, M., *Isospectral representations of dynamical systems*, Lett. Nouvo Cimento, 35(1982), no. 12, 393-397.

[14] Miron, Radu, Anastasiei, Mihai, *The geometry of Lagrange spaces: theory and applications*, Fundamental Theories of Physics no. 59, Kluwer Academic Publishers Group, Dordrecht, 1994. **MR 95f**:53120

[15] Oziewicz, Zbiegniew; Zeni, José Ricardo R., *Ordinary differential equations: symmetries and last multiplier*, in R. Alamowicz and B. Fauser(Eds.) "Clifford Algebras and their Applications in Mathematical Physics", vol. 1 (Algebra and Physics), Progr. Phys., vol. 18, Birkhauser, 2000, 425-433. MR **2001e**:34010

[16] Saltikov, N., *Lie's generalization of the theory of last multiplier* (Serbo-Croation), Glas. Srpske Akad. Nauka Od. Prirod.-Mat., 198(1950), 1-16. MR **12** 829d

[17] Terent'ev, A. M., *The existence conditions for the last Jacobi multiplier of a special form*, Differ. Equations, 29(1993), no. 5, 785-787.

Faculty of Mathematics
University "Al. I. Cuza"
Iaşi, 6600
România
e-mail: mcrasm@uaic.ro