Tzitzeica figuratrices in Hamilton Geometry

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Abstract

Tzitzeica hypersurfaces provided by figuratrices of Hamilton and generalized Hamilton spaces are studied.

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Introduction

Gheorghe Ţiţeica (1873-1938), writing in French under the name of Georges Tzitzeica, was a student of Gaston Darboux and thus a member of the second generation of classical differential geometers, after Gauss and Riemann.

Tzitzeica has introduced a class of surfaces, nowadays called *Tzitzeica surfaces*, in 1907 ([7]) and a class of curves, called *Tzitzeica curves*, in 1911. The relation between these objects is the following: for a Tzitzeica surface with negative Gaussian curvature, the asymptotic lines are Tzitzeica curves. Since their appearance these notions are a permanent subject of research, suitable for fruitful generalizations.

For example, in [3] the notion of "Tzitzeica surface" is generalized to hypersurfaces as follows. In the n-dimensional Euclidean space $I\!R^n$ let us consider a hypersurface S for which we denote by K the Gaussian curvature and by d the distance from the origin of $I\!R^n$ to the tangent space in an arbitrary point of S. Then, considering the function:

$$Tzitzeica\,(S) := \frac{K}{d^{n+1}}$$

the hypersurface S is called *Tzitzeica* if this function is a constant. Let us note that the function $Tzitzeica\,(S)$ is a centroaffine invariant of S ([9]).

In this paper we are interested in Tzitzeica hypersurfaces provided by figuratrices of Hamilton and generalized Hamilton spaces, notions introduced by R. Miron ([5]) as geometrization of Hamilton dynamics. Our results are generalizations of similar results from [8] where the Cartan (or Miron) spaces are studied.

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1 Tzitzeica figuratrices in Hamilton geometry

Let us suppose that the hypersurface S is defined implicitly by $f \in C^{\infty}(I\!R^n)$ as $S = \{x \in I\!R^n; f(x) = 0, \nabla f(x) \neq 0\}$ where ∇f denotes the gradient of f, namely $\nabla f = (f_i)$ with $f_i = \frac{\partial f}{\partial x^i}$. Suppose that the normal field of S is: $N = -\frac{\nabla f}{\|\nabla f\|}$. In [6, p. 37] and [4, p. 23] the following classical result is proved in a direct way:

Proposition 1.1 *The Gaussian curvature of the pair* (S, N) *is*:

$$K = -\frac{\begin{vmatrix} f_{ij} & f_i \\ f_j & 0 \end{vmatrix}}{\|\nabla f\|^{n+1}}. \tag{1.1}$$

Let us denote $T^*I\!R^n = (x^i, p_i)$ the cotangent bundle of $I\!R^n = (x^i)$, let $H : T^*I\!R^n \to I\!R$ be a *regular Hamiltonian*, that is, the matrix with entries $g^{ij} = \frac{1}{2}\dot{\partial}^i \dot{\partial}^j L$ is of rank n, i.e. $g = \det(g_{ij}) \neq 0$, where $\dot{\partial}^i = \frac{\partial}{\partial p_i}$,([5]). Associated with this Hamiltonian and $x \in I\!R^n$ we have *the figuratrix of* H *in* x: $F_x = \{p \in T^*_x I\!R^n; H(x, y) = 1\}$ which appears as a hypersurface defined by $f_x(y) = H(x, y) - 1$. Then, it results:

Proposition 1.2 *Let* $(I\!R^n, H)$ *be a Hamilton space and* $x \in I\!R^n$. *The Gaussian curvature* K_x *of the figuratrix* F_x *oriented in the direction* $N_x = -\frac{\dot{\nabla} H}{\|\dot{\nabla} H\|}$ *is*:

$$K_x = -\frac{\begin{vmatrix} \dot{\partial}^i \dot{\partial}^j H & \dot{\partial}^i H \\ \dot{\partial}^j H & 0 \end{vmatrix}}{\left(\sum_{i=1}^{n}\left(\dot{\partial}^i H\right)^2\right)^{\frac{n+1}{2}}}. \tag{1.2}$$

In this proposition $\dot{\nabla} H$ denotes the gradient of H with respect to $(p_i)_{1 \le i \le n}$.

Because the tangent hyperplane $T_p F_x$ in an arbitrary point $p = (p_i) \in F_x$ has the equation:

$$\dot{\partial}^i H \left(X^i - p_i\right) = 0 \tag{1.3}$$

it follows:

$$d_x = \frac{\left|\dot{\partial}^i H p_i\right|}{\left(\sum_{i=1}^{n}\left(\dot{\partial}^i H\right)^2\right)^{\frac{1}{2}}} \tag{1.4}$$

and then, from (1.2) and (1.4) we get:

Proposition 1.3 *If* $(I\!R^n, H)$ *is a Hamilton space and* $x \in I\!R^n$ *then the figuratrix* F_x *is a Tzitzeica hypersurface if and only if there exists a real number* $C(x)$ *such that*:

$$\begin{vmatrix} 2g^{ij} & \dot{\partial}^i H \\ \dot{\partial}^j H & 0 \end{vmatrix} = -C(x)\left|\dot{\partial}^i H p_i\right|^{n+1}. \tag{1.5}$$

Let us analyze the last equality. Because all derivatives are with respect to (p_i) it follows that for Hamiltonians of type $H(x, p) = A(x, p) + B(x)$ the Tzitzeica figuratrices depend only of $A(x, p)$. Therefore, considering:

$$H=\sum_{i=1}^{n}(p_i)^2$$

we get that the hyperspheres are Tzitzeica figuratrices.

Looking at the RHS of (1.5) it appears naturally to consider homogeneous Hamiltonians with respect to momenta (p_i). So, for a r-homogeneous Hamiltonian, i.e. $H(x,\lambda p)=\lambda^r H(x,p)$ for every $(x,p)\in T^*I\!R^n$ and $\lambda\in I\!R$, the Euler characterization gives:

$$\dot{\partial}^i Hp_i = rH \tag{1.6}$$

and we have:

Proposition 1.4 *Let $(I\!R^n,H)$ be a r-homogeneous Hamilton space with $r\notin\{0,1\}$ and $x\in I\!R^n$. Then F_x is a Tzitzeica figuratrix if and only if there exists a real number $C(x)$ such that*:

$$g=C(x). \tag{1.7}$$

Proof From (1.5) and (1.6) it results on F_x:

$$\begin{vmatrix} 2g^{ij} & \dot{\partial}^i H \\ \dot{\partial}^i H & 0 \end{vmatrix} = -C(x)\,|r|^{n+1}. \tag{1.8}$$

Applying $\dot{\partial}^j$ to (1.6) it follows:

$$2g^{ij}p_i=(r-1)\,\dot{\partial}^j H. \tag{1.9}$$

Multiplying the last relation with p_j and using (1.6) we get:

$$2g^{ij}p_ip_j=(r-1)\,\dot{\partial}^j Hp_i = r(r-1)H$$

which yields on F_x:

$$g^{ij}p_ip_j=\frac{r(r-1)}{2}. \tag{1.10}$$

Using (1.9) the LHS of (1.8) reads:

$$\begin{vmatrix} 2g^{ij} & \frac{2}{r-1}g^{ia}p_a \\ \frac{2}{r-1}g^{ja}p_a & 0 \end{vmatrix} = \frac{2^{n+1}}{(r-1)^2}\begin{vmatrix} g^{ij} & g^{ia}p_a \\ g^{ja} & 0 \end{vmatrix} = \frac{2^{n+1}}{(r-1)^2}(-g)\,g^{ij}p_ip_j =$$

$$\overset{on\ F_x}{=} -\frac{2^{n+1}}{(r-1)^2}g\cdot\frac{r(r-1)}{2} = -\frac{2^n r}{r-1}g. \tag{1.11}$$

From (1.8) and (1.11) it follows:

$$-\frac{2^n r}{r-1}g=-C(x)\,|r|^{n+1}$$

which imply the conclusion because n and r are constants. □

Particular cases. If K is a Cartan fundamental function ([5, p. 154]) then $H=K^2$ is a 2-homogeneous Hamiltonian and the pair (IR^n,K) is called Cartan space in

[5, p. 153] and Miron space in [8]. Tzitzeica figuratrices in this framework are studied in [8] and our result 1.4 is generalization of Theorem 4 from [8].

Let us introduce other notions. A function $f \in C^{\infty}(T^{*}I\!R)$ which does not depends on x i. e. $f = f(y)$, is called *Minkowskian function*. It results that for a Minkowskian Hamiltonian if there exists a Tzitzeica figuratrix then all figuratrices are Tzitzeica hypersurfaces.

A tensor field of (r, s)-type on $T^{*}I\!R^{n}$ with law of change, at a change of coordinates on $T^{*}I\!R^{n}$, exactly as a tensor field of (r, s)-type on $I\!R^{n}$ is called *d-tensor field of* (r, s)*-type* on $T^{*}I\!R^{n}$. We denote $T_{0}^{*}I\!R^{n}$ the cotangent bundle of $I\!R^{n}$ without the null section.

2 Tzitzeica figuratrices in generalized Hamilton geometry

A d-tensor field of $(2, 0)$-type on $T^{*}I\!R^{n}$, denoted $g = \left(g^{ij}(x, p)\right)$, is called *generalized Hamilton metric* (*GH-metric*, on short) if the following properties hold ([5]):

(i) symmetry, $g^{ij} = g^{ji}$

(ii) nondegeneracy: $\det\left(g^{ij}\right) \neq 0$

(iii) the signature of quadratic form $g(\xi) = g^{ij}\xi_{i}\xi_{j}, \xi = (\xi_{i}) \in I\!R^{n}$, is constant.

The function $\mathcal{E}(g) = g^{ij}p_{i}p_{j}$ is called *the absolute energy* of the given GH-metric.

Definition 2.1 The GH-metric is called *weak regular* if $\mathcal{E}(g)$ is a regular Hamiltonian.

It follows that for a weak Hamilton metric the d-tensor field of $(2, 0)$-type:

$$g^{*ij} = \frac{1}{2}\dot{\partial}^{i}\dot{\partial}^{j}\mathcal{E}(g) \tag{2.1}$$

is a Hamilton metric and then we can associate the figuratrix:

$$F_{x} = \{p \in T^{*}I\!R^{n}; \mathcal{E}(g)(x, p) = 1, \dot{\nabla}\mathcal{E}(g)(x, p) \neq 0\}.$$

Applying proposition 1.3 we get:

Proposition 2.2 *Let* $(I\!R^{n}, g)$ *be a weak regular GH-space and* $x \in I\!R^{n}$. *Then* F_{x} *is a Tzitzeica figuratrix if and only if there exists a real number* $C(x)$ *such that*:

$$\begin{vmatrix} 2g^{*ij} & \dot{\partial}^{i}\mathcal{E}(g) \\ \dot{\partial}^{j}\mathcal{E}(g) & 0 \end{vmatrix} = -C(x)\left|\dot{\partial}^{i}\mathcal{E}(g)p_{i}\right|^{n+1}. \tag{2.2}$$

A straightforward computation gives:

$$\begin{cases} g^{*ij} = g^{ij} + \left(\dot{\partial}^{i}\dot{\partial}^{j}g^{ab}\right)p_{a}p_{b} + \left(\dot{\partial}^{i}g^{ja} + \dot{\partial}^{j}g^{ia}\right)p_{a} \\ \dot{\partial}^{i}\mathcal{E}(g) = \left(\dot{\partial}^{i}g^{ab}\right)p_{a}p_{b} + 2g^{ia}p_{a} \end{cases}. \tag{2.3}$$

The above formulae become more simple in the following case:

Definition 2.3 A weak regular GH-metric is called *regular* if:

$$\dot{\partial}^{i}\mathcal{E}(g) = 2g^{ij}p_{j}. \tag{2.4}$$

It results:

$$g^{*ij} = g^{ij} + \left(\dot{\partial}^j g^{ik}\right) p_k \tag{2.5}$$

but the formulae is still complicated. Another approach in the regular case is provided by homogeneity. By multiplication of (2.4) with y^i we have:

$$\dot{\partial}^i \mathcal{E}(g) p_i = 2g^{ij} p_i p_j = 2\mathcal{E}(g) \tag{2.6}$$

which means that $\mathcal{E}(g)$ is 2-homogeneous i.e. $\mathcal{E}(g)$ is a Cartan function. Then we apply proposition 1.4:

Proposition 2.4 *Let $(I\!R^n, g)$ be a regular GH-space and $x \in I\!R^n$. Then F_x is a Tzitzeica figuratrix if and only if there exists a real number $C(x)$ such that*:

$$g^* = C(x). \tag{2.7}$$

where $g^* = \det\left(g^{*ij}\right)$.

3 Hamilton-Beil type metrics as examples

Let $\widetilde{g} = \left(\widetilde{g}^{ij}(x,p)\right)$ be a Cartan metric and $B = B_i(x,p)\dot{\partial}^i$ a d-covector field for which we denote $B^i = \widetilde{g}^{ij} B_j$ and $B_0 = B^i p_i$. Let also $a, b \in C^\infty(T^* I\!R^n)$. Using an idea from [1] and [2], where the Lagrangian framework is used, we consider the following GH-metric, which we call, inspired by the cited papers, *Hamilton-Beil type metric*:

$$g^{ij} = a\widetilde{g}^{ij} + bB^i B^j. \tag{3.1}$$

These GH-metrics are not Hamilton metrics. From:

$$\mathcal{E}(g) = a\mathcal{E}(\widetilde{g}) + b(B_0)^2 \tag{3.2}$$

we get:

$$\dot{\partial}^i \mathcal{E}(g) = \left(\dot{\partial}^i a\right)\mathcal{E}(\widetilde{g}) + a\left(\dot{\partial}^i \mathcal{E}(\widetilde{g})\right) + \left(\dot{\partial}^i b\right)(B_0)^2 + 2bB_0\left(\dot{\partial}^i B_0\right) \tag{3.3}$$

$$\begin{aligned} 2g^{*ij} = {} & 2a\widetilde{g}^{ij} + \dot{\partial}^i\dot{\partial}^j a\mathcal{E}(\widetilde{g}) + \dot{\partial}^i a\,\dot{\partial}^j \mathcal{E}(\widetilde{g}) + \dot{\partial}^j a\,\dot{\partial}^i \mathcal{E}(\widetilde{g}) + \dot{\partial}^i\dot{\partial}^j b(B_0)^2 + \\ & + 2B_0\left(\dot{\partial}^i b\,\dot{\partial}^j B_0 + \dot{\partial}^j b\,\dot{\partial}^i B_0 + b\,\dot{\partial}^i\dot{\partial}^j B_0\right) + 2b\,\dot{\partial}^i B_0\,\dot{\partial}^j B_0. \end{aligned} \tag{3.4}$$

Example On $T_0^* I\!R^n$ let:

$$a = \frac{1}{2}, b = \frac{1}{2\|p\|_C^2} \tag{3.5}$$

where $\|,\|_C$ is the norm induced by the Cartan metric $\widetilde{g}$ i.e. $\|p\|_C^2 = \mathcal{E}(\widetilde{g}) = \widetilde{g}^{ij} p_i p_j$. Let $B = p_i\dot{\partial}^i$ be *the Liouville covector field*, it results $B^i = \widetilde{g}^{ij} p_j \overset{denoted}{=} \widetilde{p}^i$. The associated Hamilton-Beil type metric is:

$$g^{ij} = \frac{1}{2}\widetilde{g}^{ij} + \frac{1}{2\|p\|_C^2}\widetilde{p}^i\widetilde{p}^j. \tag{3.6}$$

Thus:

$$\mathcal{E}(g) = \|p\|_C^2 = \mathcal{E}(\widetilde{g}) \tag{3.7}$$

which is 2-homogeneous and then a Cartan function. It results that the Hamilton-Beil type metric is regular GH-metric with $g^{*ij} = \widetilde{g}^{ij}$ and then the Tzitzeica figuratrices of this GH-metric are exactly the Tzitzeica figuratrices of Cartan metric $\widetilde{g}$. Recall that remarkable examples of Tzitzeica figuratrices in the Cartan approach are given in [8].

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