

Generalized Lagrange-Weyl structures and compatible connections

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Abstract

Generalized Lagrange-Weyl structures and compatible connections are introduced as a natural generalization of similar notions from Riemannian geometry. Exactly as in Riemannian case, the compatible connection is unique if certain symmetry conditions with respect to vertical and horizontal Christoffel symbols are imposed.

2000 Math. Subject Classification: 53C60.

Key words: generalized Lagrange-Weyl space; compatible connection.

Introduction

Soon after the creation of general theory of relativity, Hermann Weyl attempted in [7] an unification of gravitation and electromagnetism in a model of space-time geometry combining conformal and projective structures.

Let \mathcal{G} be a conformal structure on the smooth manifold M i.e. an equivalence class of Riemannian metrics: $g \sim \bar{g}$ if there exists a smooth function $f \in C^\infty(M)$ such that $\bar{g} = e^{2f}g$. Denoting by $\Omega^1(M)$ the $C^\infty(M)$ -module of 1-forms on M a (Riemannian) Weyl structure is a map $W : \mathcal{G} \rightarrow \Omega^1(M)$ such that $W(\bar{g}) = W(g) + 2df$. In [3] it is proved that for a Weyl manifold (M, \mathcal{G}, W) there exists a unique torsion-free linear connection ∇ on M such that for every $g \in \mathcal{G}$:

$$\nabla g = W(g) \otimes g. \quad (*)$$

The parallel transport induced by ∇ preserves the given conformal class \mathcal{G} .

*Partially supported by CNCSIS Grant 2005

A natural extension of Riemannian metrics are the generalized Lagrange metrics. These metrics, introduced by Radu Miron around 1980, are suitable in geometrical approaches of general relativity and gauge theory cf. [6].

In this paper we extend the Weyl structures and compatible connections (*) in the generalized Lagrangian framework.

1 Generalized Lagrange-Weyl manifolds

Let M be a smooth n -dimensional real manifold and $\pi : TM \rightarrow M$ the tangent bundle of M . A chart $x = (x^i)_{1 \leq i \leq n}$ of M lifts to a chart $(x, y) = (x^i, y^i)$ on TM . A tensor field of (r, s) -type on TM with law of change, at a change of charts on M , exactly as a tensor field of (r, s) -type on M , is called *d-tensor field of (r, s) -type*.

Definition 1.1([6]) A d-tensor field of $(0, 2)$ -type $g = (g_{ij}(x, y))$ on TM is called a *generalized Lagrange metric (GL-metric, on short)* if:

- (i) it is symmetric: $g_{ij} = g_{ji}$
- (ii) it is non-degenerate: $\det(g_{ij}) \neq 0$
- (iii) the quadratic form $g_{ij}(x, y) \xi^i \xi^j$ has constant signature, $\forall \xi = (\xi^i) \in \mathbb{R}^n$.

Definition 1.2 Two GL-metrics g, \bar{g} are called *conformal equivalent* if there exists $f \in C^\infty(M)$ such that $\bar{g} = e^{2f}g$.

In the following let \mathcal{G} be a conformal structure i.e. an equivalence class of conformal equivalent GL-metrics. The main notion of this paper is:

Definition 1.3 A *generalized Lagrange-Weyl structure* is a map $W : \mathcal{G} \rightarrow \Omega^1(M)$ such that for every $g, \bar{g} \in \mathcal{G}$:

$$W(\bar{g}) = W(g) + 2df. \quad (1.1)$$

The triple (M, \mathcal{G}, W) will be called *generalized Lagrange-Weyl manifold*.

Let us point that, from (1.1), if for some $g \in \mathcal{G}$ the 1-form $W(g)$ is closed (or exact) then for every $\bar{g} \in \mathcal{G}$ the 1-form $W(\bar{g})$ is closed (or exact).

On TM the map $u \in TM \rightarrow V_u TM := \ker \pi_{*,u}$ defines an integrable distribution denoted $V(TM)$ and called *the vertical distribution*. Recall that a vector field $X = X^i(x) \frac{\partial}{\partial x^i} \in \mathcal{X}(M)$ has a *vertical lift* $X^v \in V(TM)$ given by $X^v = X^i \frac{\partial}{\partial y^i}$.

Because \mathcal{G} implies the tangent bundle geometry it seems naturally the following definition: a linear connection ∇ on TM is vertical-compatible with

the generalized Lagrange-Weyl structure (M, \mathcal{G}, W) if there exists $g \in \mathcal{G}$ such that for every $X \in \mathcal{X}(M)$:

$$\nabla_{X^v} g = W(g)(X) \cdot g.$$

But this definition has a great inconvenience: the fact that ∇ is vertical-compatible with a representative of \mathcal{G} does not involve the vertical-compatibility with another representative of \mathcal{G} . Indeed, using (1.1), we have:

$$\begin{aligned} \nabla_{X^v} \bar{g} &= \nabla_{X^v} (e^{2f} g) = X^v (e^{2f}) \cdot g + e^{2f} \nabla_{X^v} g = \\ &= 0 \cdot g + e^{2f} W(g)(X) \cdot g = W(g)(X) \cdot \bar{g} \neq W(\bar{g})(X) \cdot \bar{g}. \end{aligned}$$

With this motivation we introduce the next notion, namely *nonlinear connections*, well-known in the geometry of tangent bundle.

2 Compatibility with respect to a nonlinear connection

Definition 2.1 ([6]) A distribution H on TM supplementary to the vertical distribution i.e. $TTM = H \oplus V(TM)$ is called a *nonlinear connection*.

An adapted basis for $V(TM)$ is $\left(\frac{\partial}{\partial y^i}\right)$ and an adapted basis for H has the form $\left(\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j}\right)$. The functions $\left(N_i^j(x, y)\right)$ are called *the coefficients of the nonlinear connection H* . We obtain a new lift for vector fields; namely, to $X = X^i(x) \frac{\partial}{\partial x^i} \in \mathcal{X}(M)$ we associate the *horizontal lift* $X^h = X^i \frac{\delta}{\delta x^i} \in H$.

The nonlinear connection H yields a bundle denoted $H(TM)$ and called *horizontal*. The existence of a nonlinear connection is equivalent to the reduction of the standard almost tangent structure of TM to a $D(GL(n, R))$ -structure cf [4], [1].

Definition 2.2 A $D(GL(n, R))$ -connection on TM is called *d-connection* (or *Finsler connection*).

A d-connection ∇ preserves by parallelism both the vertical and horizontal bundles. Hence, ∇ has a pair of Christoffel coefficients $\left(F_{jk}^i(x, y), C_{jk}^i(x, y)\right)$ defined by relations:

$$\left\{ \begin{array}{l} \nabla_{\frac{\delta}{\delta x^j}} \frac{\delta}{\delta x^k} = F_{jk}^i \frac{\delta}{\delta x^i}, \quad \nabla_{\frac{\delta}{\delta x^j}} \frac{\partial}{\partial y^k} = F_{jk}^i \frac{\partial}{\partial y^i} \\ \nabla_{\frac{\partial}{\partial y^j}} \frac{\delta}{\delta x^k} = C_{jk}^i \frac{\delta}{\delta x^i}, \quad \nabla_{\frac{\partial}{\partial y^j}} \frac{\partial}{\partial y^k} = C_{jk}^i \frac{\partial}{\partial y^i} \end{array} \right. .$$

It follows that ∇ yields two algorithms of covariant derivation on d-tensor fields: a horizontal one, denoted $|_h$, and a vertical one, denoted $|_v$. For example, on the d-tensor field $g = (g_{ij}(x, y))$ of $(0, 2)$ -type we have:

$$\begin{cases} g_{jk}|_h = \frac{\delta g_{jk}}{\delta x^i} - g_{ak}F_{ji}^a - g_{ja}F_{ki}^a \\ g_{jk}|_v = \frac{\partial g_{jk}}{\partial y^i} - g_{ak}C_{ji}^a - g_{ja}C_{ki}^a \end{cases} . \quad (2.1)$$

Definition 2.3 A d-connection is called:

- (i) *horizontal* if all $C_{jk}^i = 0$,
- (ii) *horizontal symmetric (h-symmetric on short)* if $F_{jk}^i = F_{kj}^i$ for all indices i, j, k ,
- (iii) *total symmetric* if it is h-symmetric and vertical symmetric i.e. $C_{jk}^i = C_{kj}^i$ for all i, j, k .

For example, if g is a Riemannian metric then the Levi-Civita connection is the unique d-connection horizontal and h-symmetric; in this case F_{jk}^i does not depend of y since they are the usual Christoffel coefficients.

It is natural to consider:

Definition 2.4 If (M, \mathcal{G}, W) is a generalized Lagrange-Weyl manifold then a d-connection ∇ is called *compatible* if there exists $g \in \mathcal{G}$ such that for every $X \in \mathcal{X}(M)$:

$$\nabla_{X^h} g = W(g)(X) \cdot g. \quad (2.1)$$

An important result is:

Proposition 2.5 *If (2.1) holds for a given $g \in \mathcal{G}$ then ∇ is compatible with the whole class \mathcal{G} .*

Proof From (1.1) we get:

$$\begin{aligned} \nabla_{X^h} \bar{g} &= \nabla_{X^h} (e^{2\sigma} \circ \pi) g = X^h (e^{2\sigma} \circ \pi) \cdot g + e^{2\sigma} \nabla_{X^h} g = \\ &= 2d\sigma(X) e^{2\sigma} g + e^{2\sigma} W(g)(X) g = \\ &= e^{2\sigma} g (2d\sigma + W(g))(X) = W(\bar{g})(X) \cdot \bar{g}. \end{aligned}$$

The pair (g, H) yields four remarkable d-connections ([2]): Cartan, Berwald, Chern-Rund and Hashiguchi. For our aim, the Chern-Rund connection, denoted ∇^{CR} , is more convenient because it satisfies ([2]):

- I) is horizontal-metrical: $\nabla_{X^h}^{CR} g = 0$ for every $X \in \mathcal{X}(M)$
- II) is total symmetric.

The main result of this paper is:

Theorem 2.6 *For every generalized Lagrange-Weyl manifold (M, \mathcal{G}, W) there exists an unique compatible d -connection which is horizontal and h-symmetric.*

Proof Let $g \in \mathcal{G}$ and the associated ∇^{CR} . For $X, Y \in \mathcal{X}(M)$ let us define $\nabla_{X^v} Y^v = 0$ and :

$$\nabla_{X^h} Y^h := \nabla_{X^h}^{CR} Y^h - \frac{1}{2} W(g)(X) \cdot Y^h - \frac{1}{2} W(g)(Y) \cdot X^h + \frac{1}{2} g(X^h, Y^h) \cdot B \quad (2.2)$$

where:

$$g(X^h, Y^h) = g_{ij} X^i Y^j, \quad X = X^i(x) \frac{\partial}{\partial x^i}, Y = Y^j(x) \frac{\partial}{\partial x^j}$$

and $B \in \mathcal{X}(TM)$ is $B = (B^i)$:

$$B^i = g^{ij} w_j, \quad W(g) = w_i dx^i.$$

Here (g^{ij}) is the inverse of (g_{ij}) . Then:

$$\nabla_{X^h} g = \nabla_{X^h}^{CR} g + W(g)(X) \cdot g \stackrel{I)}{=} W(g)(X)$$

i.e. ∇ is horizontal-compatible with g . Applying the previous result we have the conclusion.

The non-null coefficients of ∇ are:

$$F_{jk}^i = F_{jk}^{i \ CR} - \delta_j^i w_k - \delta_k^i w_j + g_{jk} w^i$$

where $\left(\begin{smallmatrix} CR \\ F \end{smallmatrix} \right)$ are the horizontal Christoffel coefficients of ∇^{CR} :

$$F_{jk}^{i \ CR} = g^{ia} \left(\frac{\delta g_{ak}}{\delta x^j} + \frac{\delta g_{ja}}{\delta x^k} - \frac{\delta g_{jk}}{\delta x^a} \right)$$

and, as usual, (w^i) is the g -contravariant version of $W(g)$ i.e. $w^i = g^{ia} w_a$.

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