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BEST APPROXIMATIONS AND ORTHOGONALITIES IN $2k$-INNER PRODUCT SPACES

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Abstract. In this paper, some characterizations of representation for continuous linear functionals on $2k$-inner product spaces in terms of best approximations and orthogonalities are given.

1. Introduction

Due to its applications, e.g., in optimization, the problem of best approximation has a long history and gives rise to a lot of notions and techniques useful in functional analysis. The usual framework of this theory consists in Banach or Hilbert spaces because the geometry of these spaces, via Birkhoff orthogonality or orthogonality with respect to the inner product, yields the support for results of existence and uniqueness for elements of best approximation. Also, the study of the dual of these spaces offers remarkable information. For a huge part of this theory the monograph of I. Singer [11] is a good reference.

In manifold setting, namely Riemannian spaces as the natural generalization of Hilbert spaces, the problem of best approximation is treated in M. Crâşmăreanu [1].

S. S. Dragomir [4], S. S. Kim, Y. J. Cho and T. D. Narang [10] have given some characterizations of best approximation theorems for continuous linear functionals in terms of norm derivatives for classical framework.

Recently, M. Crâşmareanu and S. S. Dragomir [2, 3] introduced the notion of $2k$-inner product as a generalization of classical inner products

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and obtained some properties e.g., uniformly convexity and Gâteaux differentiability.

In this paper, some characterizations of representation for continuous linear functionals on $2k$-inner product spaces in terms of best approximation and orthogonalities are given.

2. Preliminaries

Definition 2.1. [3] Let X be a real linear space and k be a natural number. A mapping $(\cdot,\dots,\cdot) : X^{2k} = \underbrace{X \times \cdots \times X}_{2k\,\text{times}} \to R$ satisfies the following conditions:

(i) $(\alpha_1 x_1 + \alpha_2 x_2, x_3, \dots, x_{2k+1}) = \alpha_1(x_1, x_3, \dots, x_{2k+1}) + \alpha_2(x_2, x_3, \dots, x_{2k+1})$ for all $\alpha_1, \alpha_2 \in R$,

(ii) $(x_{\sigma(1)}, \dots, x_{\sigma(2k)}) = (x_1, \dots, x_{2k})$ for all permutation σ of the indices $(1, \dots, 2k)$,

(iii) $(x, \dots, x) > 0$ if $x \neq 0$,

(iv) Cauchy-Buniakowski-Schwarz's inequality (in short, CBS inequality)

$$|(x_1, \dots, x_{2k})|^{2k} \leq \prod_{i=1}^{2k} (x_i, \dots, x_i)$$

with equality if and only if $x_1, \dots, x_{2k}$ are linearly dependent.

Then $(\cdot,\dots,\cdot)$ is called a $2k$-*inner product* and the pair $(X, (\cdot,\dots,\cdot))$ a $2k$-*inner product space*.

For $k = 1$ we have the usual notion of an inner product, and for $k = 2$ we obtain the concept of Q-inner product from [5]-[7]. Also, it follows that $(0, x_2, \dots, x_{2k}) = 0$ and $(\alpha x_1, \dots, \alpha x_{2k}) = \alpha^{2k}(x_1, \dots, x_{2k})$.

Examples. (1) Let $X = R^n$ and $x_j = (x_j^1, \dots, x_j^n) \in R^n$. Then

$$(x_1, \dots, x_{2k}) = \sum_{i=1}^{n} \left(\prod_{j=1}^{2k} x_j^i \right).$$

(2) Let $(\Omega, \mathcal{A}, \mu)$ be a measure space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of subsets of Ω, and a countably additive and positive measure μ on $\mathcal{A}$ with $\mu(\Omega) < \infty$. Then on $X = L^{2k}(\Omega, \mathcal{A}, \mu)$ and $x_j \in X, 1 \leq j \leq 2k$,

we have the $2k$-inner product

$$(x_1, \ldots, x_{2k}) = \int_\Omega \prod_{i=1}^{2k} x_i(t)\,d\mu(t).$$

Theorem 2.2. *If $(X, (\cdot, \ldots, \cdot))$ is a $2k$-inner product space and a mapping $\|\cdot\|_{2k} : X \to R$ is defined by $\|x\|_{2k} = (x, \ldots, x)^{1/2k}$, then $\|\cdot\|_{2k}$ is a norm on X.*

Proof. Using the property (iv) of Definition 1.1,

$$\begin{aligned}\|x+y\|_{2k}^{2k} &= \sum_{i=0}^{2k} \binom{2k}{i} \Big(\underbrace{x, \ldots, x,}_{i \text{ times}} \underbrace{y, \ldots, y}_{2k-i \text{ times}} \Big) \\ &\le \sum_{i=0}^{2k} \binom{2k}{i} \|x\|_{2k}^{i} \|y\|_{2k}^{2k-i} \\ &= (\|x\|_{2k} + \|y\|_{2k})^{2k},\end{aligned}$$

which give the triangle inequality

$$\|x+y\|_{2k} \le \|x\|_{2k} + \|y\|_{2k} \quad \text{for} \quad x, y \in X.$$

On the other hand, $\|x\|_{2k} \ge 0$ for all $x \in X$ and $\|x\|_{2k} = 0$ implies $x = 0$.

Finally, we also have

$$\|\alpha x\|_{2k} = |\alpha| \|x\|_{2k},$$

where $\alpha \in R$ and $x \in X$. Thus, $\|x\|_{2k}$ is a norm on X. This completes the proof. □

Corollary 2.3. *If $(X, (\cdot, \ldots, \cdot))$ is a $2k$-inner product space with the norm $\|x\|_{2k} = (x, \ldots, x)^{1/2k}$, then the norm $\|x\|_{2k}$ satisfies the following generalization of parallelogram identity:*

$$\|x+y\|_{2k}^{2k} + \|x-y\|_{2k}^{2k} = 2 \sum_{i=0}^{k} \binom{2k}{2(k-i)} \Big(\underbrace{x, \ldots, x,}_{2i \text{ times}} \underbrace{y, \ldots, y}_{2(k-i) \text{ times}} \Big).$$

Definition 2.4. [2] A real normed linear space is said to be a $2k$-*normed space* if its norms is defined by a $2k$-inner product.

Recall that a normed linear space $(X, \|\cdot\|)$ is said to be *uniformly convex* if given $\epsilon > 0$ there is a $\delta > 0$ such that $\|x\| \leq 1$, $\|y\| \leq 1$, and $\|x+y\| > 2-\delta$, then $\|x-y\| < \epsilon$.

Theorem 2.5. *If X is a $2k$-normed space, then X is uniformly convex.*

Proof. Let $\epsilon > 0$ be given. Then there is a $\delta > 0$ such that

$$2^{2k} - (2-\delta)^{2k} < \epsilon^{2k}.$$

Let $\|x\|_{2k} \leq 1$ and $\|y\|_{2k} \leq 1$ and $\|x+y\|_{2k} > 2-\delta$. Then, by the parallelogram identity and the CBS inequality, we have

$$\begin{aligned}\|x+y\|_{2k}^{2k} + \|x-y\|_{2k}^{2k} &= 2\sum_{i=0}^{k} \binom{2k}{2(k-i)} \Big(\underbrace{x, \ldots, x,}_{2i \text{ times}} \underbrace{y, \ldots, y}_{2(k-i) \text{ times}} \Big) \\ &\leq 2^{2k}\end{aligned}$$

and so $\|x-y\|_{2k}^{2k} \leq 2^{2k} - \|x+y\|_{2k}^{2k} < \epsilon^{2k}$. Therefore we have $\|x-y\|_{2k} < \epsilon$. This completes the proof. □

Recall that on a normed space $(X, \|\cdot\|)$ an element $x \in X$ is said to be *Birkhoff orthogonal* to an element $y \in X$ (write $x \perp_B y$) if $\|x+\lambda y\| \geq \|x\|$ for all $\lambda \in R$ ([8]).

Definition 2.6. Let $(X, (\cdot, \ldots, \cdot))$ be a $2k$-inner product space and $x, y \in X$. An element $x \in X$ is said to be $2k$-*orthogonal* to an element $y \in X$ (write $x \perp_{2k} y$) if $(x, \ldots, x, y) = 0$.

Definition 2.7. Let X be a $2k$-normed space and $\epsilon \in [0, 1)$. An element $x \in X$ is said to be ϵ-*CBS-orthogonal* to the element $y \in X$ (write $x \perp_{CBS} y(\epsilon)$) if

$$|(y, x, \ldots, x)| \leq \epsilon \|x\|_{2k}^{2k-1} \|y\|_{2k}$$

for all $x, y \in X$.

If $\epsilon = 0$, then x is $2k$-orthogonal to y.

Let X be a $2k$-inner product space. Fixed an element y in X and consider the functional $f : X \to R$ defined by $f(x) = (x, y, \ldots, y)$ for all $x \in X$. Then f is a continuous linear functional on X and

$$|f(x)| \leq \|y\|_{2k}^{2k-1} \|x\|_{2k}$$

for all $x \in X$. Also, we have $\|f\| \leq \|y\|_{2k}^{2k-1}$. On the other hand, we have $\|f\| \|y\|_{2k} \geq f(y) = \|y\|_{2k}^{2k}$. Therefore, we have $\|f\| = \|y\|_{2k}^{2k-1}$.

3. Orthogonalities and best approximations

THEOREM 3.1. *The norm $\|\cdot\|_{2k}$ of a $2k$-normed space X is a Gâteaux differentiable with*

$$(*) \qquad \tau(x,y) = \lim_{t\to 0} \frac{\|x+ty\|_{2k} - \|x\|_{2k}}{t} = \frac{(x,\ldots,x,y)}{\|x\|_{2k}^{2k-1}} \quad \textit{for} \quad x \neq 0.$$

Proof. Let $x, y \in X$ with $x \neq 0$ and t is a nonzero real number. Since

$$\frac{1}{t}(\|x+ty\|_{2k}^{2k} + \|x\|_{2k}^{2k}) = \frac{1}{t}\sum_{i=0}^{2k-1} \binom{2k}{i} \Big(\underbrace{x,\ldots,x}_{i \text{ times}}, \underbrace{ty,\ldots,ty}_{2k-i \text{ times}} \Big),$$

we have

$$\lim_{t\to 0} \frac{\|x+ty\|_{2k}^{2k} - \|x\|_{2k}^{2k}}{t} = 2k(x,\ldots,x,y).$$

Also, since

$$\begin{aligned} &\frac{1}{t}(\|x+ty\|_{2k} - \|x\|_{2k}) \\ &= \frac{1}{t}\frac{\|x+ty\|_{2k}^{2k} - \|x\|_{2k}^{2k}}{(\|x+ty\|_{2k}^{k} + \|x\|_{2k}^{k})\sum_{i=1}^{k}\|x+ty\|_{2k}^{k-i}\|x\|_{2k}^{i-1}}, \end{aligned}$$

we have

$$\lim_{t\to 0} \frac{\|x+ty\|_{2k} - \|x\|_{2k}}{t} = \frac{2k(x,\ldots,x,y)}{2\|x\|_{2k}^{k} k \|x\|_{2k}^{k-1}} = \frac{(x,\ldots,x,y)}{\|x\|_{2k}^{2k-1}}.$$

This completes the proof. □

LEMMA 3.2. *If $(X,(\cdot,\ldots,\cdot))$ is a $2k$-inner product space, then the $2k$-orthogonality is equivalent with the Birkhoff orthogonality.*

Proof. If $x\perp_{2k}y$ and $x \neq 0$, then since $\tau(x,y) = 0$, $x\perp_B y$. Conversely, if $x\perp_B y$, then since $\tau(x,y) = 0$ and a $2k$-inner product space is smooth, $x\perp_{2k}y$. This completes the proof. □

Let X be a normed linear space and G a nondense subspace of X. If $x \in X \setminus \overline{G}$ and $g_o \in G$, then an element g_o is said to be the *best approximation element* of x in G if

$$\|x - g_o\| \leq \|x - g\|$$

for all $g \in G$. The set of all elements of best approximation of x in G is denoted by $P_G(x)$. A proper linear subspace E of X is called *proximinal* in X if, for every $x \in X$, the set $P_G(x)$ contains at least one element.

The following theorems is well-known ([11]):

THEOREM 3.3. *Let $(X, \|\cdot\|)$ be a normed linear space, G a linear subspace of X, $x_o \in X \setminus \overline{G}$ and $g_o \in G$. Then $g_o \in P_G(x_o)$ if and only if $(x - g_o)\perp_B G$.*

THEOREM 3.4. *Let X be a normed linear space and H be a hyperplane in X such that $0 \in H$. Then H is proximinal in X if and only if there exists $y \in X \setminus \{0\}$ such that $y \perp_B H$.*

In [8], it is know that $x \perp_B y$ on a normed linear space if and only if there exists a continuous linear functional f such that

$$\|f\| = 1, \quad f(x) = \|x\| \quad \text{and} \quad f(y) = 0.$$

THEOREM 3.5. *Let X be $2k$-normed space, G a linear subspace in X and $x \in X \setminus \overline{G}$. If $g_o \in P_G(x)$, then there exists a continuous linear functional $f : X \to R$ such that*

(1) $f(x) = \|x - g_o\|_{2k}$.
(2) $f(g) = 0$ *for all* $g \in G$.
(3) $\|f\| = 1$.

Proof. Suppose that $g_o \in P_G(x)$. Then $\|x - g_o\|_{2k} \le \|x - g\|_{2k}$ for all $g \in G$ and there exists a continuous linear functional f_o on X such that

$$\|f_o\| = \frac{1}{\|x - g_o\|_{2k}}, \quad f_o(g) = 0 \quad \text{for all } g \in G \quad \text{and} \quad f_o(x) = 1.$$

Put $f = \|x - g_o\|_{2k} f_o$. Then f satisfies the conditions (1), (2) and (3). This completes the proof. □

COROLLARY 3.6. *Let X be a $2k$-normed product, G a linear subspace in X and $x \in X \setminus \overline{G}$. The following statement are equivalent:*

(1) $g_o \in P_G(x)$,
(2) $\tau(x - g_o, g) = 0$ *for all* $g \in G$, *where* τ *is given by* $(*)$,
(3) *There exists a unique continuous linear functional* $f : X \to R$ *with the properties:*

$$f(x - g_o) = \|x - g_o\|_{2k}, \quad f(g) = 0, \quad f(y) \le \|y\|_{2k} \quad \text{for all} \quad y \in X.$$

THEOREM 3.7. *Let $(X,(\cdot,\dots,\cdot))$ be a complete $2k$-inner product space and f be a nonzero continuous linear functional on X, $g_o \in Ker(f)$ and $x_o \in X \setminus Ker(f)$. Then the following statements are equivalent:*

(1) $g_o \in P_{Ker(f)}(x_o)$.
(2) $f(x) = \frac{f(x_o)}{\|x_o-g_o\|_{2k}^{2k}}(x, x_o - g_o, \dots, x_o - g_o)$ *for all* $x \in X$.

Proof. (1) $\Rightarrow$ (2): If $g_o \in P_{Ker(f)}(x_o)$, then $(x_o - g_o)\perp_{2k}\mathrm{Ker}(f)$. Let $w_o = x_o - g_o$. Then, since $f(x)w_o - f(w_o)x \in \mathrm{Ker}(f)$ for all $x \in X$ we have $w_o \perp_{2k} (f(x)w_o - f(w_o)x)$, which implies

$$f(x) = \frac{f(w_o)}{\|w_o\|_{2k}^{2k}}(x, w_o, \dots, w_o).$$

for all $x \in X$ and so we have (2).

(2) $\Rightarrow$ (1): Suppose that (2) holds. Then $\|f\| \le \frac{|f(x_o)|}{\|x_o-g_o\|_{2k}}$ and so

$$\|x_o - g_o\|_{2k} \le \frac{|f(x_o - g)|}{\|f\|} \le \|x_o - g\|_{2k}$$

for all $g \in Ker(f)$. Therefore $g_o \in P_{Ker(f)}(x_o)$, that is, $g_o \in P_{Ker(f)}(x_o)$. This completes the proof. □

COROLLARY 3.8. *Let $(X,(\cdot,\dots,\cdot))$ be a complete $2k$-inner product space and f be a nonzero continuous linear functional on X. Then the following statements are equivalent:*

(1) *Ker*(f) *is proximinal in* X.
(2) *There exists at least one* $y_f \in X \setminus \{0\}$ *such that* $f(x) = (x, y_f, \dots, y_f)$ *for all* $x \in X$.
(3) *There exists at least one* $v_f \in X \setminus \{0\}$ *such that* $|f(v_f)| = \|f\|\|v_f\|_{2k}^{2k-1}$.

4. Representation of a continuous linear functional

Let $(X,(\cdot,\dots,\cdot))$ be a $2k$-inner product space. If Y is a non-empty subset of X, then the set

$$Y^{\perp_{CBS}\epsilon} = \{y \in X : y\perp_{CBS}x(\epsilon) \quad \text{for all} \quad x \in Y\}$$

is called the $\epsilon - CBS$-*orthogonal complement* of Y.

The following results is given by [9]:

LEMMA 4.1. *Let X be a $2k$-inner product space, Y a non-empty closed proper subspace of X and $\epsilon \in [0, 1)$. Then $Y^{\perp_{CBS^\epsilon}}$ is non-empty.*

THEOREM 4.2. *Let X be a complete $2k$-inner product space, Y a non-empty closed subspace of X and $\epsilon \in [0, 1)$. Then we have $X = Y + Y^{\perp_{CBS^\epsilon}}$ for all $\epsilon \in [0, 1)$.*

THEOREM 4.3. *Let $(X, (\cdot, \dots, \cdot))$ be a complete $2k$-inner product space and f be a continuous linear functional on X. Then there exists $y_f \in X$ such that $f(x) = (x, y_f, \dots, y_f)$ for all $x \in X$ and $\|f\| = \|y_f\|_{2k}^{2k-1}$.*

Proof. If $f = 0$, then $y = 0$. If $f \neq 0$, then we let $x_o \in X$ with $f(x_o) \neq 0$. Since the subspace $Y = Ker(f)$ is closed in X, there is a unique $y_o \in Ker(f)$ and a unique $z_o \in Ker(f)^{\perp_{2k}}$ such that $x_o = y_o + z_o$. It results that $z_o \notin Ker(f)$. Let $\lambda \in R$ with

$$\lambda^{2k-1} = \frac{f(x_o)}{\|z_o\|_{2k}^{2k}}$$

and $y = \lambda z_o$. Since $f(x)z_o - f(z_o)x \in Ker(f)$ for all $x \in X$ we have $z_o \perp_{2k} (f(x)z_o - f(z_o)x)$, that is,

$$(f(x)z_o - f(z_o)x, z_o, \dots, z_o) = 0,$$

which implies

$$\begin{aligned} f(x) = \frac{f(z_o)}{\|z_o\|_{2k}^{2k}}(x, z_o, \dots, z_o) &= \lambda^{2k-1}(x, z_o, \dots, z_o) \\ &= (x, \lambda z_o, \dots, \lambda z_o) \\ &= (x, y, \dots, y) \end{aligned}$$

for all $x \in X$.

Now, using CBS inequality, $|f(x)| \leq \|x\|_{2k}\|y\|_{2k}^{2k-1}$ and so $\|f\| \leq \|y\|_{2k}^{2k-1}$. And letting $x = \frac{y}{\|y\|_{2k}}$, we have

$$\begin{aligned} |f(x)| &= (\frac{y}{\|y\|_{2k}^{2k}}, y, \dots, y) \\ &= \frac{1}{\|y\|_{2k}^{2k}}(y, y, \dots, y) \\ &= \|y\|_{2k}^{2k-1} \end{aligned}$$

Thus, we have $\|f\| = \|y_f\|_{2k}^{2k-1}$. This completes the proof. $\square$

COROLLARY 4.4. *Let* $(X, (\cdot, \dots, \cdot))$ *be a complete* $2k$*-inner product space and* f *be a continuous linear functional on* X *such that* $S_{Ker(f)} = \{g \in Ker(f) : \|g\|_{2k} \leq 1\}$ *is weakly sequentially compact in* X. *Then there exists* $y_f \in X$ *such that* $f(x) = (x, y_f, \dots, y_f)$ *for all* $x \in X$ *and* $\|f\| = \|y_f\|_{2k}^{2k-1}$.

THEOREM 4.5. *Let* X *be a complete* $2k$*-inner product space,* Y *a non-empty closed subspace of* X *and* $X \neq Y$. *Then for all* $\epsilon > 0$, *there exists a continuous linear functional* f_ϵ *such that*

$$\|f_\epsilon\| \leq 1 \quad \textit{and} \quad \|f_\epsilon\|_Y \leq \epsilon$$

where $\|f_\epsilon\|_Y = \sup\{|f_\epsilon(x)| : \|x\|_{2k} = 1 \quad \textit{for all} \quad x \in Y\}$.

Proof. Let $\epsilon \geq 1$. Then the statement is trivial. Assume that $\epsilon \in (0, 1)$. By Lemma 4.1, there exists a nonzero element $y_\epsilon \in Y^{\perp_{CBS}\epsilon}$ such that

$$|(x, y_\epsilon, \dots, y_\epsilon)| \leq \epsilon \|x\|_{2k} \|y_\epsilon\|_{2k}^{2k-1}$$

for all $x \in X$. Taking $x_\epsilon = y_\epsilon / \|y_\epsilon\|$, we have $\|x_\epsilon\|_{2k} \leq 1$. The functional $f_\epsilon : X \to R$ is defined by

$$f_\epsilon(x) = (x, x_\epsilon, \dots, x_\epsilon) = \frac{1}{\|y\|_{2k}^{2k-1}} (x, y_\epsilon, \dots, y_\epsilon).$$

Then we have $\|f_\epsilon\| \leq 1$ and $\|f_\epsilon\|_Y \leq \epsilon$. This completes the proof. $\square$

THEOREM 4.7. *Let* X *be a complete* $2k$*-inner product space,* f *be a non-zero continuous linear functional on* X. *Then for all* $\epsilon > 0$, *there exists a non-zero element* $x_\epsilon \in X$ *such that*

$$|f(x) - (x, x_\epsilon, \dots, x_\epsilon)| \leq \epsilon \|x\|_{2k}$$

for all $x \in X$.

Proof. Since f is a non-zero continuous linear functional on X, the subspace $Y = \mathrm{Ker}(f)$ is closed in X and $Y \neq X$. Let $\epsilon > 0$ and $\delta(\epsilon) = \epsilon / (2\|f\|)$.

If $\delta(\epsilon) \geq 1$, then by Lemma 4.1, there exists en element $y_\epsilon \in Y^{\perp_{CBS}(\epsilon)}$ such that

$$(**) \qquad |(x, y_\epsilon, \dots, y_\epsilon)| \leq \delta(\epsilon) \|y\|_{2k} \|y_\epsilon\|_{2k}^{2k-1}$$

for all $x \in X$. Let $0 < \delta(\epsilon) < 1$. Then there exists an element $y_\epsilon \in Y^{\perp_{CBS^\epsilon}}$ such that $(**)$ holds. Putting $w_\epsilon = y_\epsilon / \|y_\epsilon\|_{2k}$, we have $f(x)w_\epsilon - f(w_\epsilon)x \in Y$ and by $(**)$

$$\begin{aligned} |(f(x)w_\epsilon - f(w_\epsilon)x, w_\epsilon, \dots, w_\epsilon)| &\le \delta(\epsilon)\|f(x)w_\epsilon - f(w_\epsilon)x\|_{2k}\|w_\epsilon\|_{2k}^{2k-1} \\ &\le 2\delta(\epsilon)\|f\|\|x\|_{2k} \\ &= \epsilon\|x\|_{2k} \end{aligned}$$

for all $x \in X$. On the other hand, we have

$$\begin{aligned} &(f(x)w_\epsilon - f(w_\epsilon)x, w_\epsilon, \dots, w_\epsilon) \\ &= f(x)\|w_\epsilon\|_{2k}^{2k} - f(w_\epsilon)(x, w_\epsilon, \dots, w_\epsilon) \\ &= f(x) - \frac{f(y_\epsilon)}{\|y_\epsilon\|_{2k}^{2k}}(x, y_\epsilon, \dots, y_\epsilon). \end{aligned}$$

Putting $x_\epsilon^{2k-1} = \frac{f(y_\epsilon)}{\|y_\epsilon\|_{2k}^{2k}} y_\epsilon$, we have

$$(f(x)w_\epsilon - f(w_\epsilon)x, w_\epsilon, \dots, w_\epsilon) = f(x) - (x, x_\epsilon, \dots, x_\epsilon)$$

and

$$|f(x) - (x, x_\epsilon, \dots, x_\epsilon)| \le \epsilon\|x\|_{2k}.$$

This completes the proof. □

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