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On Leibniz algebra bundles

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Abstract. Derivations and linear connections on a Leibniz algebra vector bundle which are compatible with the Leibniz bracket are studied.

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Introduction

Vector bundles, namely bundles whose fibres are linear spaces, are main objects in differential geometry and in various applications of geometry in theoretical physics. In addition, if the fibre is an algebra then special features appear and important notions in the geometry of bundles, e.g. derivations and linear connections, have new remarkable characterizations.

So, in this program the particular case of Lie algebras was studied in [4] using ideas from [3] and [11]. In [1] the general case of algebraic bundles was treated using the Yamaguti's notion of *homogeneous system* and examples of Lie algebra bundles are presented.

The class of Leibniz algebras was introduced by Jean-Louis Loday in [8] and [9] as a non-commutative version of Lie algebras. These algebras are intensively studied since then ([2], [10]) and applications of Leibniz algebroids, which are particular examples of Leibniz algebra bundles, to homology and cohomology theories for Nambu-Poisson manifolds are pointed out in [5] and [6].

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The purpose of this work is to present some properties of derivations and linear connections compatible with the bracket on a Leibniz algebra bundle. In the first section the Leibniz algebras are revisited and the objects of our framework namely Leibniz algebra bundles are defined. Characterizations for derivations and connections compatible with the Leibniz bracket are the theme of following two sections. An example relating Lie and Leibniz brackets from [7] is used to obtain, by means of a Lie structure on bundle of (1,1)-type tensor fields, a Leibniz algebra vector bundle. Moreover, a derivation compatible in Lie setting induces a derivation compatible with Leibniz bracket.

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1. Leibniz algebra vector bundles and Leibniz algebra bundles

Definition 1.1 (i) A *Leibniz algebra* over the field K is a K-vector space $\mathfrak{g}$ endowed with an K-bilinear mapping $\{,\} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that the Leibniz identity holds:

$$\{a_1, \{a_2, a_3\}\} = \{\{a_1, a_2\}, a_3\} + \{a_2, \{a_1, a_3\}\} \tag{1.1}$$

for all $a_1, a_2, a_3 \in \mathfrak{g}$. If the bracket is skew-symmetric we recover the notion of Lie algebra.

(ii) If $f : (\mathfrak{g}_1, \{,\}_1) \to (\mathfrak{g}_2, \{,\}_2)$ is an K-(iso)morphism of vector spaces between two Leibniz algebras then f is called (iso)morphism of Leibniz algebras if:

$$f(\{a_1, a_2\}_1) = \{f(a_1), f(a_2)\}_2 \tag{1.2}$$

for every $a_1, a_2 \in \mathfrak{g}_1$.

Example 1.2([7]) Let g be a Lie algebra. Then $g \otimes g$ is a Leibniz algebra with respect to the bracket:

$$[x \otimes y, a \otimes b] = [x, [a, b]] \otimes y + x \otimes [y, [a, b]]. \tag{1.3}$$

For a smooth, paracompact and connected differentiable manifold M_n let $C^\infty(M)$ be the ring of real functions, $\mathcal{D}^r_s(M)$ the $C^\infty(M)$–module of tensor fields of type (r, s) and $\mathcal{D}(M)$ the tensorial algebra. For a vector bundle $\xi = (E, \pi, M)$ of rank m we denote by $\mathcal{D}^r_s(\xi)$ the $C^\infty(M)$–module of tensor fields of type (r, s) and by $\mathcal{D}(\xi)$ its tensorial algebra.

Definition 1.3 (i) ([1, p. 80]) An *algebraic vector bundle* is a pair (ξ, C) with ξ a vector bundle and $C \in \mathcal{D}^1_2(\xi)$.

(ii) A *Leibniz algebra vector bundle* is an algebraic vector bundle such that Leibniz identity holds:

$$C(u, C(v, w)) = C(C(u, v), w) + C(v, C(u, w)). \tag{1.4}$$

(iii) ([4, p. 45]) A *Lie algebra vector bundle* is an algebraic vector bundle which is skew–symmetric and satisfies the Jacobi identity, that is:

$$C(u, v) + C(v, u) = 0 \tag{1.5'}$$

$$C(u, C(v, w)) + C(v, C(w, u)) + C(w, C(u, v)) = 0 \tag{1.5''}$$

$\forall u, v, w \in \mathcal{D}^1_0(\xi)$.

It results:

$$\{\text{Lie vector bundles}\} \subset \{\text{Leibniz vector bundles}\} \subset \{\text{algebraic vector bundles}\}.$$

In the following we restrict our work to Leibniz case. Setting for each $x \in M$ and $u_x, v_x \in E_x$:

$$\{u_x, v_x\}_x = C_x(u_x, v_x) \tag{1.6}$$

we obtain an $\mathbb{R}$–Leibniz algebra structure on the fibre E_x with C_x as structural tensor. Putting then, for $u, v \in \mathcal{D}^1_0(\xi)$:

$$\{u, v\} = C(u, v) \tag{1.7}$$

we obtain a $C^\infty(M)$-Leibniz algebra structure on the module $\mathcal{D}^1_0(\xi)$ denoted by $\mathcal{D}^1_0(\xi, C)$. The vector field $\{u, v\}$ is called the *Leibniz product* or the *bracket* of u and v.

Definition 1.4 A Leibniz algebra vector bundle is called *Leibniz algebra bundle* if there exists an $\mathbb{R}$-Leibniz algebra L and for each $x_0 \in M$ a vectorial chart $(\mathcal{U}, \varphi)$ such that for any $x \in \mathcal{U}$ the map $t_x : L \longrightarrow E_x$ given by $t_x(\vec{y}) = \varphi^{-1}(x, \vec{y})$ is a Leibniz algebra isomorphism.

It follows that for a basis $\{\ell_a\} \subset L$, $a = 1, 2, ..., m = rank\, L$ and the corresponding basis $\{e_a(x)\} = \{t_x(\ell_a)\} \subset E_x$ one obtains the structure equations:

$$\{e_b(x), e_c(x)\} = C^a_{bc} e_a(x), \; b, c = 1, 2, ..., m, \tag{1.8}$$

where C^a_{bc} are constant on $\mathcal{U}$. In this case all the fibres of ξ are isomorphic to L as $I\!R$-Leibniz algebras.

Example 1.5 Let $\mathcal{T}^1_1(M)$ be the bundle of tensor fields of $(1,1)$-type on M. Conform [4] this bundle is a Lie algebra bundle with respect to:

$$[T, S] = T \circ S - S \circ T$$

having the general Lie algebra $gl(n, I\!R)$ as typical fibre. Then, using Example 1.2, we have a Leibniz structure on $\mathcal{T}^1_1(M) \otimes \mathcal{T}^1_1(M)$ given by the bracket (1.3).

2. Derivations compatible with the bracket

By a derivation in the vector bundle $\xi = (E, \pi, M)$ we shall understand an $I\!R$–derivation in the tensorial algebra $\mathcal{D}(\xi)$ which preserves the type and commutes with the contractions. It is uniquely determined by its values on $C^\infty(M)$ and $\mathcal{D}^1_0(\xi)$. The set of these derivations is denoted by $Der(\xi)$ and it is a $C^\infty(M)$–module and an $I\!R$–Lie algebra. For every derivation $D \in Der(\xi)$, we shall denote by $res^r_s(D)$ its restriction to $\mathcal{D}^r_s(\xi)$. In particular, $res^0_0(D)$ is a vector field on M. There is an isomorphism between the algebra $\mathcal{D}^1_1(\xi)$ and the subalgebra of the derivations on ξ with the restriction to $\mathcal{D}^0_0(\xi) = C^\infty(M)$ equal to zero. This isomorphism associates to each $S \in \mathcal{D}^1_1(\xi)$ the derivation $D = i(S)$ given by $i(S)(f) = 0$ if $f \in C^\infty(M)$ and $i(S)(u) = S(u)$ if $u \in \mathcal{D}^1_0(\xi)$.

If (ξ, C) is a Leibniz algebra vector bundle, then setting for each $u \in \mathcal{D}^1_0(\xi, C)$:

$$ad_u(v) = \{u, v\} \tag{2.1}$$

it comes out that ad_u is a tensor field of type $(1,1)$ on ξ and that the map $ad : \mathcal{D}^1_0(\xi, C) \longrightarrow \mathcal{D}^1_1(\xi)$ is a $C^\infty(M)$-Leibniz algebra morphism. The kernel of ad is the center K of $\mathcal{D}^1_0(\xi, C)$ and its image is an ideal in $\mathcal{D}^1_1(\xi)$, which is isomorphic to the factor algebra $\mathcal{D}^1_0(\xi, C)/K$.

Definition 2.1 One says that a derivation $D \in Der(\xi)$ is *compatible* with the bracket on the Leibniz algebra vector bundle (ξ, C) if it is a derivation in the $I\!R$-Leibniz algebra $\mathcal{D}^1_0(\xi, C)$.

In other words $D \in Der(\xi)$ is compatible with the bracket on ξ if and only if

$$D\{u, v\} = \{Du, v\} + \{u, Dv\}, \ \forall u, v \in \mathcal{D}^1_0(\xi). \tag{2.2}$$

It follows from here:

Proposition 2.2 *A derivation $D \in Der(\xi)$ is compatible with the bracket on the Leibniz algebra vector bundle (ξ, C) if and only if the derivative of the structural tensor field C is zero*:

$$DC = 0. \tag{2.3}$$

Remarks 2.3 (i) The set $Der(\xi, C)$ of the derivations compatible with the bracket on ξ is a $C^\infty(M)$–submodule and a $I\!R$–Lie subalgebra in $Der(\xi)$.

(ii) The set $i(\mathcal{D}^1_1(\xi, C))$ of the derivations of the form $D = i(S)$ with $S \in \mathcal{D}^1_1(\xi)$, compatible with the bracket on ξ, is an $C^\infty(M)$–submodule and a subalgebra of $Der(\xi, C)$.

(iii) All derivations of the form $D = i(ad_u)$ with $u \in \mathcal{D}^1_0(\xi)$, called *inner derivations*, are compatible with the bracket on ξ and their set $In(\mathcal{D}^1_0(\xi, C)) = Im(i \circ ad)$ is a $C^\infty(M)$–submodule and an ideal in $Der(\xi, C)$.

Example 2.4 Let us consider the Lie algebra bundle $\mathcal{T}^1_1(M)$ given in Example 1.5. Then every derivation on this bundle compatible with Lie bracket yields a derivation compatible with the Leibniz bracket on $\mathcal{T}^1_1(M) \otimes \mathcal{T}^1_1(M)$.

3. Linear connections compatible with the bracket

A linear connection ∇ in the bundle $\xi = (E, \pi, M)$ may be defined as a map $\nabla : \mathcal{D}^1_0(M) \longrightarrow Der(\xi)$ which is $C^\infty(M)$–linear and satisfies the condition:

$$res^0_0 \circ \nabla = 1_{\mathcal{D}^1_0(M)}. \tag{3.1}$$

The set of linear connections in the bundle ξ is a $C^\infty(M)$–affine module([3]), which we denote by $\mathcal{C}(\xi)$. Considering the exact sequence of $C^\infty(M)$–modules:

$$0 \to \mathcal{D}^1_1(\xi) \stackrel{i}{\to} Der(\xi) \stackrel{res^0_0}{\to} \mathcal{D}^1_0(M) \to 0 \tag{3.2}$$

a linear connection ∇ in the vector bundle ξ is a right splitting for this sequence. It follows from here:

Proposition 3.1 *Given a linear connection ∇ in the vector bundle ξ every derivation $D \in Der(\xi)$ can be decomposed uniquely as:*

$$D = i(S) + \nabla_X \tag{3.3}$$

where $X = res_0^0(D)$ and $S = res_0^1(D - \nabla_X)$.

Hence, the connection ∇ determines a decomposition of the $C^\infty(M)$–module $Der(\xi)$ in the direct sum: $Der(\xi) = i(\mathcal{D}_1^1(\xi)) \oplus \nabla(\mathcal{D}_0^1(M))$.

Definition 3.2 One says that a linear connection ∇ in the Leibniz algebra vector bundle (ξ, C) is *compatible* with the bracket if for each $X \in \mathcal{D}_0^1(M)$ the derivation ∇_X is in $Der(\xi, C)$.

From Proposition 2.2 it follows:

Proposition 3.3 *The linear connection ∇ in the Leibniz algebra vector bundle (ξ, C) is compatible with the bracket if and only if:*

$$\nabla_X C = 0, \quad \forall X \in \mathcal{D}_0^1(M). \tag{3.4}$$

For this condition we may give the following geometrical characterization:

Proposition 3.4 *The linear connection ∇ in the Leibniz algebra vector bundle ξ is compatible with the bracket if and only if for each curve on the manifold M the parallel transport defined by ∇ establishes a Leibniz algebra isomorphism between the fibres of ξ along the curve.*

Using the partition of unity and the formula (1.8) one obtains:

Proposition 3.5 *Every Leibniz algebra bundle admits a linear connection compatible with the bracket.*

Conversely, from Proposition 3.5 it follows that any Leibniz algebra vector bundle which admits a linear connection compatible with the bracket is a Leibniz algebra bundle.

The set of these connections is a $C^\infty(M)$–affine submodule of $\mathcal{C}(\xi)$ denoted by $\mathcal{C}(\xi, C)$.

Remark. If ∇ is a linear connection compatible with the bracket in the Leibniz algebra bundle (ξ, C) then from the formula (3.2) it follows that the sequence of $C^\infty(M)$–modules:

$$0 \to \mathcal{D}_1^1(\xi, C) \xrightarrow{i} Der(\xi, C) \xrightarrow{res_0^0} \mathcal{D}_0^1(M) \to 0 \tag{3.5}$$

is an exact one and that every linear connection $\nabla' \in \mathcal{C}(\xi, C)$ is a right splitting of this sequence.

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