ON SOME INVARIANT SUBMANIFOLDS IN A RIEMANNIAN MANIFOLD WITH GOLDEN STRUCTURE

BY

C.-E. HRETČANU and M. CRĂȘMAREANU∗

Dedicated to Academician Radu Miron at his 80th anniversary

Abstract. The purpose of this paper is to study invariant submanifolds of a Riemannian manifold endowed with a golden structure. An m-dimensional Riemannian manifold \((\tilde{M}, \tilde{g}, \tilde{P})\) is called a golden Riemannian manifold if the \((1,1)\)-tensor field \(\tilde{P}\) on \(\tilde{M}\) is a golden structure (i.e. \(\tilde{P}^2 = \tilde{P} + \text{Id}\)) and \(\tilde{g}(\tilde{PU}, V) = \tilde{g}(U, \tilde{PV})\) for every tangent vector fields \(U, V \in \chi(\tilde{M})\).


Key words: Riemannian manifold, Golden structures.

1. Introduction. The idea to construct a new structure on a Riemannian manifold, named by us the golden structure, is based on several results regarding geometrical structures constructed on Riemannian manifolds ([1],[8],[10]). Goldberg and Yano ([5]) introduced the notion of polynomial structures on a manifold. Our structure was inspired by the Golden Ratio, which was described by Johannes Kepler (1571 – 1630) as one of the “two great treasures of geometry” (the other one is the Theorem of Pythagoras). The Golden Ratio arises as a result of the solution regarding the division problem of the line segment AB with a point C (which belongs to the segment AB) in the ratio \(\frac{AC}{CB} = \frac{AB}{AC}\). The first known written definition of the Golden Ratio is given by Euclid of Alexandria (around 300 BC): "A straight line is said to have been cut in extreme and mean ratio

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when, as the whole line is to the greater segment, so is the greater to the lesser” ([4]).

In this paper, we study the properties of the golden structure on a m-dimensional Riemannian manifold \((\tilde{M}, \tilde{g})\), defined as a polynomial structure ([5]) with the structure polynomial \(f(x) = x^2 - x - \text{Id} = 0\). This structure is determined by an \((1,1)\) tensor field \(\tilde{P}\) on a \(m\)-dimensional Riemannian manifold \((\tilde{M}, \tilde{g})\) which verifies a similar equation to that satisfied by the Golden Ratio (i.e. \(x^2 = x + 1\)). In Section 2 we establish several properties of the induced structure on a submanifold in a Golden Riemannian manifold and in Section 3 we find some properties of the induced structure on an invariant submanifold in a golden Riemannian manifold.

We called, in [6], a golden structure on a \(m\)-dimensional Riemannian manifold \((\tilde{M}, \tilde{g})\) an \((1,1)\)-tensor field \(\tilde{P}\) which satisfies the equation:

\[
\tilde{P}^2 = \tilde{P} + \text{Id}
\]  

where \(\text{Id}\) is the identity on the Lie algebra of vector fields on \(\tilde{M}, \chi(\tilde{M})\).

A Riemannian manifold \((\tilde{M}, \tilde{g})\), endowed with a golden structure \(\tilde{P}\) such that

\[
\tilde{g}(\tilde{P}U, V) = \tilde{g}(U, \tilde{P}V),
\]

we say that the metric \(\tilde{g}\) is \(\tilde{P}\)-compatible and \((\tilde{P}, \tilde{g})\) is named a golden Riemannian structure.

2. Properties of induced structures on submanifolds in golden Riemannian manifolds. Let \(M\) be a \(n\)-dimensional submanifold of codimension \(r\), immersed in a Riemannian manifold \((\tilde{M}, \tilde{g}, \tilde{P})\), with a Riemannian metric \(\tilde{g}\) and a golden structure \(\tilde{P}\) such that the metric \(\tilde{g}\) is \(\tilde{P}\)-compatible (the compatibility is provided by the relation (1.2)). We denote by \(T_xM\) the tangent space of \(M\) in \(x \in M\) and by \(T^+_xM\) the normal space of \(M\) in \(x\), for every \(x \in M\). Let \(i_*\) the differential of immersion \(i : M \to \tilde{M}\).

The induced Riemannian metric \(g\) of \(M\) is given by

\[
g(X, Y) = \tilde{g}(i_*X, i_*Y),
\]

for all \(X, Y \in \chi(M)\). We consider a local orthonormal basis \(\{N_1, ..., N_r\}\) of the normal space \(T_x(M)^\perp\) at every point \(x \in M\). We suppose that the range indices \(\alpha, \beta, \gamma\) is in \(1, 2, ..., r\) and \(i, j, k \in \{1, ..., n\}\).
For any $X \in T_xM$, $\tilde{P}_iX$ and $\tilde{P}N_\alpha$ can be decomposed in tangential and normal components at $M$ in the forms:

\begin{equation}
\tilde{P}_iX = i_*PX + \sum_{\alpha} u_\alpha(X)N_\alpha, \quad (\forall) X \in \chi(M)
\end{equation}

and

\begin{equation}
\tilde{P}N_\alpha = \varepsilon i_\ast \xi_\alpha + \sum_{\beta} a_{\alpha\beta}N_\beta, \quad (\varepsilon = \pm 1)
\end{equation}

where $P$ is an $(1,1)$-tensor field, $\xi_\alpha$ are tangent vector fields on submanifold $M$, $u_\alpha$ are 1-forms on $M$ and $a := (a_{\alpha\beta})_r$ is a $r \times r$ matrix of real functions on $M$. Thus, we obtain a structure $(P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ induced on $M$ by $(\tilde{P}, \tilde{g})$. The Gauss and Weingarten formulae are:

\begin{equation}
\tilde{\nabla}_X Y = \nabla_X Y + \sum_{a=1}^r h_a(X,Y)N_\alpha, \quad \tilde{\nabla}_X N_\alpha = -A_\alpha X + \nabla^\perp_X N_\alpha,
\end{equation}

respectively, where $h_a(X,Y) = g(A_\alpha X, Y)$, for every $X,Y \in \chi(M)$.

If $\{N_1, ..., N_r\}$ and $\{N_1', ..., N_r'\}$ are two local orthonormal basis on a normal space $T_x^\perp M$ then, the decomposition of $N'_\alpha$ in the basis $\{N_1, ..., N_r\}$ is the following

\begin{equation}
N'_\alpha = \sum_{\gamma=1}^r k_{\alpha\gamma}N_\gamma,
\end{equation}

for any $\alpha \in \{1, ..., r\}$, where $(k_{\alpha\gamma}^r)$ is an $r \times r$ orthogonal matrix and we have (from [2]): $u'_\alpha = \sum_\gamma k_{\alpha\gamma}u_\gamma$, $\xi'_\alpha = \sum_\gamma k_{\alpha\gamma}\xi_\gamma$ and $a'_{\alpha\beta} = \sum_\gamma k_{\alpha\gamma}a_{\gamma\delta}k_{\beta\delta}$. Thus, if $\xi_1, ..., \xi_r$ are linearly independent vector fields, then $\xi'_1, ..., \xi'_r$ are also linearly independent. Furthermore, because $a_{\alpha\beta}$ is symmetric in $\alpha$ and $\beta$, under a suitable transformation, we can find that $a_{\alpha\beta}$ can be reduced to $a'_{\alpha\beta} = \lambda_\alpha \delta_{\alpha\beta}$, where $\lambda_\alpha$ ($\alpha \in \{1,2, ..., r\}$) are eigenvalues of the matrix $(a_{\alpha\beta})_r$ and in this case we have: $u'_{\beta}(\xi_\alpha) = \varepsilon \delta_{\alpha\beta}(1 + \lambda_\alpha - \lambda_\alpha \lambda_\beta)$ and from this we obtain $u'_\alpha(\xi_\alpha) = \varepsilon(1 + \lambda_\alpha - \lambda^2_\alpha)$.

**Proposition 2.1.** If we suppose that $\xi_1, ..., \xi_r$ are linearly independent tangent vector fields on $M$, it follows that the 1-forms $u_1, ..., u_r$ are also linearly independent.
Proof. The equality $\sum_{\alpha=1}^{r} \mu^{\alpha} u_{\alpha}(X) = 0$ is equivalent with

$$0 = \sum_{\alpha} \mu^{\alpha} g(X, \xi_{\alpha}) = g(X, \sum_{\alpha} \mu^{\alpha} \xi_{\alpha}), \quad (\forall) X \in \chi(M)$$

thus, we have $\sum_{\alpha=1}^{r} \mu^{\alpha} \xi_{\alpha} = 0 \Rightarrow \mu^{\alpha} = 0$ so, $u_{1}, ..., u_{r}$ are linearly independent on $M$.

**Proposition 2.2.** If $M$ is a $n$-dimensional submanifold of codimension $r$, isometrically immersed in a golden Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$, from (1.4) and (1.5) we obtain that the elements $P, g, u_{\alpha}, \varepsilon \xi_{\alpha}, (a_{\alpha \beta})_{r}$ of the induced structure on $M$ by the golden structure $\tilde{P}$, verify the following equalities:

(2.6) \[ P^{2}X = PX + X - \varepsilon \sum_{\alpha} u_{\alpha}(X) \xi_{\alpha}, \]

(2.7) \[ u_{\alpha}(PX) = (1 - a_{\alpha \alpha}) u_{\alpha}(X) = u_{\alpha}(X) - a_{\alpha \beta}(X), \]

(2.8) \[ a_{\alpha \beta} = a_{\beta \alpha}, \]

(2.9) \[ u_{\beta}(\xi_{\alpha}) = \varepsilon (\delta_{\alpha \beta} + a_{\alpha \beta} - \sum_{\gamma} a_{\alpha \gamma} a_{\gamma \beta}), \]

(2.10) \[ P \xi_{\alpha} = \xi_{\alpha} - \sum_{\beta} a_{\alpha \beta} \xi_{\beta}, \]

and the connections between the $(1,1)$ tensor field $P$ on $M$ and the induced metric $g$ on the submanifold $M$ are as follows:

(2.11) \[ u_{\alpha}(X) = \varepsilon g(X, \xi_{\alpha}), \]

(2.12) \[ g(PX, Y) = g(X, PY), \]

and

(2.13) \[ g(PX, PY) = g(PX, Y) + g(X, Y) - \sum_{\alpha} u_{\alpha}(X) u_{\alpha}(Y), \]

for any $X, Y \in \chi(M)$. 
Remark 2.1. If $M$ is a non-invariant $n$-dimensional submanifold of codimension $r$, immersed in a golden Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$ so that the tangent vector fields $\xi_1, \xi_2, \ldots, \xi_r$ are linearly independent then, from (2.9) and (2.11) we obtain:

\begin{equation}
\|\xi_\alpha\|^2 = 1 + a_{\alpha\alpha} - \sum_\gamma a_{\alpha\gamma}^2
\end{equation}

and, for $\alpha \neq \beta$ we have

\begin{equation}
\sum_\gamma a_{\alpha\gamma} a_{\gamma\beta} = a_{\alpha\beta}.
\end{equation}

For the normal connection $\nabla^\perp_X N_\alpha$, we have the decomposition

\begin{equation}
\nabla^\perp_X N_\alpha = \sum_{\beta=1}^r l_{\alpha\beta}(X)N_\beta,
\end{equation}

for every $X \in \chi(M)$. Therefore, we obtain an $r \times r$ matrix $(l_{\alpha\beta}(X))_r$ of 1-forms on $M$. From $\tilde{g}(N_\alpha, N_\beta) = \delta_{\alpha\beta}$ we get $\tilde{g}(\nabla^\perp_X N_\alpha, N_\beta) + \tilde{g}(N_\alpha, \nabla^\perp_X N_\beta) = 0$ which is equivalent with $\tilde{g}(\sum_\gamma l_{\alpha\gamma}(X)N_\gamma, N_\beta) + \tilde{g}(N_\alpha, \sum_\gamma l_{\beta\gamma}(X)N_\gamma) = 0$, for any $X \in \chi(M)$. Thus, we obtain

\begin{equation}
l_{\alpha\beta} = -l_{\beta\alpha},
\end{equation}

for any $\alpha, \beta \in \{1, \ldots, r\}$.

Let $N_{\tilde{P}}(X, Y)$ be the Nijenhuis torsion tensor field of $\tilde{P}$, defined by ([7]):

\begin{equation}
N_{\tilde{P}}(X, Y) = [\tilde{P}X, \tilde{P}Y] + \tilde{P}^2[X, Y] - \tilde{P}[\tilde{P}X, Y] - \tilde{P}[X, \tilde{P}Y].
\end{equation}

Remark 2.2. If $(M, g)$ is a Riemannian manifold endowed with an (1,1) tensor field $P$, then the Nijenhuis tensor of $P$ verifies that

\begin{equation}
N_{\tilde{P}}(X, Y) = (\tilde{\nabla}_{P X} \tilde{P})(Y) - (\tilde{\nabla}_{P Y} \tilde{P})(X)
\end{equation}

\begin{equation}
-\tilde{P}[(\tilde{\nabla}_X \tilde{P})(Y) - (\tilde{\nabla}_Y \tilde{P})(X)],
\end{equation}

for any $X, Y \in \chi(\tilde{M})$, where $\tilde{\nabla}$ is the Levi-Civita connection on $\tilde{M}$. 
Remark 2.3. If we suppose that \((\tilde{M}, \tilde{g}, \tilde{P})\) is a locally product Riemannian manifold (i.e. \(\tilde{P}\) is parallel with respect to the Levi-Civita connection \(\tilde{\nabla}\) of \(\tilde{g}\), \(\tilde{\nabla}\tilde{P} = 0\), from (2.19) we obtain the integrability of the structure \(\tilde{P}\) (which is equivalent with the vanishing of the Nijenhuis torsion tensor field of \(\tilde{P}\)).

We denoted by \(P\) the (1,2)-tensor field on \(\tilde{M}\), such that
\[
(2.20) \quad P(\tilde{X}, \tilde{Y}) = (\tilde{\nabla}_X \tilde{P})(\tilde{Y}) := \tilde{\nabla}_X(\tilde{P}\tilde{Y}) - \tilde{P}(\tilde{\nabla}_X \tilde{Y}),
\]
for any \(\tilde{X}, \tilde{Y} \in \chi(\tilde{M})\).

Let \(P(\tilde{X}, \tilde{Y})^\top\) and \(P(\tilde{X}, \tilde{Y})^\perp\) be tangential and respectively normal components on \(M\) of \(P(\tilde{X}, \tilde{Y})\), for any \(\tilde{X}, \tilde{Y} \in \chi(\tilde{M})\).

Theorem 2.1. If \(M\) is an \(n\)-dimensional submanifold of codimension \(r\) in a golden Riemannian manifold \((\tilde{M}, \tilde{g}, \tilde{P})\), then the structure \((P, g, u_\alpha, \varepsilon \xi_\alpha, (a_{\alpha\beta})_r)\) induced on \(M\) by the structure \(\tilde{P}\) has the following properties:
\[
(2.21) \quad \begin{cases}
(i)(\nabla_X P)(Y) = P(X, Y)^\top + \varepsilon \sum_\alpha h_\alpha(X, Y)\xi_\alpha + \sum_\alpha u_\alpha(Y)A_\alpha X, \\
(ii)(\nabla_X u_\alpha)(Y) = \tilde{g}(P(X, Y), N_\alpha) - h_\alpha(X, PY) + \sum_\beta (u_\beta(Y)l_{\alpha\beta}(X) + h_\beta(X, Y)a_{\beta\alpha})
\end{cases}
\]
\[
(iii) \nabla_X \xi_\alpha = P(X, N_\alpha)^\top - \varepsilon P(A_\alpha X) + \varepsilon \sum_\beta a_{\alpha\beta}A_\beta X
+ \sum_\beta l_{\alpha\beta}(X)\xi_\beta,
\]
\[
(iv) X(a_{\alpha\beta}) = \tilde{g}(P(X, N_\alpha), N_\beta) - \varepsilon u_\alpha(A_\beta X) - u_\beta(A_\alpha X)
+ \sum_\gamma [l_{\alpha\gamma}(X)a_{\gamma\beta} + l_{\beta\gamma}(X)a_{\alpha\gamma}]
\]
for any \(X, Y \in \chi(M)\).

Proof. From \(\tilde{\nabla}_X (\tilde{P}Y) = \nabla_X PY - \sum_\alpha u_\alpha(Y)A_\alpha X + \sum_\alpha [h_\alpha(X, PY + X(u_\alpha(Y))) + \sum_\beta u_\beta(Y)l_{\beta\alpha}(X)]N_\alpha\) and
\[
\tilde{P}(\tilde{\nabla}_X Y) = P(\nabla_X Y) + \varepsilon \sum_\alpha h_\alpha(X, Y)\xi_\alpha
+ \sum_\alpha [u_\alpha(\nabla_X Y) + \sum_\beta h_\beta(X, Y)a_{\beta\alpha}]N_\alpha
\]
we obtain
\[ \mathcal{P}(X, Y) = (\nabla_X P)(Y) - \sum_{\alpha} u_{\alpha}(Y)A_{\alpha}X - \varepsilon \sum_{\alpha} h_{\alpha}(X, Y)\xi_{\alpha} \]
\[ + \sum_{\alpha} [h_{\alpha}(X, PY) + (\nabla_X u_{\alpha})(Y)] \]
\[ + \sum_{\beta} u_{\beta}(Y)l_{\beta\alpha}(X) - \sum_{\beta} h_{\beta}(X, Y)a_{\beta\alpha}]N_{\alpha}. \]

Thus, identifying the tangential part and respectively the normal part in the last equality, we obtain (i) and (ii) from (2.21).

From
\[ \tilde{\nabla}_X(\tilde{P}N_{\alpha}) = \varepsilon \nabla_X \xi_{\alpha} - \sum_{\beta} a_{\alpha\beta}A_{\beta}X + \sum_{\beta} [X(a_{\alpha\beta}) + \varepsilon h_{\beta}(X, \xi_{\alpha})] \]
\[ + \sum_{\gamma} a_{\alpha\gamma} \cdot l_{\gamma\beta}(X)\] and
\[ \tilde{P}(\tilde{\nabla}_X N_{\alpha}) = -P(A_{\alpha}X) + \varepsilon \sum_{\beta} l_{\alpha\beta}(X)\xi_{\beta} - \sum_{\beta} [u_{\beta}(A_{\alpha}X)] \]
\[ - \varepsilon \sum_{\gamma} a_{\gamma\beta}l_{\alpha\gamma}(X)\] we obtain
\[ \mathcal{P}(X, N_{\alpha}) = \varepsilon \nabla_X \xi_{\alpha} + P(A_{\alpha}X) - \varepsilon \sum_{\beta} l_{\alpha\beta}(X)\xi_{\beta} - \sum_{\beta} a_{\alpha\beta}A_{\beta}X \]
\[ + \sum_{\beta} [X(a_{\alpha\beta}) + \varepsilon h_{\beta}(X, \xi_{\alpha}) + u_{\beta}(A_{\alpha}X)] \]
\[ - \sum_{\gamma} (a_{\gamma\beta}l_{\alpha\gamma}(X) - a_{\alpha\gamma} \cdot l_{\gamma\beta}(X))\]N_{\beta}.

Thus, identifying the tangential part and respectively the normal part in the last equality, we obtain (iii) and (iv) from (2.21). □

We can find, in a similar way like in [7], the following property:
Theorem 2.2. Let $M$ be a $n$-dimensional submanifold of codimension $r$ in a golden Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$ with $\tilde{\nabla} \tilde{P} = 0$. If $(P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ is the induced structure on $M$ by $(\tilde{P}, \tilde{g})$ and $\nabla$ is the Levi-Civita connection defined on $M$ with respect to $g$ then, the Nijenhuis torsion tensor field of $P$ has the form:

$$N_P(X, Y) = \sum_\alpha g((PA_\alpha - A_\alpha P)(X), Y)\xi_\alpha$$

$$- \sum_\alpha g(Y, \xi_\alpha)(PA_\alpha - A_\alpha P)(X)$$

$$+ \sum_\alpha g(X, \xi_\alpha)(PA_\alpha - A_\alpha P)(Y)$$

(2.22)

for any $X, Y \in \chi(M)$.

Corollary 2.1. Let $M$ be a $n$-dimensional submanifold of codimension $r$ in a golden Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$ and let $(P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ be the induced structure on $M$ by $(\tilde{P}, \tilde{g})$. If $\tilde{\nabla} \tilde{P} = 0$ and the $(1,1)$ tensor field $P$ on $M$ commutes with the Weingarten operators $A_\alpha$ (that is $PA_\alpha = A_\alpha P$, for any $\alpha \in \{1, ..., r\}$) then, the Nijenhuis torsion tensor field of $P$ vanishes on $M$ (that is $N_P(X, Y) = 0$, for any $X, Y \in \chi(M)$).

Remark 2.4. Using the model for an almost paracontact structure ([9]), we can compute the components $N^{(1)}, N^{(2)}, N^{(3)}$ and $N^{(4)}$ of the Nijenhuis torsion tensor field of $P$ for the $(P, g, \xi_\alpha, u_\alpha, (a_{\alpha\beta})_r)$ induced structure on a $n$-dimensional submanifold $M$ of codimension $r$ in a golden Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$:

$$\begin{align*}
(i) \quad N^{(1)}(X, Y) &= N_P(X, Y) - 2 \sum_{\alpha=1}^r du_\alpha(X, Y)\xi_\alpha, \\
(ii) \quad N^{(2)}_\alpha(X, Y) &= (L_{P_X}u_\alpha)Y - (L_{P_Y}u_\alpha)X, \\
(iii) \quad N^{(3)}_\alpha(X) &= (L_{\xi_\alpha}P)X, \\
(iv) \quad N^{(4)}_{\alpha\beta}(X) &= (L_{\xi_\alpha}u_\beta)X,
\end{align*}$$

(2.23)

for any $X, Y \in \chi(M)$ and $\alpha, \beta \in \{1, ..., r\}$, where $N_P$ is the Nijenhuis torsion tensor field of $P$ and $L_X$ means the Lie derivative with respect to $X$.

Proposition 2.3. Let $M$ be a $n$-dimensional submanifold of codimension $r$ in a golden Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$ with $\tilde{\nabla} \tilde{P} = 0$ and let
(P, g, uα, ξα, (aαβ)r) be the induced structure on M. If the normal connection ∇⊥ on the normal bundle T⊥M vanishes identically (∇⊥ = 0 ⇐⇒ lαβ = 0), then the components N(1), N(2), N(3) and N(4) of the Nijenhuis torsion tensor field of P for the structure (P, g, ξα, uα, (aαβ)r) induced on M have the forms:

\[ \begin{align*}
(i) \quad & N(1)(X, Y) = \sum_\alpha g(X, \xi_\alpha)(PA_\alpha - A_\alpha P)(Y) \\
& - \sum_\alpha g(Y, \xi_\alpha)(PA_\alpha - A_\alpha P)(X), \\
(ii) \quad & N(2)_\alpha(X, Y) = \sum_\beta a_{\alpha\beta}g((PA_\beta - A_\beta P)(X), Y) \\
& + \sum_\beta a_{\alpha\beta}u_\beta([X, Y]) + \sum_\beta [u_\beta(X)u_\alpha(A_\beta Y) - u_\beta(Y)u_\alpha(A_\beta X)], \\
(iii) \quad & N(3)_\alpha(X) = \sum_\beta a_{\alpha\beta}(PA_\beta - A_\beta P)(X) \\
& - P(PA_\alpha - A_\alpha P)(X) + \sum_\beta [u_\alpha(A_\beta X)\xi_\beta + u_\beta(X)A_\beta \xi_\alpha], \\
(iv) \quad & N(4)_{\alpha\beta}(X) = -u_\alpha(A_\beta PX) - u_\beta(PA_\alpha X) \\
& + \sum_\gamma [a_{\alpha\gamma}u_\beta(A_\gamma X) + a_{\gamma\beta}u_\alpha(A_\gamma X)]
\end{align*} \]

for any X, Y ∈ χ(M).

**Corollary 2.2.** Under the assumptions of the last proposition, if P and the Weingarten operators Aα commute (i.e. PAα = Aα P, α ∈ {1, ..., r}) then we obtain

\[ \begin{align*}
(i) \quad & N(1)(X, Y) = 0, \\
(ii) \quad & N(2)_\alpha(X, Y) = \sum_\beta (a_{\alpha\beta}u_\beta([X, Y]) + u_\beta(X)u_\alpha(A_\beta Y)) \\
& - u_\beta(Y)u_\alpha(A_\beta X), \\
(iii) \quad & N(3)_\alpha(X) = \sum_\beta [u_\alpha(A_\beta X)\xi_\beta + u_\beta(X)A_\beta \xi_\alpha], \\
(iv) \quad & N(4)_{\alpha\beta}(X) = 2 \sum_\gamma a_{\alpha\gamma}u_\alpha(A_\gamma X) - 2u_\alpha(PA_\alpha X)
\end{align*} \]

We denote by

\[ D_x = \{X_x \in T_x M : u_\alpha(X_x) = 0\}, \text{ for any } \alpha \in \{1, ..., r\}. \]
If \( \xi_1, ..., \xi_r \) are linearly independent, we remark that \( D_x \) is an \((n - r)\)-dimensional subspace in \( T_x M \) and the function \( D : x \mapsto D_x, (\forall)x \in M \) is a distribution locally defined on \( M \). If \( X \in D \), we have that \( u_\alpha(PX) = 0 \) for any \( X \in D \), then \( PX \in D \). Therefore \( D \) is an invariant distribution with respect to \( P \).

If \( D^\perp_x \) is the orthogonal supplement of \( D_x \) in \( T_x M \), then we obtain the distribution \( D^\perp : x \mapsto D^\perp_x \). Furthermore, we have the decomposition of \( T_x M = D_x \oplus D^\perp_x \), the vector fields \( \xi_\alpha \neq 0 \) are orthogonal on \( D_x \) and \( \xi_\alpha \in D^\perp_x \). Thus, if \( \xi_\alpha \neq 0 \) for any \( \alpha \in \{1, ..., r\} \), then \( D^\perp_x \) is generated by \( \xi_1, ..., \xi_r \) and \( D^\perp_x \) is \( r \)-dimensional in \( T_x M \). We remark that the space \( D_x \) is \( P \)-invariant and \( P \) is a golden Riemannian structure on \( D \) and its eigenvalues are \( \frac{1 \pm \sqrt{5}}{2} \).

3. Properties of invariant submanifolds in golden Riemannian manifolds

Definition 3.1. ([3]) If on \( M \) there exist two complementary and orthogonal distributions \( D \) and \( D^\perp \), satisfying the conditions:

\[
(3.1) \quad \tilde{P}(D_x) = D_x; \quad \tilde{P}(D^\perp_x) \subset T_x(M)^\perp
\]

for each \( x \in M \), then \( M \) is called a semi-invariant submanifold of the locally product Riemannian manifold \( \tilde{M} \).

Particularly, we have ([2]):

(i) if \( \text{dim}D_x = \text{dim}T_x(M) \) for each \( x \in M \) then \( M \) is an invariant submanifold of \( \tilde{M} \); in this case we have that

\[
(3.2) \quad \tilde{P}(T_x(M)) \subset T_x(M)
\]

and

\[
(3.3) \quad \tilde{P}(T_x(M)^\perp) \subset T_x(M)^\perp
\]

for every \( x \in M \).

(ii) if \( \text{dim}D_x = 0 \), for each \( x \in M \) then \( M \) is an anti-invariant submanifold of \( \tilde{M} \); in this case we have that

\[
(3.4) \quad \tilde{P}(T_x(M)) \subset T_x(M)
\]

for every \( x \in M \).
Remark 3.1. If \( M \) is a \( n \)-dimensional invariant submanifold of codimension \( r \), immersed in a golden Riemannian manifold \((\tilde{M}, \tilde{g}, \tilde{P})\), then \( \xi_\alpha \) (\( \alpha \in \{1, 2, \ldots, r\} \)) are zero vector fields and 1-forms \( u_\alpha \) vanishes identically on \( M \) (that is \( u_\alpha(X) = g(X, \xi_\alpha) = 0 \)). Consequently, (2.2) and (2.3) are respectively written as follows:

\[
(3.5) \quad \tilde{P}i_*X = i_*PX, \quad \tilde{P}N_\alpha = \sum_\beta a_{\alpha\beta}N_\beta, \quad (\forall) X \in \chi(M), \alpha \in \{1, 2, \ldots, r\}
\]

Thus, from Proposition 2.2, we obtain that the element of the structure \((P, g, u_\alpha, \varepsilon\xi_\alpha, (a_{\alpha\beta})_r)\), induced on \( M \) by structure \((\tilde{g}, \tilde{P})\), verifies these equalities:

\[
(i) \quad \nabla X P(Y) = \mathcal{P}(X, Y)^T, \\
(ii) \quad (\xi_\alpha)^\sharp(g(P(X, Y), N_\alpha)) - \delta_\alpha(X, PY) + \sum_\beta a_{\alpha\beta}h_\beta(X, Y) = 0 \\
(iii) \quad \mathcal{P}(X, N_\alpha)^T - \varepsilon\mathcal{P}(A\alpha X) + \varepsilon\sum_\beta a_{\alpha\beta}A\beta X = 0, \\
(iv) \quad \sum_\gamma [l_{\alpha\gamma}(X)\alpha\gamma + l_{\beta\gamma}(X)\alpha\gamma] = 0
\]

for any \( X, Y \in \chi(M) \).

From Theorem 2.1 we obtain:

Proposition 3.1. Let \( M \) be a \( n \)-dimensional submanifold of codimension \( r \), isometrically immersed in a golden Riemannian manifold \((M, \tilde{g}, \tilde{P})\) and let \((P, g, u_\alpha, \varepsilon\xi_\alpha, (a_{\alpha\beta})_r)\) be the induced structure on \( M \) by structure \((\tilde{g}, \tilde{P})\). A necessary and sufficient condition for \( M \) to be invariant is that the induced structure \((P, g)\) on \( M \) is a golden Riemannian structure, whenever \( P \) is non-trivial.

Proposition 3.2. Let \( M \) be an \( n \)-dimensional invariant submanifold of codimension \( r \), isometrically immersed in a golden Riemannian manifold \((M, \tilde{g}, \tilde{P})\) and let \((P, g, u_\alpha, \varepsilon\xi_\alpha, (a_{\alpha\beta})_r)\) be the induced structure on \( M \) by structure \((\tilde{g}, \tilde{P})\). Then

\[
(3.7) \quad \begin{cases} 
(i) \quad (\nabla X P)(Y) = \mathcal{P}(X, Y)^T, \\
(ii) \quad \tilde{g}(P(X, Y), N_\alpha) - h_\alpha(X, PY) + \sum_\beta a_{\alpha\beta}h_\beta(X, Y) = 0 \\
(iii) \quad \mathcal{P}(X, N_\alpha)^T - \varepsilon\mathcal{P}(A\alpha X) + \varepsilon\sum_\beta a_{\alpha\beta}A\beta X = 0, \\
(iv) \quad X(a_{\alpha\beta}) = \tilde{g}(P(X, N_\alpha), N_\beta) + \sum_\gamma [l_{\alpha\gamma}(X)a_{\gamma\beta} + l_{\beta\gamma}(X)a_{\alpha\gamma}] 
\end{cases}
\]

for any \( X, Y \in \chi(M) \).
Proposition 3.3. If \( M \) is a \( n \)-dimensional invariant submanifold of codimension \( r \) in a golden Riemannian manifold \((\tilde{M}, \tilde{g}, \tilde{P})\) with \( \tilde{\nabla} \tilde{P} = 0 \) and \((P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)\) is the induced structure on \( M \) by \((\tilde{P}, \tilde{g})\) (where \( \tilde{\nabla} \) is the Levi-Civita connection defined on \( M \) with respect to \( g \)) then, the Nijenhuis torsion tensor field of \( P \) vanishes identically on \( M \) (i.e. \( N_P(X, Y) = 0 \), for any \( X, Y \in \chi(M) \)).

Proposition 3.4. Let \( M \) be a \( n \)-dimensional invariant submanifold of codimension \( r \) in a golden Riemannian manifold \((\tilde{M}, \tilde{g}, \tilde{P})\) with \( \tilde{\nabla} \tilde{P} = 0 \) and let \((P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)\) be the induced structure on \( M \). If the normal connection \( \nabla^\perp \) on the normal bundle \( T^\perp M \) vanishes identically \((l_{\alpha\beta} = 0)\), then the components \( N^{(1)}, N^{(2)}, N^{(3)} \) and \( N^{(4)} \) of the Nijenhuis torsion tensor field of \( P \) for the structure \((P, g, \xi_\alpha, u_\alpha, (a_{\alpha\beta})_r)\) induced on \( M \) have the forms:

\[
\begin{align*}
(i) \quad N^{(1)}(X, Y) &= N^{(4)}_{\alpha\beta}(X) = 0, \\
(ii) \quad N^{(2)}_{\alpha}(X, Y) &= -\sum_{\beta} a_{\alpha\beta} g((PA_\beta - A_\beta P)(X), Y) \\
(iii) \quad N^{(3)}_{\alpha}(X) &= \sum_{\beta} a_{\alpha\beta} (PA_\beta - A_\beta P)(X) - P(\alpha PA_\alpha - A_\alpha P)(X),
\end{align*}
\]

for any \( X, Y \in \chi(M) \).

Remark 3.2. In conditions of the last proposition, if \( PA_\alpha = A_\alpha P \) for every \( \alpha \in \{1, 2, \ldots, r\} \), then the components \( N^{(1)}, N^{(2)}, N^{(3)} \) and \( N^{(4)} \) vanishes identically on \( M \) (i.e. \( N^{(1)} = N^{(2)} = N^{(3)} = N^{(4)} = 0 \)).

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Ștefan cel Mare University, Suceava, ROMÂNIA

"Al.I. Cuza" University, Faculty of Mathematics, Iași, ROMÂNIA