ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I. CUZA" DIN IAŞI (S.N.)
MATEMATICĂ, Tomul LIII, 2007, Supliment

ON SOME INVARIANT SUBMANIFOLDS IN A RIEMANNIAN MANIFOLD WITH GOLDEN STRUCTURE

BY

C.-E. HREŢCANU and M. CRÂŞMAREANU*

Dedicated to Academician Radu Miron at his 80th anniversary

Abstract. The purpose of this paper is to study invariant submanifolds of a Riemannian manifold endowed with a golden structure. An m-dimensional Riemannian manifold $(\widetilde{M}, \widetilde{g}, \widetilde{P})$ is called a golden Riemannian manifold if the (1,1)-tensor field $\widetilde{P}$ on $\widetilde{M}$ is a golden structure (i.e. $\widetilde{P}^2 = \widetilde{P} + Id$) and $\widetilde{g}(\widetilde{P}U, V) = \widetilde{g}(U, \widetilde{P}V)$ for every tangent vector fields $U, V \in \chi(\widetilde{M})$.

Mathematics Subject Classification 2000: 53B25, 53C25, 53C15.
Key words: Riemannian manifold, Golden structures.

1. Introduction. The idea to construct a new structure on a Riemannian manifold, named by us the golden structure, is based on several results regarding geometrical structures constructed on Riemannian manifolds ([1],[8],[10]). Goldberg and Yano ([5]) introduced the notion of polynomial structures on a manifold. Our structure was inspired by the Golden Ratio, which was described by Johannes Kepler $(1571 - 1630)$ as one of the "two great treasures of geometry" (the other one is the Theorem of Pythagoras). The Golden Ratio arises as a result of the solution regarding the division problem of the line segment AB with a point C (which belongs to the segment AB) in the ratio $\frac{AC}{CB} = \frac{AB}{AC}$. The first known written definition of the Golden Ratio is given by Euclid of Alexandria (around 300 BC): "A straight line is said to have been cut in extreme and mean ratio

*Partially supported by the CEx Grant 05-D11-84

when, as the whole line is to the greater segment, so is the greater to the lesser" ([4]).

In this paper, we study the properties of the golden structure on a m-dimensional Riemannian manifold $(\widetilde{M}, \widetilde{g})$, defined as a polynomial structure ([5]) with the structure polynomial $f(x) = x^2 - x - Id = 0$. This structure is determined by an (1,1) tensor field $\widetilde{P}$ on a m-dimensional Riemannian manifold $(\widetilde{M}, \widetilde{g})$ which verifies a similar equation to that satisfied by the Golden Ratio (i.e. $x^2 = x + 1$). In Section 2 we establish several properties of the induced structure on a submanifold in a Golden Riemannian manifold and in Section 3 we find some properties of the induced structure on an invariant submanifold in a golden Riemannian manifold.

We called, in [6], a golden structure on a m-dimensional Riemannian manifold $(\widetilde{M}, \widetilde{g})$ an (1,1)-tensor field $\widetilde{P}$ which satisfies the equation:

$$\widetilde{P}^2 = \widetilde{P} + Id \tag{1.1}$$

where Id is the identity on the Lie algebra of vector fields on $\widetilde{M}$, $\chi(\widetilde{M})$.

A Riemannian manifold $(\widetilde{M}, \widetilde{g})$, endowed with a golden structure $\widetilde{P}$ such that

$$\widetilde{g}(\widetilde{P}U, V) = \widetilde{g}(U, \widetilde{P}V), \tag{1.2}$$

we say that the metric $\widetilde{g}$ is $\widetilde{P}$-compatible and $(\widetilde{P}, \widetilde{g})$ is named a golden Riemannian structure.

2. Properties of induced structures on submanifolds in golden Riemannian manifolds. Let M be a n-dimensional submanifold of codimension r, immersed in a Riemannian manifold $(\widetilde{M}, \widetilde{g}, \widetilde{P})$, with a Riemannian metric $\widetilde{g}$ and a golden structure $\widetilde{P}$ such that the metric $\widetilde{g}$ is $\widetilde{P}$-compatible (the compatibility is provided by the relation (1.2)). We denote by T_xM the tangent space of M in $x \in M$ and by $T_x^{\perp}M$ the normal space of M in x, for every $x \in M$. Let i_* the differential of immersion $i : M \to \widetilde{M}$. The induced Riemannian metric g of M is given by

$$g(X, Y) = \widetilde{g}(i_*X, i_*Y), \tag{2.1}$$

for all $X, Y \in \chi(M)$. We consider a local orthonormal basis $\{N_1, ..., N_r\}$ of the normal space $T_x(M)^{\perp}$ at every point $x \in M$. We suppose that the range indices α, β, γ is in $1, 2, ..., r$ and $i, j, k \in \{1, ..., n\}$.

For any $X \in T_xM$, $\widetilde{P}i_*X$ and $\widetilde{P}N_\alpha$ can be decomposed in tangential and normal components at M in the forms:

$$\widetilde{P}i_*X = i_*PX + \sum_\alpha u_\alpha(X)N_\alpha,\ (\forall)X \in \chi(M) \tag{2.2}$$

and

$$\widetilde{P}N_\alpha = \varepsilon i_*\xi_\alpha + \sum_\beta a_{\alpha\beta}N_\beta, \quad (\varepsilon = \pm 1) \tag{2.3}$$

where P is an (1,1)-tensor field, ξ_α are tangent vector fields on submanifold M, u_α are 1-forms on M and $a := (a_{\alpha\beta})_r$ is a $r \times r$ matrix of real functions on M. Thus, we obtain a structure $(P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ induced on M by $(\widetilde{P}, \widetilde{g})$. The Gauss and Weingarten formulae are:

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sum_{\alpha=1}^{r} h_\alpha(X,Y)N_\alpha, \quad \widetilde{\nabla}_X N_\alpha = -A_\alpha X + \nabla^\perp_X N_\alpha, \tag{2.4}$$

respectively, where $h_\alpha(X,Y) = g(A_\alpha X, Y)$, for every $X, Y \in \chi(M)$.

If $\{N_1, ..., N_r\}$ and $\{N'_1, ..., N'_r\}$ are two local orthonormal basis on a normal space $T^\perp_x M$ then, the decomposition of N'_α in the basis $\{N_1, ..., N_r\}$ is the following

$$N'_\alpha = \sum_{\gamma=1}^{r} k^\gamma_\alpha N_\gamma, \tag{2.5}$$

for any $\alpha \in \{1, ..., r\}$, where (k^γ_α) is an $r \times r$ orthogonal matrix and we have (from [2]): $u'_\alpha = \sum_\gamma k^\gamma_\alpha u_\gamma$, $\xi'_\alpha = \sum_\gamma k^\gamma_\alpha \xi_\gamma$ and $a'_{\alpha\beta} = \sum_\gamma k^\gamma_\alpha a_{\gamma\delta} k^\delta_\beta$. Thus, if $\xi_1, ..., \xi_r$ are linearly independent vector fields, then $\xi'_1, ..., \xi'_r$ are also linearly independent. Furthermore, because $a_{\alpha\beta}$ is symmetric in α and β, under a suitable transformation, we can find that $a_{\alpha\beta}$ can be reduced to $a'_{\alpha\beta} = \lambda_\alpha \delta_{\alpha\beta}$, where λ_α ($\alpha \in \{1, 2, ..., r\}$) are eigenvalues of the matrix $(a_{\alpha\beta})_r$ and in this case we have: $u'_\beta(\xi_\alpha) = \varepsilon\delta_{\alpha\beta}(1 + \lambda_\alpha - \lambda_\alpha\lambda_\beta)$ and from this we obtain $u'_\alpha(\xi_\alpha) = \varepsilon(1 + \lambda_\alpha - \lambda^2_\alpha)$.

Proposition 2.1. *If we suppose that $\xi_1, ..., \xi_r$ are linearly independent tangent vector fields on M, it follows that the 1-forms $u_1, ..., u_r$ are also linearly independent.*

Proof. The equality $\sum_{\alpha=1}^{r}\mu^{\alpha}u_{\alpha}(X)=0$ is equivalent with

$$0=\sum_{\alpha}\mu^{\alpha}g(X,\xi_{\alpha})=g(X,\sum_{\alpha}\mu^{\alpha}\xi_{\alpha}),\ (\forall)X\in\chi(M)$$

thus, we have $\sum_{\alpha=1}^{r}\mu^{\alpha}\xi_{\alpha}=0\Rightarrow\mu^{\alpha}=0$ so, $u_1,...,u_r$ are linearly independent on M.

Proposition 2.2. *If M is a n-dimensional submanifold of codimension r, isometrically immersed in a golden Riemannian manifold $(\widetilde{M},\widetilde{g},\widetilde{P})$, from* (1.4) *and* (1.5) *we obtain that the elements $P,g,u_{\alpha},\varepsilon\xi_{\alpha},(a_{\alpha\beta})_r$ of the induced structure on M by the golden structure $\widetilde{P}$, verify the following equalities:*

$$P^2X=PX+X-\varepsilon\sum_{\alpha}u_{\alpha}(X)\xi_{\alpha}, \tag{2.6}$$

$$u_{\alpha}(PX)=(1-a_{\alpha\alpha})u_{\alpha}(X)=u_{\alpha}(X)-a_{\alpha\beta}(X), \tag{2.7}$$

$$a_{\alpha\beta}=a_{\beta\alpha}, \tag{2.8}$$

$$u_{\beta}(\xi_{\alpha})=\varepsilon(\delta_{\alpha\beta}+a_{\alpha\beta}-\sum_{\gamma}a_{\alpha\gamma}a_{\gamma\beta}), \tag{2.9}$$

$$P\xi_{\alpha}=\xi_{\alpha}-\sum_{\beta}a_{\alpha\beta}\xi_{\beta}, \tag{2.10}$$

and the connections between the (1,1) tensor field P on M and the induced metric g on the submanifold M are as follows:

$$u_{\alpha}(X)=\varepsilon g(X,\xi_{\alpha}), \tag{2.11}$$

$$g(PX,Y)=g(X,PY), \tag{2.12}$$

and

$$g(PX,PY)=g(PX,Y)+g(X,Y)-\sum_{\alpha}u_{\alpha}(X)u_{\alpha}(Y), \tag{2.13}$$

for any $X,Y\in\chi(M)$.

Remark 2.1. If M is a non-invariant n-dimensional submanifold of codimension r, immersed in a golden Riemannian manifold $(\widetilde{M}, \widetilde{g}, \widetilde{P})$ so that the tangent vector fields $\xi_1, \xi_2, ..., \xi_r$ are linearly independent then, from (2.9) and (2.11) we obtain:

$$\|\xi_\alpha\|^2 = 1 + a_{\alpha\alpha} - \sum_\gamma a_{\alpha\gamma}^2 \tag{2.14}$$

and, for $\alpha \neq \beta$ we have

$$\sum_\gamma a_{\alpha\gamma} a_{\gamma\beta} = a_{\alpha\beta}. \tag{2.15}$$

For the normal connection $\nabla_X^\perp N_\alpha$, we have the decomposition

$$\nabla_X^\perp N_\alpha = \sum_{\beta=1}^r l_{\alpha\beta}(X) N_\beta, \tag{2.16}$$

for every $X \in \chi(M)$. Therefore, we obtain an $r \times r$ matrix $(l_{\alpha\beta}(X))_r$ of 1-forms on M. From $\widetilde{g}(N_\alpha, N_\beta) = \delta_{\alpha\beta}$ we get $\widetilde{g}(\nabla_X^\perp N_\alpha, N_\beta) + \widetilde{g}(N_\alpha, \nabla_X^\perp N_\beta) = 0$ which is equivalent with $\widetilde{g}(\sum_\gamma l_{\alpha\gamma}(X) N_\gamma, N_\beta) + \widetilde{g}(N_\alpha, \sum_\gamma l_{\beta\gamma}(X) N_\gamma) = 0$, for any $X \in \chi(M)$. Thus, we obtain

$$l_{\alpha\beta} = -l_{\beta\alpha}, \tag{2.17}$$

for any $\alpha, \beta \in \{1, ..., r\}$.

Let $N_{\widetilde{P}}(X, Y)$ be the Nijenhuis torsion tensor field of $\widetilde{P}$, defined by ([7]):

$$N_{\widetilde{P}}(X, Y) = [\widetilde{P}X, \widetilde{P}Y] + \widetilde{P}^2[X, Y] - \widetilde{P}[\widetilde{P}X, Y] - \widetilde{P}[X, \widetilde{P}Y]. \tag{2.18}$$

Remark 2.2. If (M, g) is a Riemannian manifold endowed with an (1,1) tensor field P, then the Nijenhuis tensor of P verifies that

$$\begin{aligned} N_{\widetilde{P}}(X, Y) &= (\widetilde{\nabla}_{\widetilde{P}X} \widetilde{P})(Y) - (\widetilde{\nabla}_{\widetilde{P}Y} \widetilde{P})(X) \\ &\quad - \widetilde{P}[(\widetilde{\nabla}_X \widetilde{P})(Y) - (\widetilde{\nabla}_Y \widetilde{P})(X)], \end{aligned} \tag{2.19}$$

for any $X, Y \in \chi(\widetilde{M})$, where $\widetilde{\nabla}$ is the Levi-Civita connection on $\widetilde{M}$.

Remark 2.3. If we suppose that $(\widetilde{M}, \widetilde{g}, \widetilde{P})$ is a locally product Riemannian manifold (i.e. $\widetilde{P}$ is parallel with respect to the Levi-Civita connection $\widetilde{\nabla}$ of $\widetilde{g}$, $\widetilde{\nabla}\widetilde{P} = 0$), from (2.19) we obtain the integrability of the structure $\widetilde{P}$ (which is equivalent with the vanishing of the Nijenhuis torsion tensor field of $\widetilde{P}$).

We denoted by $\mathcal{P}$ the (1,2)-tensor field on $\widetilde{M}$, such that

$$\mathcal{P}(\widetilde{X}, \widetilde{Y}) = (\widetilde{\nabla}_{\widetilde{X}}\widetilde{P})(\widetilde{Y}) := \widetilde{\nabla}_{\widetilde{X}}(\widetilde{P}\widetilde{Y}) - \widetilde{P}(\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y}), \tag{2.20}$$

for any $\widetilde{X}, \widetilde{Y} \in \chi(\widetilde{M})$.

Let $\mathcal{P}(\widetilde{X}, \widetilde{Y})^{\top}$ and $\mathcal{P}(\widetilde{X}, \widetilde{Y})^{\perp}$ be tangential and respectively normal components on M of $\mathcal{P}(\widetilde{X}, \widetilde{Y})$, for any $\widetilde{X}, \widetilde{Y} \in \chi(\widetilde{M})$.

Theorem 2.1. *If M is an n-dimensional submanifold of codimension r in a golden Riemannian manifold $(\widetilde{M}, \widetilde{g}, \widetilde{P})$, then the structure $(P, g, u_{\alpha}, \varepsilon\xi_{\alpha}, (a_{\alpha\beta})_r)$ induced on M by the structure $\widetilde{P}$ has the following properties:*

$$\begin{cases} (i)(\nabla_X P)(Y) = \mathcal{P}(X,Y)^{\top} + \varepsilon \sum_{\alpha} h_{\alpha}(X,Y)\xi_{\alpha} + \sum_{\alpha} u_{\alpha}(Y) A_{\alpha}X, \\ (ii)(\nabla_X u_{\alpha})(Y) = \widetilde{g}(\mathcal{P}(X,Y), N_{\alpha}) - h_{\alpha}(X, PY) \\ \qquad + \sum_{\beta} (u_{\beta}(Y) l_{\alpha\beta}(X) + h_{\beta}(X,Y) a_{\beta\alpha}) \\ (iii)\nabla_X \xi_{\alpha} = \mathcal{P}(X, N_{\alpha})^{\top} - \varepsilon P(A_{\alpha}X) + \varepsilon \sum_{\beta} a_{\alpha\beta} A_{\beta} X \\ \qquad + \sum_{\beta} l_{\alpha\beta}(X)\xi_{\beta}, \\ (iv) X(a_{\alpha\beta}) = \widetilde{g}(\mathcal{P}(X, N_{\alpha}), N_{\beta}) - \varepsilon u_{\alpha}(A_{\beta}X) - u_{\beta}(A_{\alpha}X) \\ \qquad + \sum_{\gamma} [l_{\alpha\gamma}(X) a_{\gamma\beta} + l_{\beta\gamma}(X) a_{\alpha\gamma}] \end{cases} \tag{2.21}$$

for any $X, Y \in \chi(M)$.

Proof. From $\widetilde{\nabla}_X(\widetilde{P}Y) = \nabla_X PY - \sum_{\alpha} u_{\alpha}(Y) A_{\alpha} X + \sum_{\alpha}[h_{\alpha}(X, PY) + X(u_{\alpha}(Y)) + \sum_{\beta} u_{\beta}(Y) l_{\beta\alpha}(X)] N_{\alpha}$ and

$$\begin{aligned} \widetilde{P}(\widetilde{\nabla}_X Y) &= P(\nabla_X Y) + \varepsilon \sum_{\alpha} h_{\alpha}(X,Y)\xi_{\alpha} \\ &\quad + \sum_{\alpha} [u_{\alpha}(\nabla_X Y) + \sum_{\beta} h_{\beta}(X,Y) a_{\beta\alpha}] N_{\alpha} \end{aligned}$$

we obtain

$$\begin{aligned}\mathcal{P}(X,Y) &= (\nabla_X P)(Y) - \sum_\alpha u_\alpha(Y) A_\alpha X - \varepsilon \sum_\alpha h_\alpha(X,Y)\xi_\alpha \\ &+ \sum_\alpha [h_\alpha(X,PY) + (\nabla_X u_\alpha)(Y) \\ &+ \sum_\beta u_\beta(Y) l_{\beta\alpha}(X) - \sum_\beta h_\beta(X,Y) a_{\beta\alpha}] N_\alpha.\end{aligned}$$

Thus, identifying the tangential part and respectively the normal part in the last equality, we obtain (i) and (ii) from (2.21).

From

$$\begin{aligned}\widetilde{\nabla}_X(\widetilde{P}N_\alpha) &= \varepsilon \nabla_X \xi_\alpha - \sum_\beta a_{\alpha\beta} A_\beta X + \sum_\beta [X(a_{\alpha\beta}) + \varepsilon h_\beta(X,\xi_\alpha) \\ &+ \sum_\gamma a_{\alpha\gamma} \cdot l_{\gamma\beta}(X)] N_\beta\end{aligned}$$

and

$$\begin{aligned}\widetilde{P}(\widetilde{\nabla}_X N_\alpha) &= -P(A_\alpha X) + \varepsilon \sum_\beta l_{\alpha\beta}(X)\xi_\beta - \sum_\beta [u_\beta(A_\alpha X) \\ &- \sum_\gamma a_{\gamma\beta} l_{\alpha\gamma}(X)] N_\beta\end{aligned}$$

we obtain

$$\begin{aligned}&\mathcal{P}(X,N_\alpha) = \varepsilon \nabla_X \xi_\alpha + P(A_\alpha X) - \varepsilon \sum_\beta l_{\alpha\beta}(X)\xi_\beta - \sum_\beta a_{\alpha\beta} A_\beta X \\ &+ \sum_\beta [X(a_{\alpha\beta}) + \varepsilon h_\beta(X,\xi_\alpha) + u_\beta(A_\alpha X) \\ &- \sum_\gamma (a_{\gamma\beta} l_{\alpha\gamma}(X) - a_{\alpha\gamma} \cdot l_{\gamma\beta}(X))] N_\beta.\end{aligned}$$

Thus, identifying the tangential part and respectively the normal part in the last equality, we obtain (iii) and (iv) from (2.21). □

We can find, in a similar way like in [7], the following property:

Theorem 2.2. *Let M be a n-dimensional submanifold of codimension r in a golden Riemannian manifold $(\widetilde{M},\widetilde{g},\widetilde{P})$ with $\widetilde{\nabla}\widetilde{P}=0$. If $(P,g,u_\alpha,\xi_\alpha,(a_{\alpha\beta})_r)$ is the induced structure on M by $(\widetilde{P},\widetilde{g})$ and ∇ is the Levi-Civita connection defined on M with respect to g then, the Nijenhuis torsion tensor field of P has the form:*

$$\begin{aligned} N_P(X,Y) &= -\sum_\alpha g((PA_\alpha - A_\alpha P)(X),Y)\xi_\alpha \\ &-\sum_\alpha g(Y,\xi_\alpha)(PA_\alpha - A_\alpha P)(X) \\ &+\sum_\alpha g(X,\xi_\alpha)(PA_\alpha - A_\alpha P)(Y) \end{aligned} \tag{2.22}$$

for any $X,Y\in\chi(M)$.

Corollary 2.1. *Let M be a n-dimensional submanifold of codimension r in a golden Riemannian manifold $(\widetilde{M},\widetilde{g},\widetilde{P})$ and let $(P,g,u_\alpha,\xi_\alpha,(a_{\alpha\beta})_r)$ be the induced structure on M by $(\widetilde{P},\widetilde{g})$. If $\widetilde{\nabla}\widetilde{P}=0$ and the (1,1) tensor field P on M commutes with the Weingarten operators A_α (that is $PA_\alpha=A_\alpha P$, for any $\alpha\in\{1,...,r\}$) then, the Nijenhuis torsion tensor field of P vanishes on M (that is $N_P(X,Y)=0$, for any $X,Y\in\chi(M)$).*

Remark 2.4. Using the model for an almost paracontact structure ([9]), we can compute the components $N^{(1)}$, $N^{(2)}$, $N^{(3)}$ and $N^{(4)}$ of the Nijenhuis torsion tensor field of P for the $(P,g,\xi_\alpha,u_\alpha,(a_{\alpha\beta})_r)$ induced structure on a n-dimensional submanifold M of codimension r in a golden Riemannian manifold $(\widetilde{M},\widetilde{g},\widetilde{P})$:

$$\begin{cases} (i) \quad N^{(1)}(X,Y) = N_P(X,Y) - 2\sum_{\alpha=1}^{r} du_\alpha(X,Y)\xi_\alpha, \\ (ii) \ N^{(2)}_\alpha(X,Y) = (\mathcal{L}_{PX}u_\alpha)Y - (\mathcal{L}_{PY}u_\alpha)X, \\ (iii) \ N^{(3)}_\alpha(X) = (\mathcal{L}_{\xi_\alpha}P)X, \\ (iv) \ N^{(4)}_{\alpha\beta}(X) = (\mathcal{L}_{\xi_\alpha}u_\beta)X, \end{cases} \tag{2.23}$$

for any $X,Y\in\chi(M)$ and $\alpha,\beta\in\{1,...,r\}$, where N_P is the Nijenhuis torsion tensor field of P and $\mathcal{L}_X$ means the Lie derivative with respect to X.

Proposition 2.3. *Let M be a n-dimensional submanifold of codimension r in a golden Riemannian manifold $(\widetilde{M},\widetilde{g},\widetilde{P})$ with $\widetilde{\nabla}\widetilde{P}=0$ and let*

$(P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ be the induced structure on M. If the normal connection $\nabla^\perp$ on the normal bundle $T^\perp M$ vanishes identically ($\nabla^\perp = 0 \Longleftrightarrow l_{\alpha\beta} = 0$), then the components $N^{(1)}$, $N^{(2)}$, $N^{(3)}$ and $N^{(4)}$ of the Nijenhuis torsion tensor field of P for the structure $(P, g, \xi_\alpha, u_\alpha, (a_{\alpha\beta})_r)$ induced on M have the forms:

$$(2.24)\quad \begin{cases} (i)\ N^{(1)}(X,Y) = \sum_\alpha g(X,\xi_\alpha)(PA_\alpha - A_\alpha P)(Y) \\ -\sum_\alpha g(Y,\xi_\alpha)(PA_\alpha - A_\alpha P)(X), \\ (ii)\ N^{(2)}_\alpha(X,Y) = -\sum_\beta a_{\alpha\beta} g((PA_\beta - A_\beta P)(X), Y) \\ +\sum_\beta a_{\alpha\beta} u_\beta([X,Y]) + \sum_\beta [u_\beta(X) u_\alpha(A_\beta Y) - u_\beta(Y) u_\alpha(A_\beta X)], \\ (iii)\ N^{(3)}_\alpha(X) = \sum_\beta a_{\alpha\beta}(PA_\beta - A_\beta P)(X) \\ -P(PA_\alpha - A_\alpha P)(X) + \sum_\beta [u_\alpha(A_\beta X)\xi_\beta + u_\beta(X) A_\beta \xi_\alpha], \\ (iv)\ N^{(4)}_{\alpha\beta}(X) = -u_\alpha(A_\beta P X) - u_\beta(PA_\alpha X) \\ +\sum_\gamma [a_{\alpha\gamma} u_\beta(A_\gamma X) + a_{\gamma\beta} u_\alpha(A_\gamma X)] \end{cases}$$

for any $X, Y \in \chi(M)$.

Corollary 2.2. *Under the assumptions of the last proposition, if P and the Weingarten operators A_α commute (i.e. $PA_\alpha = A_\alpha P, \alpha \in \{1, ..., r\}$) then we obtain*

$$(2.25)\quad \begin{cases} (i)\ N^{(1)}(X,Y) = 0, \\ (ii) N^{(2)}_\alpha(X,Y) = \sum_\beta (a_{\alpha\beta} u_\beta([X,Y]) + u_\beta(X) u_\alpha(A_\beta Y) \\ -u_\beta(Y) u_\alpha(A_\beta X)), \\ (iii) N^{(3)}_\alpha(X) = \sum_\beta [u_\alpha(A_\beta X)\xi_\beta + u_\beta(X) A_\beta \xi_\alpha], \\ (iv)\ N^{(4)}_{\alpha\alpha}(X) = 2\sum_\gamma a_{\alpha\gamma} u_\alpha(A_\gamma X) - 2u_\alpha(PA_\alpha X) \end{cases}$$

We denote by

$$(2.26)\qquad D_x = \{X_x \in T_x M\ :\ u_\alpha(X_x) = 0\},\ \text{for any}\ \alpha \in \{1, ..., r\}.$$

If $\xi_1, ..., \xi_r$ are linearly independent, we remark that D_x is an $(n-r)$-dimensional subspace in T_xM and the function $D : x \mapsto D_x$, $(\forall)x \in M$ is a distribution locally defined on M. If $X \in D$, we have that $u_\alpha(PX) = 0$ for any $X \in D$, then $PX \in D$. Therefore D is an invariant distribution with respect to P.

If $D_x^\perp$ is the orthogonal supplement of D_x in T_xM, then we obtain the distribution $D^\perp : x \mapsto D_x^\perp$. Furthermore, we have the decomposition of $T_xM = D_x \oplus D_x^\perp$, the vector fields $\xi_\alpha \neq 0$ are orthogonal on D_x and $\xi_\alpha \in D_x^\perp$. Thus, if $\xi_\alpha \neq 0$ for any $\alpha \in \{1, ..., r\}$, then $D_x^\perp$ is generated by $\xi_1, ..., \xi_r$ and $D_x^\perp$ is r-dimensional in T_xM. We remark that the space D_x is P-invariant and P is a golden Riemannian structure on D and its eigenvalues are $\frac{1\pm\sqrt{5}}{2}$.

3. Properties of invariant submanifolds in golden Riemannian manifolds

Definition 3.1. ([3])*If on M there exist two complementary and orthogonal distributions D and $D^\perp$, satisfying the conditions:*

$$\widetilde{P}(D_x) = D_x; \quad \widetilde{P}(D_x^\perp) \subset T_x(M)^\perp \tag{3.1}$$

for each $x \in M$, then M is called a semi-invariant submanifod of the locally product Riemannian manifold $\widetilde{M}$.

Particulary, we have ([2]):
(i) if $dimD_x = dimT_x(M)$ for each $x \in M$ then M is an invariant submanifold of $\widetilde{M}$; in this case we have that

$$\widetilde{P}(T_x(M)) \subset T_x(M) \tag{3.2}$$

and

$$\widetilde{P}(T_x(M)^\perp) \subset T_x(M)^\perp \tag{3.3}$$

for every $x \in M$.
(ii) if $dimD_x = 0$, for each $x \in M$ then M is an anti-invariant submanifold of $\widetilde{M}$; in this case we have that

$$\widetilde{P}(T_x(M)) \subset T_x(M) \tag{3.4}$$

for every $x \in M$.

Remark 3.1. If M is a n-dimensional invariant submanifold of codimension r, immersed in a golden Riemannian manifold $(\widetilde{M}, \widetilde{g}, \widetilde{P})$, then ξ_α $(\alpha \in \{1, 2, ..., r\})$ are zero vector fields and 1-forms u_α vanishes identically on M (that is $u_\alpha(X) = g(X, \xi_\alpha) = 0$). Consequently, (2.2) and (2.3) are respectively written as follows:

$$\widetilde{P} i_* X = i_* P X, \quad \widetilde{P} N_\alpha = \sum_\beta a_{\alpha\beta} N_\beta, \ (\forall) X \in \chi(M), \alpha \in \{1, 2, ..., r\} \tag{3.5}$$

Thus, from Proposition 2.2, we obtain that the element of the structure $(P, g, u_\alpha, \varepsilon\xi_\alpha, (a_{\alpha\beta})_r)$, induced on M by structure $(\widetilde{g}, \widetilde{P})$, verifies these equalities:

$$\begin{cases} (i) \quad P^2 X = PX + X, \\ (ii) \quad a_{\alpha\beta} = a_{\beta\alpha}, \\ (iii) \sum_\gamma a_{\alpha\gamma} a_{\gamma\beta} = a_{\alpha\beta} + \delta_{\alpha\beta}, \\ (iv) \quad g(PX, Y) = g(X, PY), \\ (v) \quad g(PX, PY) = g(PX, Y) + g(X, Y), \end{cases} \tag{3.6}$$

for every $X, Y \in \chi(M)$ and $\alpha, \beta \in \{1, 2, ..., r\}$.

Proposition 3.1. *Let M be a n-dimensional submanifold of codimension r, isometrically immersed in a golden Riemannian manifold $(\widetilde{M}, \widetilde{g}, \widetilde{P})$ and let $(P, g, u_\alpha, \varepsilon\xi_\alpha, (a_{\alpha\beta})_r)$ be the induced structure on M by structure $(\widetilde{g}, \widetilde{P})$. A necessary and sufficient condition for M to be invariant is that the induced structure (P, g) on M is a golden Riemannian structure, whenever P is non-trivial.*

From Theorem 2.1 we obtain:

Proposition 3.2. *Let M be an n-dimensional invariant submanifold of codimension r, isometrically immersed in a golden Riemannian manifold $(\widetilde{M}, \widetilde{g}, \widetilde{P})$ and let $(P, g, u_\alpha, \varepsilon\xi_\alpha, (a_{\alpha\beta})_r)$ be the induced structure on M by structure $(\widetilde{g}, \widetilde{P})$. Then*

$$\begin{cases} (i) \ (\nabla_X P)(Y) = \mathcal{P}(X, Y)^\top, \\ (ii) \ \widetilde{g}(\mathcal{P}(X, Y), N_\alpha) - h_\alpha(X, PY) + \sum_\beta a_{\alpha\beta} h_\beta(X, Y) = 0 \\ (iii) \ \mathcal{P}(X, N_\alpha)^\top - \varepsilon P(A_\alpha X) + \varepsilon \sum_\beta a_{\alpha\beta} A_\beta X = 0, \\ (iv) \ X(a_{\alpha\beta}) = \widetilde{g}(\mathcal{P}(X, N_\alpha), N_\beta) + \sum_\gamma [l_{\alpha\gamma}(X) a_{\gamma\beta} + l_{\beta\gamma}(X) a_{\alpha\gamma}] \end{cases} \tag{3.7}$$

for any $X, Y \in \chi(M)$.

Proposition 3.3. *If M is a n-dimensional invariant submanifold of codimension r in a golden Riemannian manifold $(\widetilde{M}, \widetilde{g}, \widetilde{P})$ with $\widetilde{\nabla}\widetilde{P} = 0$ and $(P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ is the induced structure on M by $(\widetilde{P}, \widetilde{g})$ (where ∇ is the Levi-Civita connection defined on M with respect to g) then, the Nijenhuis torsion tensor field of P vanishes identically on M (i.e. $N_P(X, Y) = 0$, for any $X, Y \in \chi(M)$).*

Proposition 3.4. *Let M be a n-dimensional invariant submanifold of codimension r in a golden Riemannian manifold $(\widetilde{M}, \widetilde{g}, \widetilde{P})$ with $\widetilde{\nabla}\widetilde{P} = 0$ and let $(P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ be the induced structure on M. If the normal connection $\nabla^\perp$ on the normal bundle $T^\perp M$ vanishes identically ($l_{\alpha\beta} = 0$), then the components $N^{(1)}$, $N^{(2)}$, $N^{(3)}$ and $N^{(4)}$ of the Nijenhuis torsion tensor field of P for the structure $(P, g, \xi_\alpha, u_\alpha, (a_{\alpha\beta})_r)$ induced on M have the forms:*

$$(3.8) \begin{cases} (i)\ N^{(1)}(X, Y) = N^{(4)}_{\alpha\beta}(X) = 0, \\ (ii)\ N^{(2)}_\alpha(X, Y) = -\sum_\beta a_{\alpha\beta} g((PA_\beta - A_\beta P)(X), Y) \\ (iii)\ N^{(3)}_\alpha(X) = \sum_\beta a_{\alpha\beta}(PA_\beta - A_\beta P)(X) - P(PA_\alpha - A_\alpha P)(X), \end{cases}$$

for any $X, Y \in \chi(M)$.

Remark 3.2. In conditions of the last proposition, if $PA_\alpha = A_\alpha P$ for every $\alpha \in \{1, 2, ..., r\}$, then the components $N^{(1)}$, $N^{(2)}$, $N^{(3)}$ and $N^{(4)}$ vanishes identicaly on M (i.e. $N^{(1)} = N^{(2)} = N^{(3)} = N^{(4)} = 0$).

REFERENCES

1. Anastasiei, M. – *Some Riemannian almost product structures on tangent manifold*, Algebras Groups Geom. 17 (2000), no. 3, 253-262.

2. Adati, T. – *Submanifolds of an almost product Riemannian manifold*, Kodai Math. J., 4-2 (1981), 327–343.

3. Bejancu, A. – *Semi-invariant submanifolds of locally product Riemannian manifolds*, Ann. Univ. Timisoara, s. Math., XXII (1984), 3–11.

4. Ghyka, M. – *The Geometry of Art and Life*, Dover Publications; 2 edition, (1977), 7–19.

5. Goldberg, S.I.; Yano, K. – *Polinomial structures on manifolds*, Kodai Math. Sem. Rep. 22 (1970), 199–218.

6. Hreţcanu, C.E. – *Submanifolds in Riemannian manifold with golden structure*, "Workshop on Finsler Geometry and its Applications" - Hungary, (2007).
7. Hreţcanu, C.E. – *Induced structures on submanifolds in almost product Riemannian manifolds*, arXiv.org, math/0608533, (2006).
8. Ianuş, S. – *Some almost product structures on manifolds with linear connection*, Kodai Math. Sem. Rep. 23 (1971), 305–310.
9. Mihai, I.; Roşca, R.; Verstraelen, L. – *Some aspects of the differential geometry of vector fields*, Katholicke Universiteit Leuven, PADGE, vol 2, (1996).
10. Miron, R. – *Spaces with Higher Order Metric Structures.* Tensor N.S., 53, (1993), 1–23.

Received: 15.X.2007

Ştefan cel Mare University,
Suceava,
ROMÂNIA

"Al.I. Cuza" University,
Faculty of Mathematics,
Iaşi,
ROMÂNIA