$(a,1)f$ structures on product of spheres

Cristina-Elena Hreţcanu
Ştefan cel Mare University, Suceava, România
Mircea Crasmareanu
Al. I. Cuza University, Iaşi, România

Abstract

Our aim in this paper is to give some examples of $(a,1)f$ Riemannian structures (a generalization of an r-paracontact structure) induced on product of spheres of codimension r ($r \in \{1,2\}$) in an m-dimensional Euclidean space ($m > 2$), endowed with an almost product structure.

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Introduction

A classification of smooth structures on product of spheres of the form $S^k \times S^p$, where $2 \leq k \leq p$ and $k + p \geq 6$, was given by R. De Sapio [13]. In [2], S.O.Ajala extended De Sapios result to smooth structures on $S^p \times S^q \times S^r$ where $2 \leq p \leq q \leq r$. Also, a complete classification of smooth structures on a generalized product of spheres was given in [3].

By studying properties of some structures constructed on Riemannian manifolds [1, 4, 7, 11, 12, 14], we obtain a generalization of r-paracontact structure, constructed as an induced structure on a submanifold in an almost product Riemannian manifold.

In this paper we show that, if M is a submanifold of codimension 1, isometrically immersed in $\overline{M}$, and $\overline{M}$ is also of codimension 1 and isometrically immersed in an n-dimensional almost product Riemannian manifold $(\widetilde{M}, \widetilde{g}, \widetilde{P})$ ($n > 2$), so that $(M, g) \hookrightarrow (\overline{M}, \overline{g}) \hookrightarrow (\widetilde{M}, \widetilde{g})$ then, the induced $(a,1)f$ structure on M by the structure $(\widetilde{P}, \widetilde{g})$ from $\widetilde{M}$ is a structure of type $(P, g, u_1, u_2, \xi_1, \xi_2^{\top}, (a_{\alpha\beta}))$, which is the same that one induced on M by the structure $(\overline{P}, \overline{g}, u_2, \xi_2, a_{22})$ induced on $\overline{M}$ by the structure from $\widetilde{M}$.

Finally we give some examples for induced $(a,1)f$ Riemannian structures on product of spheres of codimension r ($r \in \{1,2\}$) in an Euclidean space of dimension $m > 2$ endowed with an almost product structure.

1 Submanifolds in almost product Riemannian manifolds

Let $\widetilde{M}$ be an m-dimensional Riemannian manifold endowed with a pair $(\widetilde{P}, \widetilde{g})$ where $\widetilde{g}$ is a Riemannian metric and $\widetilde{P}$ is an $(1,1)$ tensor field so that $\widetilde{P}^2 = \varepsilon Id$ for $\varepsilon \in \{1, -1\}$. We suppose that $\widetilde{g}$ and $\widetilde{P}$ verify the compatibility condition $\widetilde{g}(\widetilde{P}U, \widetilde{P}V) = \widetilde{g}(U, V)$ for every $U, V \in \chi(\widetilde{M})$ where $\chi(\widetilde{M})$ is the Lie algebra of the vector fields on $\widetilde{M}$. This conditions is equivalent with $\widetilde{g}(\widetilde{P}U, V) = \varepsilon\widetilde{g}(U, \widetilde{P}V)$ for every $U, V \in \chi(\widetilde{M})$.

For $\varepsilon = 1$, we obtain that $(\widetilde{M}, \widetilde{g}, \widetilde{P})$ is an *almost product Riemannian manifold*.

Let M be an n-dimensional submanifold of codimension r $(n, r \in \mathbb{N}^*)$ in an almost product Riemannian manifold $(\widetilde{M}, \widetilde{g}, \widetilde{P})$ and let g be the Riemannian metric induced on M by $\widetilde{g}$.

If $(N_1, ..., N_r) := (N_\alpha)$ is a local orthonormal basis in the normal space of M in x, denoted $T_x^\perp(M)$, for every $x \in M$ (with $\alpha \in \{1, ..., r\}$) then, decompositions of the vector fields $\widetilde{P}X$ and $\widetilde{P}N_\alpha$, respectively, in the tangential and normal components on the submanifold M in $\widetilde{M}$ are as follows:

$$\widetilde{P}X = PX + \sum_{\alpha=1}^{r} u_\alpha(X) N_\alpha, \tag{1.1}$$

and

$$\widetilde{P}N_\alpha = \varepsilon\xi_\alpha + \sum_{\beta=1}^{r} a_{\alpha\beta} N_\beta \tag{1.2}$$

for every $X \in \chi(M)$ and $\alpha \in \{1, ..., r\}$.

We called in [9] an $(a, \varepsilon)f$ Riemannian structure on M, induced by $\widetilde{P}$ from $(\widetilde{M}, \widetilde{g})$, the following data which results from the relations (1.1) and (1.2): $(P, g, \varepsilon\xi_\alpha, u_\alpha, (a_{\alpha\beta})_r)$ where a is a notation for the matrix $(a_{\alpha\beta})_r$ and $f := P$. Here, P is an $(1,1)$-tensor field on M, ξ_α are tangent vector fields on M, u_α are 1-forms on M and $(a_{\alpha\beta})_r$ is a $r \times r$ matrix of real functions on M. Some properties of $(a, \varepsilon)f$ Riemannian structures are given by the first author in [9, 10]. The $(a, 1)f$ Riemannian structure generalizes the Riemannian almost r-paracontact structure [6] obtained from $(a, 1)$f structure for $a = 0$, and it was also considered by T. Adati in [1]. A similar structure induced on M by an almost Hermitian structure on $\widetilde{M}$ was studied by K. Yano and M. Okumura [14].

2 $(a, 1)f$ induced structures on submanifolds in submanifolds of almost product Riemannian manifolds

In the following statements we suppose that $(\widetilde{M}, \widetilde{g}, \widetilde{P})$ is an n-dimensional $(n > 2)$ almost product Riemannian manifold and $(\overline{M}, \overline{g})$ is a submanifold of codimension 1, isometrically immersed in $\widetilde{M}$ (with the induced metric $\overline{g}$ on $\overline{M}$ by $\widetilde{g}$). Let N_1 be an unit vector field, normal on $\overline{M}$ in $\widetilde{M}$. Then, we suppose that (M, g) is a submanifold of codimension 1 isometrically immersed in $\overline{M}$ and let N_2 be an unit vector field, normal on M in $\overline{M}$. Thus, (M, g) is a submanifold of codimension 2 in $(\widetilde{M}, \widetilde{g})$ and we have the following isometric immersions between two Riemannian manifolds: $(M, g) \hookrightarrow (\overline{M}, \overline{g}) \hookrightarrow (\widetilde{M}, \widetilde{g})$ and (N_1, N_2) is a local orthonormal basis in $T_x^{\perp}(M)$ for every $x \in M$.

From the decompositions in tangential and normal components at $\overline{M}$ in $\widetilde{M}$ of vector fields $\widetilde{P}\overline{X}$ $(\overline{X} \in \chi(\overline{M}))$ and $\widetilde{P}N_1$ respectively, we obtain:

$$\widetilde{P}\overline{X} = \overline{P}\,\overline{X} + u_1(\overline{X})N_1, \quad \widetilde{P}N_1 = \xi_1 + a_{11}N_1, \tag{2.1}$$

for any $\overline{X} \in \chi(\overline{M})$ where $\overline{P}$ is an $(1,1)$ tensor field on $\overline{M}$, u_1 is an 1-form on $\overline{M}$, ξ_1 is a tangent vector field on $\overline{M}$ and a_{11} is a real function on $\overline{M}$.

Lemma 2.1. *The almost product Riemannian structure $(\widetilde{P}, \widetilde{g})$ on a manifold $\widetilde{M}$ induces, on any submanifold $\overline{M}$ of codimension 1 in $\widetilde{M}$, an $(\overline{a}, 1)f$ Riemannian structure, which is a $(\overline{P}, \overline{g}, u_1, \xi_1, a_{11})$ Riemannian structure, (with $\overline{a} := a_{11}$ and $f := \overline{P}$), where $\overline{P}$ is an $(1,1)$ tensor field on $\overline{M}$, u_1 is an 1-form on $\overline{M}$, ξ_1 is a tangent vector field on $\overline{M}$ and a_{11} is a real function on $\overline{M}$. This structure has the following properties:*

$$\begin{cases} (i) \quad \overline{P}^2\overline{X} = \overline{X} - u_1(\overline{X})\xi_1, \ (\forall)\overline{X} \in \chi(\overline{M}), \\ (ii) \quad u_1(\overline{P}\,\overline{X}) = -a_{11}u_1(\overline{X}), \ (\forall)\overline{X} \in \chi(\overline{M}), \\ (iii) \quad u_1(\xi_1) = 1 - a_{11}^2, \\ (iv) \quad \overline{P}\xi_1 = -a_{11}\xi_1, \end{cases} \tag{2.2}$$

and

$$\begin{cases} (i) \quad u_1(\overline{X}) = \overline{g}(\overline{X}, \xi_1), \ (\forall)\overline{X} \in \chi(\overline{M}), \\ (ii) \quad \overline{g}(\overline{P}\,\overline{X}, \overline{Y}) = \overline{g}(\overline{X}, \overline{P}\,\overline{Y}), \ (\forall)\overline{X}, \overline{Y} \in \chi(\overline{M}), \\ (iii) \quad \overline{g}(\overline{P}\,\overline{X}, \overline{P}\,\overline{Y}) = \overline{g}(\overline{X}, \overline{Y}), \ (\forall)\overline{X}, \overline{Y} \in \chi(\overline{M}). \quad \Box \end{cases} \tag{2.3}$$

The decompositions in tangential and normal components on M in $\overline{M}$ of the vector fields $\overline{P}X$ $(X \in \chi(M))$ and $\overline{P}N_2$ are, respectively, as follows:

$$\overline{P}X = PX + u_2(X)N_2, \quad \overline{P}N_1 = \xi_2 + a_{22}N_2, \tag{2.4}$$

for any $X \in \chi(M)$, where P is an $(1,1)$ tensor field on M, u_2 is an 1-form on M, ξ_2 is a tangential vector field on M and a_{22} is a real function on M.

On the other hand, we remark that the decomposition of the vector field $\xi_1 \in \chi(\overline{M})$ in tangential and normal components on M in $\overline{M}$ has the form $\xi_1 = \xi_1^{\top} + \xi_1^{\perp}$ and $\xi_1^{\perp}$ and N_2 are collinear.

Lemma 2.2. *The decompositions in the tangential and normal parts on M in $\widetilde{M}$ of vector fields $\widetilde{P}X$ $(X \in \chi(M))$, $\widetilde{P}N_1$ and $\widetilde{P}N_2$ are, respectively, as follows:*

$$\begin{cases} (i) \quad \widetilde{P}X = PX + u_1(X)N_1 + u_2(X)N_2, \ (\forall)X \in \chi(M) \\ (ii) \quad \widetilde{P}N_1 = \xi_1^{\top} + a_{11}N_1 + a_{12}N_2, \\ (iii) \quad \widetilde{P}N_2 = \xi_2 + a_{21}N_1 + a_{22}N_2, \end{cases} \tag{2.5}$$

where P is an $(1,1)$ tensor field on M, u_1, u_2 are 1-forms on M, $\xi_1^{\top}, \xi_2$ are tangent vector fields on M, $(a_{\alpha\beta})$ (with $\alpha, \beta \in \{1,2\}$) is an 2×2 matrix, and its entries a_{11}, a_{22} and $a_{12} = a_{21} = \widetilde{g}(\xi_1^{\perp}, N_2)$ are real functions on M.

Lemma 2.3. *The structure $(\overline{P}, \overline{g}, \xi_2, u_2, a_{22})$ (induced on a submanifold $(\overline{M}, \overline{g})$ of codimension 1 in a n-dimensional $(n > 2)$ almost product Riemannian manifold $(\widetilde{M}, \widetilde{g}, \widetilde{P})$) also induces, on a submanifold (M, g) of codimension 1 in $\overline{M}$, a Riemannian structure $(P, g, u_1, u_2, \xi_1, \xi_2^{\top}, (a_{\alpha\beta}))$ (where $P, u_1, u_2, \xi_1, \xi_2^{\top}$, $(a_{\alpha\beta})$ were defined in the last two propositions) which has the following properties:*

$$\begin{cases} (i) \quad P^2X = X - u_1(X)\xi_1 - u_2(X)\xi_2^{\top}, \ (\forall)X \in \chi(M) \\ (ii) \quad u_1(PX) = -a_{11}u_1(X) - a_{12}u_2(X), \ (\forall)X \in \chi(M) \\ (iii) \quad u_2(PX) = -a_{21}u_1(X) - a_{22}u_2(X), \ (\forall)X \in \chi(M) \\ (iv) \quad u_1(\xi_1) = 1 - a_{11}^2 - a_{12}^2, \\ (v) \quad u_2(\xi_1) = -a_{11}a_{12} - a_{12}a_{22}, \\ (vi) \quad u_1(\xi_2^{\top}) = -a_{11}a_{12} - a_{12}a_{22}, \\ (vii) \quad u_2(\xi_2^{\top}) = 1 - a_{12}^2 - a_{22}^2, \\ (viii) \quad P(\xi_1) = -a_{11}\xi_1 - a_{12}\xi_2^{\top}, \\ (ix)\, quad(\xi_2^{\top}) = -a_{12}\xi_1 - a_{22}\xi_2^{\top}, \end{cases} \tag{2.6}$$

and the properties which depends on the metric g are:

$$\begin{cases} (i) \quad u_1(X) = g(X, \xi_1), \\ (ii) \quad u_2(X) = g(X, \xi_2^{\top}), \\ (iii) \quad g(PX, Y) = g(X, PY), \\ (iv) \quad g(PX, PY) = g(X, Y) - u_1(X)u_1(Y) - u_2(X)u_2(Y), \end{cases} \tag{2.7}$$

for any $X, Y \in \chi(M)$.

Proof. From $\widetilde{P}(\widetilde{P}X) = X$ it follows that:

$$\widetilde{P}(PX + u_1(X)N_1 + u_2(X)N_2) = X$$

, thus we have:

$$P^2X + u_1(PX)N_1 + u_2(PX)N_2 + u_1(X)(\xi_1 + a_{11}N_1 + a_{12}N_2) +$$

$$+u_2(X)(\xi_2^{\top} + a_{12}N_1 + a_{22}N_2) = X$$

Identifying the tangential and respectively, normal components on M from the last equality, we obtain (i), (ii) and (iii) from (2.6).

On the other hand, from $\widetilde{P}(\widetilde{P}N_1) = N_1$ we derive:

$$N_1 = \widetilde{P}(\widetilde{P}N_1) = \widetilde{P}(\xi_1 + a_{11}N_1 + a_{12}N_2) =$$

$$= P\xi_1 + u_1(\xi_1)N_1 + u_2(\xi_1)N_2 + a_{11}(\xi_1 + a_{11}N_1 + a_{12}N_2) + a_{12}(\xi_2^{\top} + a_{21}N_1 + a_{22}N_2)$$

. Identifying the tangential and, respectively, normal components on M we obtain (iv), (v) and (viii) from (2.6). In the same manner, it result (vi), (vii) and (ix) from (2.6) using $\widetilde{P}(\widetilde{P}N_2) = N_2$.

From $g(PX, Y) = \widetilde{g}(\widetilde{P}X - u_1N_1 - u_2N_2, Y) = \widetilde{g}(\widetilde{P}X, Y) = \widetilde{g}(X, \widetilde{P}Y) =$ $= \widetilde{g}(X, PY + u_1(Y) + u_2(Y)N_2) = g(X, PY)$ we get: the equality (iii) from (2.7). From $\widetilde{g}(\widetilde{P}X, N_1) = \widetilde{g}(X, \widetilde{P}N_1)$ we have

$$\widetilde{g}(PX + u_1(X)N_1 + u_2(X)N_2, N_1) = \widetilde{g}(X, \xi_1 + a_{11}N_1 + a_{12}N_2)$$

. Thus, $u_1(X) = \widetilde{g}(X, \xi_1) = g(X, \xi_1)$ and this yields the equality (i) from (2.7). In the same manner, using $\widetilde{g}(\widetilde{P}X, N_2) = \widetilde{g}(X, \widetilde{P}N_2)$, we obtain (ii) from (2.7).

From $g(PX, Y) = g(X, PY)$, replacing Y with PY we have:

$$g(PX, PY) = g(X, P^2Y) = g(X, Y - u_1(Y)\xi_1 - u_2(Y)\xi_2^{\top}).$$

and from this it results (iv) from (2.7). □

From Lemma 1 and Lemma 3 we obtain:

Theorem 2.1. *Let M be an n-dimensional submanifold of codimension 1 isometrically immersed in $\overline{M}$, which is also a submanifold of codimension 1 and isometrically immersed in an almost product Riemannian manifold $(\widetilde{M}, \widetilde{g}, \widetilde{P})$. Then, the induced structure on M by the structure $(\widetilde{P}, \widetilde{g})$ from $\widetilde{M}$ is an $(a, 1)f$ Riemannian structure, determined by $(P, g, u_1, u_2, \xi_1^{\top}, \xi_2, (a_{\alpha\beta})_2)$, (where $a := (a_{\alpha\beta})_2$ and $f := P$) which is the same that one induced on M by the structure $(\overline{P}, \overline{g}, u_1, \xi_1, a_{11})$ (induced on $\overline{M}$ by the almost product structure $\widetilde{P}$ from $\widetilde{M}$).*

We can give a generalization of the Theorem 2.1 as follows:

Let $M := M_r$ be an n-dimensional submanifold of codimension r (with $r \geq 2$) in an almost product Riemannian manifold $(\widetilde{M}, \widetilde{g}, \widetilde{P})$. We make the following notations: $\widetilde{M} := M_0$, $\widetilde{g} := g^0$, $\widetilde{P} := P_0$, such that we have the sequence of Riemannian immersions given by:

$$(M_r, g^r) \hookrightarrow (M_{r-1}, g^{r-1}) \hookrightarrow ... \hookrightarrow (M_1, g^1) \hookrightarrow (\widetilde{M}, \widetilde{g}, \widetilde{P})$$

where g^i is an induced metric on M^i by the metric g^{i-1} from M_{i-1}, ($i \in \{1, ..., r\}$) and each one of (M_i, g^i) is a submanifold of codimension 1, isometric immersed in the manifold (M_{i-1}, g^{i-1}) ($i \in \{1, ..., r\}$). Let $i \in \{1, ..., r\}$ and $\alpha_i, \beta_i \in \{1, ..., i\}$). In this condition we obtain:

Theorem 2.2. *The $(a,1)f$ Riemannian structure, determined by the induced structure $(P_r, g^r, \xi^r_{\alpha_r}, u^r_{\alpha_r}, (a^r_{\alpha_r\beta_r}))$ on an n-dimensional submanifold $M := M_r$ of codimension r (with $r \geq 2$) in an almost product Riemannian manifold $(\widetilde{M}, \widetilde{g}, \widetilde{P})$, is the same that one induced on M by any structures $(P_i, g^i, \xi^i_{\alpha_i}, u_{\alpha_i}, (a^i_{\alpha_i\beta_i}))$ $(i < r)$ induced on M_i by the almost product structure $\widetilde{P}$ on $\widetilde{M}$, where $f := P_r$ is the tangential component of P_i on M, the vector fields $\xi^r_{\alpha_i}$ on M_r are the tangential components on M of the tangent vector fields $\xi^i_{\alpha_i}$ from M_i, the 1-forms $u^r_{\alpha_i}$ are the restrictions on M of the 1-forms $u^r_{\alpha_i}$ from M_i (for $i < r$), the entries of the $r \times r$ matrix $a := (a^r_{\alpha_r\beta_r})$ are defined by $a^r_{\alpha_r,\beta_r} = a^r_{\beta_r,\alpha_r} = g^r(P_{r-1}(N_{\alpha_r}), N_{\beta_r})$.*

3 Examples of $(a,1)f$ Riemannian structures

Example 1. Let E^{2p+q} be the $(2p+q)$-dimensional Euclidean space ($p, q \in \mathbb{N}^*$). In this example, we construct an $(\overline{a}, 1)f$-structure on the sphere $S^{2p+q-1}(R) \hookrightarrow E^{2p+q}$.

For any point of E^{2p+q} we have its coordinates:

$$(x^1, ..., x^p, y^1, ..., y^p, z^1, ..., z^q) := (x^i, y^i, z^j)$$

where $i \in \{1, ..., p\}$ and $j \in \{1, ..., q\}$. The tangent space $T_x(E^{2p+q})$ is isomorphic with E^{2p+q}.

Let $\widetilde{P} : E^{2p+q} \to E^{2p+q}$ an almost product structure on E^{2p+q} so that:

(3.1)
$$\widetilde{P}(x^1, ..., x^p, y^1, ..., y^p, z^1, ..., z^q) = (\nu_1 y^1, ..., \nu_p y^p, \nu_1 x^1, ..., \nu_p x^p, \varepsilon_1 z^1, ..., \varepsilon_q z^q)$$

and we use the notation:

$$(\nu_1 y^1, ..., \nu_p y^p, \nu_1 x^1, ..., \nu_p x^p, \varepsilon_1 z^1, ..., \varepsilon_q z^q) := (\nu_i y^i, \nu_i x^i, \varepsilon_j z^j)$$

where $\nu_i^2 = \varepsilon_j^2 = 1$ for every $i \in \{1, ..., p\}$ and $j \in \{1, ..., q\}$.

The equation of the sphere $S^{2p+q-1}(R)$ is:

$$\sum_{i=1}^{p} (x^i)^2 + \sum_{i=1}^{p} (y^i)^2 + \sum_{j=1}^{q} (z^j)^2 = R^2 \tag{3.2}$$

where R is its radius and $(x^1, ..., x^p, y^1, ..., y^p, z^1, ..., z^q) := (x^i, y^i, z^j)$ are the coordinates of any point of $S^{2p+q-1}(R)$.

We use the following notations:

$$\sum_{i=1}^{p}(x^i)^2 = r_1^2, \quad \sum_{i=1}^{p}(y^i)^2 = r_2^2, \quad \sum_{j=1}^{q}(z^j)^2 = r_3^2$$

and $r_1^2 + r_2^2 = r^2$. Thus we have $r^2 + r_3^2 = R^2$.

We remark that an unit normal vector field on sphere $S^{2p+q-1}(R)$ has the form:

$$N_1 := \frac{1}{R}(x^i, y^i, z^j), \tag{3.3}$$

for $i \in \{1, ..., p\}$ and $j \in \{1, ..., q\}$ and we have $\widetilde{P}N_1 = \frac{1}{R}(\nu_i y^i, \nu_i x^i, \varepsilon_j z^j)$.

For any tangent vector field:

$$\overline{X} = (X^1, ..., X^p, Y^1, ..., Y^p, Z^1, ..., Z^q) := (X^i, Y^i, Z^j)$$

on $S^{2p+q-1}(R)$ we have:

$$\sum_{i=1}^{p} x^i X^i + \sum_{i=1}^{p} y^i Y^i + \sum_{j=1}^{q} z^j Z^j = 0, \tag{3.4}$$

From (1.1) and (1.2) we have the decompositions of $\widetilde{P}\overline{X}$ and $\widetilde{P}N_1$ in tangential and normal components, respectively, at the sphere $S^{2p+q-1}(R)$.

In the following issue we use the notations $\overline{a} := a_{11}$ and $f := \overline{P}$:

$$\sigma = \sum_{i=1}^{p} \nu_i x^i y^i, \quad \tau = \sum_{j=1}^{q} \varepsilon_j (z^j)^2, \tag{3.5}$$

$$\gamma = \sum_{i=1}^{p} \nu_i (x^i Y^i + y^i X^i), \quad \mu = \sum_{j=1}^{q} \varepsilon_j z^j Z^j \tag{3.6}$$

for any point (x^i, y^i, z^j) of $S^{2p+q-1}(R)$ and for any tangent vector field $\overline{X} = (X^i, Y^i, Z^j)$ ($i \in \{1, ..., p\}$ and $j \in \{1, ..., q\}$). Using the first Lemma, we obtain an $(\overline{a}, 1)f$ structure on the sphere $S^{2p+q-1}(R) \hookrightarrow E^{2p+q}$ (with $\overline{g} :=<>$), determined by $(\overline{P}, <>, \xi_1, u_1, a_{11})$ which has the elements as follows:

$$a_{11} = \frac{2\sigma + \tau}{R^2}, \tag{3.7}$$

$$u_1(\overline{X}) = \gamma + \tau, \tag{3.8}$$

$$\xi_1 = \frac{1}{R}(\nu_i y^i - a_{11}x^i, \nu_i x^i - a_{11}y^i, (\varepsilon_j - a_{11})z^j), \tag{3.9}$$

and:

$$\overline{P}(\overline{X}) = (\nu_i Y^i - \frac{u_1(\overline{X})}{R}x^i, \nu_i X^i - \frac{u_1(\overline{X})}{R}y^i, \varepsilon_j Z^j - \frac{u_1(\overline{X})}{R}z^j). \tag{3.10}$$

Example 2. In this example, we construct an $(a, 1)f$-structure on the product of spheres $S^{2p-1}(r) \times S^{q-1}(r_3)$. Let E^{2p+q} $(p, q \in \mathbb{N}^*)$ be the Euclidean space $(p, q \in \mathbb{N}^*)$ endowed with the almost product Riemannian structure $\widetilde{P}$ defined in (3.1). We set $E^{2p+q} = E^{2p} \times E^q$ and in each of spaces E^{2p} and E^q respectively, we consider the spheres:

$$S^{2p-1}(r) = \{(x^1, ..., x^p, y^1, ..., y^p), \sum_{i=1}^{p}((x^i)^2 + (y^i)^2) = r^2\}$$

and respectively:

$$S^{q-1}(r_3) = \{(z^1, ..., z^q), \sum_{j=1}^{q}(z^j)^2 = r_3^2\}$$

where $r^2 + r_3^2 = R^2$. Any point of the product manifold $S^{2p-1}(r) \times S^{q-1}(r_3)$ has the coordinates $(x^1, ..., x^p, y^1, ..., y^p, z^1, ..., z^q) := (x^i, y^i, z^j)$ which verify (3.2). Thus $S^{2p-1}(r) \times S^{q-1}(r_3)$ is a submanifold of codimension 2 in E^{2p+q}. Furthermore, $S^{2p-1}(r) \times S^{q-1}(r_3)$ is a submanifold of codimension 1 in $S^{2p+q-1}(R)$. Therefore, we have:

$$S^{2p-1}(r) \times S^{q-1}(r_3) \hookrightarrow S^{2p+q-1}(R) \hookrightarrow E^{2p+q}$$

. The tangent space in a point (x^i, y^i, z^j) at the product of spheres $S^{2p-1}(r) \times S^{q-1}(r_3)$ is $T_{(x^1,...,x^p,y^1,...,y^p,\underbrace{o,...,o}_{q})}S^{2p-1}(r) \oplus T_{(\underbrace{o,...,o}_{2p},z^1,...,z^q)}S^{q-1}(r_3)$.

A vector $(X^1, ..., X^p, Y^1, ..., Y^p)$ from $T_{(x^1,...,x^p,y^1,...,y^p)}E^{2p}$ is tangent to $S^{2p-1}(r)$ if and only if:

$$\sum_{i=1}^{p} x^i X^i + \sum_{i=1}^{p} y^i Y^i = 0 \tag{3.11}$$

and it can be identified with $(X^1, ..., X^p, Y^1, ..., Y^p, \underbrace{0, ..., 0}_{q})$ from E^{2p+q}. A vector $(Z^1, ..., Z^q)$ from $T_{(z^1,...,z^q)}E^q$ is tangent to $S^{q-1}(r_3)$ if and only if:

$$\sum_{j=1}^{q} z^j Z^j = 0 \tag{3.12}$$

and it can be identified with $(\underbrace{0,...,0}_{2p}, Z^1, ..., Z^q)$ from E^{2p+q}.

Consequently, for any point $(x^i, y^i, z^j) \in S^{2p-1}(r) \times S^{q-1}(r_3)$ we have $(X^i, Y^i, Z^j) \in T_{(x^1,...,x^p,y^1,...,y^p,z^1,...,z^q)}(S^{2p-1}(r) \times S^{q-1}(r_3))$ if and only if the equations (3.11) and (3.12) are satisfied. Furthermore, we remark that (X^i, Y^i, Z^j) is a tangent vector field at $S^{2p+q-1}(R)$ and from this it follows that:

$$T_{(x^i,y^i,z^j)}(S^{2p-1}(r) \times S^{q-1}(r_3)) \subset T_{(x^i,y^i,z^j)}S^{2p+q-1}(R),$$

for any point $(x^i, y^i, z^j) \in S^{2p-1}(r) \times S^{q-1}(r_3)$.

The normal unit vector field N_1 at $S^{2p+q-1}(R)$ given by (3.3) is also a normal vector field at $(S^{2p-1}(r) \times S^{q-1}(r_3))$ when it is considered in its points. We construct an unit vector field N_2 on S^{2p+q-1} as follows:

$$N_2 = \frac{1}{R}(\frac{r_3}{r}x^i, \frac{r_3}{r}y^i, -\frac{r}{r_3}z^j) \tag{3.13}$$

It is obvious that (N_1, N_2) defined in (3.3) and (3.13) is a local orthonormal basis in $T^{\perp}_{(x^i,y^i,z^j)}S^{2p-1}(r) \times S^{q-1}(r_3)$ in any point $(x^i, y^i, z^j) \in S^{2p-1}(r) \times S^{q-1}(r_3)$. Using Lemma 2 and Lemma 3, we obtain the structure $(\widehat{P}, <>, \widehat{\xi}_1, \widehat{\xi}_2, \widehat{u}_1, \widehat{u}_2, \widehat{a})$ on the product of spheres $S^{2p-1}(r) \times S^{q-1}(r_3)$, induced by the almost product Riemannian structure $(\widetilde{P}, <,>)$ as follows:

· the matrix $a := (a_{\alpha\beta})_2$ is given by:

$$a := \begin{pmatrix} \frac{2\sigma+\varepsilon r_3^2}{R^2} & \frac{(2\sigma-\varepsilon r^2)r_3}{rR^2} \\ \frac{(2\sigma-\varepsilon r^2)r_3}{rR^2} & \frac{2\sigma r_3^2+\varepsilon r^4}{r^2R^2} \end{pmatrix}, \tag{3.14}$$

· the tangent vector fields have the form:

$$\xi_1 = \frac{1}{R}(\nu_i y^i - \frac{2\sigma}{r^2}x^i, \nu_i x^i - \frac{2\sigma}{r^2}y^i, (\varepsilon_j - \frac{\tau}{r_3^2})z^j), \tag{3.15}$$

and:

$$\xi_2 = \frac{1}{R}(\frac{r_3}{r}(\nu_i y^i - \frac{2\sigma}{r^2}x^i), \frac{r_3}{r}(\nu_i x^i - \frac{2\sigma}{r^2}y^i), -\frac{r}{r_3}((\varepsilon_j - \frac{\tau}{r_3^2})z^j), \tag{3.16}$$

· the 1-forms are given by:

$$u_1(X) = \frac{1}{R}(\gamma + \mu), \quad u_2(X) = \frac{1}{R}(\frac{r_3}{r}\gamma - \frac{r}{r_3}\mu), \tag{3.17}$$

and the $(1,1)$ tensor field P has the form:

$$P(X) = (\nu_i Y^i - \frac{\gamma}{r^2}x^i, \nu_i X^i - \frac{\gamma}{r^2}y^i, \varepsilon_j Z^j - \frac{\mu}{r_3^2}z^j) \tag{3.18}$$

for any tangent vector field $X := (X^i, Y^i, Z^j) \in T_{(x^1,...,x^p,y^1,...,y^p,z^1,...,z^q)}(S^{2p-1}(r) \times S^{q-1}(r_3))$ and any point $(x^i, y^i, z^j) \in S^{2p-1}(r) \times S^{q-1}(r_3)$. For $a := (a_{\alpha\beta})_2$ and $f := P$, the structure $(\widehat{P}, <>, \widehat{\xi}_1, \widehat{\xi}_2, \widehat{u}_1, \widehat{u}_2, \widehat{a})$ is an $(a,1)f$ Riemannian structure induced on the on the product of spheres $S^{2p-1}(r) \times S^{q-1}(r_3)$ which is a submanifold of codimension 2 in the Euclidean space E^{2p+q}.

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