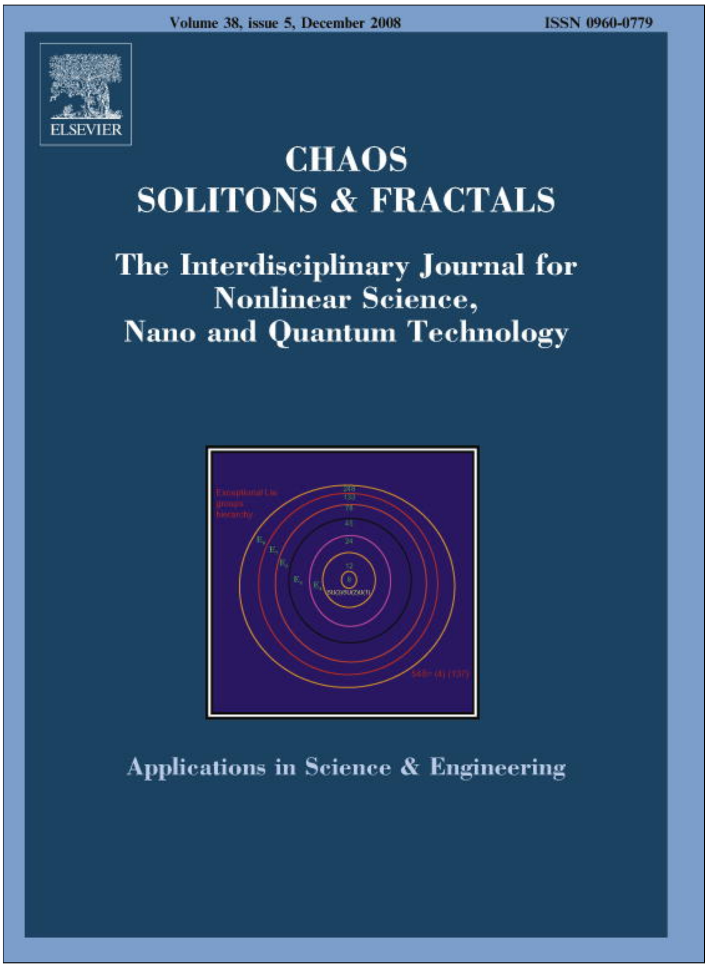
Volume 38, issue 5, December 2008
ISSN 0960-0779
ELSEVIER
CHAOS
SOLITONS & FRACTALS
The Interdisciplinary Journal for
Nonlinear Science,
Nano and Quantum Technology
Applications in Science & Engineering

Chaos, Solitons and Fractals 38 (2008) 1229–1238

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

journal homepage: www.elsevier.com/locate/chaos

Golden differential geometry ☆

Mircea Crasmareanu [a], Cristina-Elena Hreţcanu [b,*]

[a] Al. I. Cuza University, Iaşi, Romania
[b] Ştefan cel Mare University, Suceava, Romania

ARTICLE INFO

Article history:
Accepted 3 April 2008

Dedicated to M.S. El Naschie on the occasion of his 65th birthday.

Communicated by Prof. L. Marek-Crnjac

ABSTRACT

A research on the properties of the Golden structure (i.e. a polynomial structure with the structure polynomial $Q(X) = X^2 - X - I$) is carried out in this article. The Golden proportion plays a central role in this paper. The geometry of the Golden structure on a manifold is investigated by using a corresponding almost product structure.

0. Introduction

The Golden proportion, also called the Golden ratio, Divine ratio, Golden section or Golden mean, has been well known since the time of Euclid. Many objects alive in the natural world that possess pentagonal symmetry, such as inflorescence of many flowers and phyllotaxis objects have a numerical description given by the Fibonacci numbers which are themselves based on the Golden proportion. The Golden proportion has also been found in the structure of musical compositions, in the ratios of harmonious sound frequencies and in dimensions of the human body [19]. From ancient times it has played an important role in architecture and visual arts. The Golden proportion and the Golden rectangle (which is spanned by two sides in the Golden proportion) have been found in the harmonious proportions of temples, churches, statues, paintings, pictures and fractals [10,15].

Let us recall that Golden proportion partitions a line segment into a major subsegment and a minor subsegment in such a way that both the ratio of whole segment and the major subsegment and the ratio of major subsegment and the minor subsegment must equal the number ϕ (the Phidias number), which is the real positive root of the equation $x^2 - x - 1 = 0$ (thus, $\phi = \frac{1+\sqrt{5}}{2} = 1.618, \ldots$). This number is often encountered when taking the ratios of distances in simple geometric figures such as the pentagram, decagon and dodecagon.

In the last few years, the Golden proportion has played an increasing role in modern physical research [6–9,26] and it has a unique significant role in atomic physics [16]. The Golden proportion is found to govern the transition from Newtons physics to relativistic mechanics and the Golden rectangle has been used to derive the dilation of time intervals and the Lorentz contraction of lengths in special relativity [25]. The Golden proportion has also interesting properties in topology of four-manifolds, in conformal field theory, in mathematical probability theory and in Cantorian spacetime [21,22] as well as in the El Naschie's field theory [20].

The aim of the present paper is to investigate the geometry of the Golden structure on a manifold by using a corresponding almost product structure. The Golden structure is seen by us as a polynomial structure [11,12] with the structure polynomial $Q(X) = X^2 - X - I$. Thus, a tangent and a complex analog of the previous structure is introduced into Section 1 and the

☆ Partially supported by the CEex Grant 05-D11-84.

* Corresponding author.
E-mail addresses: mcrasm@uaic.ro (M. Crasmareanu), cristinah@usv.ro (C.-E. Hreţcanu).
URL: http://www.math.uaic.ro/~mcrasm (M. Crasmareanu).

0960-0779/$ - see front matter © 2008 Elsevier Ltd. All rights reserved.
doi:10.1016/j.chaos.2008.04.007

corresponding complex number, called by us *complex Golden ratio*, is studied. Let us remark that a complex version of Fibonacci numbers appears in [14].

In Section 2, several frameworks in which almost product structures (and their associated splitting) are natural, are treated in our language of Golden structure. For example, one obtain a two-parametric family of two-dimensional Golden structures, Golden Clifford identities and symplectic Golden structures.

Section 3 is devoted to treatment of connections in principal and tangent bundles in terms of Golden structures.

In Section 4 we obtain fundamental formulae, more precisely equivalent relations related to the parallelism with respect to Schouten and Vrănceanu connections [1].

Section 5 connects Golden structures with a main geometrical object, namely Riemannian metrics.

1. Golden structures on manifolds

In order to state the main results of this paper we need some more definitions and notations.

Definition 1.1 [11]. Let M be a C^∞-differentiable real manifold. A tensor field F of type $(1,1)$ on M is said to define a *polynomial structure* if F satisfies the algebraic equation

$$Q(x) = x^n + a_n x^{n-1} + \cdots + a_2 x + a_1 I = 0$$

where I is the identity $(1,1)$ tensor field and $F^{n-1}(p), F^{n-2}(p), \ldots, F(p), I$ are linearly independent for every point $p \in M$. The polynomial $Q(X)$ is called *the structure polynomial.*

Remark 1.1. For $Q(X) = X^2 + I$ (respectively, $Q(X) = X^2 - I$) we obtain an *almost complex structure* J (respectively, *an almost product structure* P). Let us recall that the existence of an almost complex structure implies a condition on the dimension of M, namely it is even. For $Q(X) = X^2$ we obtain the notion of *almost tangent structure* T [23].

Definition 1.2 [17]. A $(1,1)$-tensor field Φ which satisfies the equation

$$\Phi^2 = \Phi + I \tag{1.1}$$

is called a *Golden structure* on M.

Let us point out some properties of these structures:

Proposition 1.1. *A Golden structure on the manifold M has the power*

$$\Phi^n = F_n \Phi + F_{n-1} I \tag{1.2}$$

for any integer number n, where $(F_n)_n$ *is the Fibonacci sequence.*

Using an explicit expression for the Fibonacci sequence namely the *Binet's formula* from [19]

$$F_n = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}} \tag{1.3}$$

we obtain a new form for the equality (1.2)

$$\Phi^n = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}} \Phi + \frac{\phi^{n-1} - (1-\phi)^{n-1}}{\sqrt{5}} I \tag{1.4}$$

A straightforward computation yields:

Proposition 1.2

(i) *The eigenvalues of a Golden structure* Φ *are the Golden ratio* ϕ *and* $1-\phi$.
(ii) *A Golden structure* Φ *is an isomorphism on the tangent space of the manifold,* T_xM, *for every* $x \in M$.
(iii) *It follows that* Φ *is invertible and its inverse* $\widehat{\Phi} = \Phi^{-1}$ *satisfies:*

$$\widehat{\Phi}^2 = -\widehat{\Phi} + I \tag{1.5}$$

Remark 1.2. An important remark is that Golden structures appear in pairs; namely if Φ is a Golden structure then $\widetilde{\Phi} = I - \Phi$ is also a Golden structure. But, so is the case for almost tangent structures (T and $-T$), almost complex structures (J and $-J$) and for almost product structures (P and $-P$). It is natural to seek for a connection between Golden and product structures:

Theorem 1.1. *An almost product structure P induces a Golden structure as follows:*

$$\Phi = \frac{1}{2}(I + \sqrt{5}P) \tag{1.6}$$

Conversely, any Golden structure Φ yields an almost product structure

$$P = \frac{1}{\sqrt{5}}(2\Phi - I) \tag{1.7}$$

Remark 1.3. In the above correspondence $\Phi \leftrightarrow P$ we have

$$\widetilde{\Phi} = I - \Phi \leftrightarrow \widetilde{P} = -P$$

Copying the scheme (1.6) of the previous structure let us introduce:

(I) Let M be endowed with an almost tangent structure T. We say that

$$\Phi_t = \frac{1}{2}(I + \sqrt{5}T) \tag{1.8}$$

is a *tangent Golden structure* on (M, T). It follows:

$$\Phi_t^2 - \Phi_t + \frac{1}{4}I = 0 \tag{1.9}$$

the equation verified by a tangent Golden structure. Considering the associate equation in the real field $\mathbb{R}$, i.e. $x^2 - x + \frac{1}{4} = 0$ we have the *tangent real Golden ratio* $\phi_t = \frac{1}{2}$!

(II) Let (M, J) be an almost complex manifold. The tensor field Φ_c defined by

$$\Phi_c = \frac{1}{2}(I + \sqrt{5}J) \tag{1.10}$$

is called *the complex Golden structure* on (M, J).
The polynomial equation satisfied by Φ_c is

$$\Phi_c^2 - \Phi_c + \frac{3}{2}I = 0 \tag{1.11}$$

Returning to $M = \mathbb{R}^2$ we arrive at the equation

$$x^2 - x + \frac{3}{2} = 0 \tag{1.12}$$

with solutions $x_1 = \frac{1}{2} + \frac{1}{2}\mathrm{i}\sqrt{5}, x_2 = \bar{x}_1 = \frac{1}{2} - \frac{1}{2}\mathrm{i}\sqrt{5}$. Let us remark that $2x_1$ and $2x_2$ appear in the non-unique factorization property of $Z[\sqrt{-5}]$ through $6 = 2 \cdot 3 = 2x_1 \cdot 2x_2$.
In the following definition, let us introduce a new notion, called by us a *complex Golden ratio*:

Definition 1.3. The complex number

$$\phi_c = \frac{1}{2} + \frac{\sqrt{5}}{2}\mathrm{i} \tag{1.13}$$

will be called *complex Golden ratio.*

Remark 1.4. This number admits the trigonometric expression

$$\phi_c = \sqrt{\frac{3}{2}}\left(\frac{1}{\sqrt{6}} + \mathrm{i}\sqrt{\frac{5}{6}}\right) \tag{1.14}$$

which implies:

Definition 1.4. The complex number

$$\phi_c^u = \frac{1}{\sqrt{6}} + \mathrm{i}\sqrt{\frac{5}{6}} \tag{1.15}$$

will be called *unitary complex Golden ratio.*

The equation satisfied by ϕ_c^u is

$$x^2 - \frac{\sqrt{6}}{3}x + 1 = 0 \tag{1.16}$$

Using Maple for the quadratic matrix associated to ϕ_c

$$\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{5}}{2} \\ \frac{\sqrt{5}}{2} & \frac{1}{2} \end{pmatrix}$$

we obtain

(1) The diagonal expression

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{i}{2} & \frac{i}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2}+\frac{\sqrt{5}}{2}\mathrm{i} & 0 \\ 0 & \frac{1}{2}-\frac{\sqrt{5}}{2}\mathrm{i} \end{pmatrix} \begin{pmatrix} 1 & \mathrm{i} \\ 1 & -\mathrm{i} \end{pmatrix} \tag{1.17}$$

(2) The QR-factorization

$$\begin{pmatrix} -.40825 & .40825\sqrt{5} \\ -.40825\sqrt{5} & -.40825 \end{pmatrix} \begin{pmatrix} -1.2247 & 3\times 10^{-10}\sqrt{5} \\ -2\times 10^{-10}\sqrt{5} & -1.2247 \end{pmatrix} \tag{1.18}$$

(3) The rational canonical form

$$\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} 0 & -\frac{3}{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{\sqrt{5}}{5} \\ 0 & \frac{2\sqrt{5}}{5} \end{pmatrix} \tag{1.19}$$

As a conclusion of this section we get a *Trinity concept*

$$\begin{array}{ccc} & \text{Harmony} & \\ \swarrow & \downarrow & \searrow \\ \text{total} & \text{Golden} & \text{complex} \\ \phi_t = \frac{1}{2} & \phi = \frac{\sqrt{5}+1}{2} & \phi_c = \frac{1}{2}+\frac{\sqrt{5}}{2}\mathrm{i} = \\ & & = \phi_t + \mathrm{i}(\phi-\phi_t) \end{array} \tag{1.20}$$

2. Examples of Golden structures

We state now some examples of our Golden structure:

Example 2.1 (*Clifford algebras*). Using the notations of [24, p. 130] let $C'(n)$ be the real Clifford algebra of the positive definite form $\sum_{i=1}^{n}(x^i)^2$ of $\mathbb{R}^n$. Fixing a basis $\{e_1,\dots,e_n\}$ of $\mathbb{R}^n$ it is well known that the defining relations of $C'(n)$ are

$$\begin{cases} e_i^2 = 1 \\ e_ie_j + e_je_i = 0, \quad i \neq j \end{cases} \tag{2.1}$$

Therefore, introducing $\Phi_i = \frac{1}{2}(1+\sqrt{5}e_i)$ we derive new presentations relations of $C'(n)$:

$$\begin{cases} \Phi_i = \text{Golden structure } (\Phi_i^2 = \Phi_i + 1) \\ \Phi_i\Phi_j + \Phi_j\Phi_i = \Phi_i + \Phi_j - \frac{1}{2}, \quad i \neq j \end{cases} \tag{2.2}$$

In the cited book is treated $C'(2)$ with

$$1 = I_2, \quad e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.3}$$

and hence

$$\begin{cases} \text{(i)} \ \ \Phi_1 = \frac{1}{2}(I_2 + \sqrt{5}e_1) = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix} = \begin{pmatrix} \phi & 0 \\ 0 & 1-\phi \end{pmatrix} \\ \text{(ii)} \ \ \Phi_2 = \frac{1}{2}(I_2 + \sqrt{5}e_2) = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{5} \\ \sqrt{5} & 1 \end{pmatrix} \end{cases} \tag{2.4}$$

Example 2.2 (*2D Golden matrices*). In dimension two we obtain a two-parametric family of Golden structures by solving (1.1)

$$\Phi_{a,b} = \begin{pmatrix} a & -\frac{1}{b}(a^2-a-1) \\ b & 1-a \end{pmatrix}, \quad a \in \mathbb{R}, \quad b \in \mathbb{R}^* \tag{2.5}$$

plus the separate solutions $\Phi_1, I_2 - \Phi_1$ from (2.4)(i).

For example

$$\Phi_{\phi,b} = \begin{pmatrix} \phi & 0 \\ b & 1-\phi \end{pmatrix} \tag{2.6}$$

captures on its principal diagonal the both roots of the characteristic equation $x^2 - x - 1 = 0$ of the classical Golden ratio exactly as Φ_1 (hence, Φ_1 can be thought as $\lim_{b\to 0}\Phi_{\phi,b}$) while $\Phi_2 = \Phi_{\frac{1}{2},\frac{\sqrt{5}}{2}}$.

Returning to the general case (2.5) let us point out:

(i) The determinant of $\Phi_{a,b}$ is independent of parameters a and b, namely is (-1). Searching if $\Phi_{\cos t,\sin t}$ belongs to $O^-(2) = \{S \in O(2); \det S = -1\}$ we obtain

$$\Phi_{\cos t,\sin t} = \begin{pmatrix} \cos t & \sin t + \cot t \\ \sin t & 1 - \cos t \end{pmatrix} \tag{2.7}$$

and then the answer is negative. From (2.7) results the example

$$\Phi_{0,1} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \tag{2.8}$$

(ii) The trace of $\Phi_{a,b}$ is independent of parameters a and b, namely is (+1); a new motivation for $\Phi_{\cos t,\sin t} \notin O^-(2)$ since the trace of an element of this set is 0. Therefore, the sequence of traces $\left(\mathrm{Tr}\Phi^k_{a,b}\right)_{k\geqslant 0}$ is the Fibonacci sequence: $2, 1, 3, 4, 7, \ldots$.

(iii) The sum of solutions of equation $x^2 - x - 1 = 0$ is +1. Verifying this identity on the family (2.5) we arrive at

$$I_2 - \Phi_{a,b} = \Phi_{1-a,b} \tag{2.9}$$

To end this example, let us note that the general quadratic equation

$$\mathscr{L}(x) = x^2 + \alpha x + \beta$$

admits the 2D solution

$$\Phi_{a,b} = \begin{pmatrix} a & -\frac{1}{b}\mathscr{L}(a) \\ b & -\alpha - a \end{pmatrix}, \quad a \in \mathbb{R}, \quad b \in \mathbb{R}^*$$

Example 2.3 (*Golden reflections*). Recall, after [24, p. 314], that in an Euclidean space $(E, \langle,\rangle)$ the reflection with respect to a hyperplane H with the normal $v \in E \setminus \{0\}$ has the formula

$$r_v : x \in E \to r_v(x) = x - \frac{2\langle x, \rangle}{\langle v, v\rangle} v \tag{2.10}$$

and obviously, $r_v^2 = I_E$ = the identity on E. Hence, we can define the *Golden reflection* with respect to v as

$$\Phi_v = \frac{1}{2}\left(I_E + \sqrt{5} r_v\right) \tag{2.11}$$

and then v is an eigenvector of Φ_v with the corresponding eigenvalue $1 - \phi$. Also, the Lemma from the cited book yields that for $X \in O(E, \langle,\rangle)$ = the orthogonal group of E

$$X\Phi_v X^{-1} = \Phi_{X(v)} \tag{2.12}$$

An explicit expression of this linear transformation is

$$\Phi_v(x) = \phi x - \frac{\sqrt{5}\langle x, v\rangle}{\langle v, v\rangle} v \tag{2.13}$$

Example 2.4 (*Triple structures in terms of Golden structures*). Let, after [3], F and P be two $(1,1)$-tensor fields on the manifold M. With the triple $(F, P, J = P \circ F)$ we can form the following four structures:

(1) $F^2 = P^2 = I$ and $P \circ F - F \circ P = 0$; then $J^2 = I$,
(2) $F^2 = P^2 = I$ and $P \circ F + F \circ P = 0$; then $J^2 = -I$,
(3) $F^2 = P^2 = -I$ and $P \circ F - F \circ P = 0$; then $J^2 = I$,
(4) $F^2 = P^2 = -I$ and $P \circ F + F \circ P = 0$; then $J^2 = -I$,

called, respectively, *almost hyperproduct* (ahp), *almost biproduct complex* (abpc), *almost product bicomplex* (apbc), and *almost hypercomplex* (ahc).

Let us associate Φ_F, Φ_P, Φ_J after the algorithm (1.6); hence

$$\sqrt{5}\Phi_J = 2\Phi_P\Phi_F - \Phi_P - \Phi_F + \phi I \tag{2.14}$$

and the triple (Φ_F, Φ_P, Φ_J) is:

(1′) An (ahp)-structure if and only if: Φ_F, Φ_P are Golden structures and $\Phi_P\Phi_F = \Phi_F\Phi_P$; then Φ_J is a Golden structure.
(2′) An (abpc)-structure if and only if: Φ_F, Φ_P are Golden structures and $4(\Phi_P\Phi_F + \Phi_F\Phi_P) = 2(\Phi_P + \Phi_F) - I$; then Φ_J is a complex Golden structure.

(3′) An (apbc)-structure if and only if: Φ_F, Φ_P are complex Golden structures and $\Phi_P\Phi_F = \Phi_F\Phi_P$; then Φ_J is a Golden structure.
(4′) An (ahc)-structure if and only if: Φ_F, Φ_P are complex Golden structures and $4(\Phi_P\Phi_F + \Phi_F\Phi_P) = 2(\Phi_P + \Phi_F) - I$; then Φ_J is a complex Golden structure.

Example 2.5 (*Golden structures from symplectic distributions*). Recall, after [4, p. 272], that if W is a subspace of a symplectic vector space (V, σ) then $V = W + W^\sigma$, where $W^\sigma = \{v \in V; \sigma(v, w) = 0 \text{ for every } w \in W\}$. Also [4, p. 273] the given subspace W is called *symplectic* if $\sigma|_{W\times W}$ is non-degenerate and then [4, p. 274] $W \cap W^\sigma = \{0\}$. In conclusion, given any symplectic distribution R on a symplectic manifold (M, σ) (i.e. R_x is a symplectic subspace of T_xM, for all $x \in M$), we get another symplectic distribution $S = R^\sigma$, complementary to R. Denoting r, s the corresponding projectors one has that $P = r - s$ is an almost product structure and then there exists an associated *symplectic Golden structure* (1.6)

$$\Phi = \Phi_R = \phi r + (1 - \phi)s \tag{2.15}$$

3. Connections as Golden structures

3.1. Connections in principal fibre bundles

Let $P(M, G)$ be a G-principal fibre bundle with $V = \ker \pi_*$ ($\pi : P \to M$ being the fibre projection) the vertical distribution on P and fix a connection on P given by a complementary distribution H, i.e. $TP = V \oplus H$ and H is G-invariant. Then, denoting by v and h the corresponding projectors, the $(1, 1)$ tensor field

$$F = v - h \tag{3.1}$$

is an almost product structure on P. In [2] a converse result is proved: an almost product structure F on P is associated to a connection if and only if the following relations hold:

(1) $F(X) = X \Longleftrightarrow X \in V$,
(2) $dR_a \circ F_u = F_{ua} \circ dR_a$ for every $a \in G$ and $u \in P$.

Considering the process (1.6) we derive:

Proposition 3.1. *A Golden structure Φ on P represents a connection if and only if the following conditions are satisfied:*

(1′) *$X \in \mathcal{X}(P)$ (=the Lie algebra of vector fields on P) is an eigenvalue of Φ with respect to the eigenvalue $\phi \Longleftrightarrow X$ is a vertical vector field.*
(2′) *$dR_a \circ \Phi_u = \Phi_{ua} \circ dR_a$ for every $a \in G$ and $u \in P$ with I the identity on $\mathcal{X}(P)$.*

Let $\omega \in \Lambda^1(P, \mathfrak{g})$ be the connection 1-form of H, where $\mathfrak{g}$ is the Lie algebra of G. Since ω is an eigenform of F with respect to the eigenvalue +1, [2], it results that ω is an eigenform of $\Phi = \Phi_F$ with respect to the eigenvalue ϕ. Also, taking into account $\Omega \in \Lambda^2(P, \mathfrak{g})$ the curvature form of ω, the remarkable relation

$$\Omega(X, Y) = -\frac{1}{2}\omega(N_F(X, Y)) \tag{3.2}$$

holds [2], where N_F is the Nijenhuis tensor of F. Since $N_I = 0$ it results

$$N_F = \frac{4}{5}N_\Phi \tag{3.3}$$

and then

$$\Omega(X, Y) = -\frac{2}{5}\omega(N_\Omega(X, Y)) \tag{3.4}$$

which implies

Proposition 3.2. *The connection is flat ($\Omega \equiv 0$) if and only if the associated Golden structure is integrable ($N_\Omega \equiv 0$).*

The given connection yields a lift $l_\omega : \mathcal{X}(M) \to \mathcal{X}(P)$ satisfying

$$[l_\omega X^*, l_\omega Y] - l_\omega[X^*, Y^*] = N_F(l_\omega X^*, l_\omega Y^*)$$

for every $X^*, Y^* \in \mathcal{X}(M)$ [2, relation (27)], and therefore:

Proposition 3.3. *The lift defined by ω is a morphism of Lie algebras if and only if the associated Golden structure is integrable.*

3.2. Connections in tangent bundles

Let $\pi : TM \to M$ be the tangent bundle of the manifold M. The kernel of π_* is denoted $V(M)$ and is called *the vertical distribution* of M. For an atlas on TM with local coordinates $(x,y) = (x^i, y^i)_{1\leqslant i\leqslant n}$ with (x^i) coordinates on the base M the almost tangent structure of TM is $J = \frac{\partial}{\partial y^i} \otimes dx^i$, i.e.

$$J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}, \quad J\left(\frac{\partial}{\partial y^i}\right) = 0 \tag{3.5}$$

Definition 3.1. (1) A $(1,1)$-tensor field $v : \mathcal{X}(M) \to \mathcal{X}(M)$ satisfying

$$\begin{cases} J \circ v = 0 \\ v \circ J = J \end{cases} \tag{3.6}$$

is called *vertical projector*.

(2) A complementary distribution N to the vertical distribution $V(M)$

$$\mathcal{X}(M) = N \oplus V(M) \tag{3.7}$$

is called *normalization* or *horizontal distribution* or *non-linear connection*.

Since a vertical projector v is $C^\infty(M)$-linear with $\operatorname{im} v = V(M)$ we have:

Proposition 3.4. *A vertical projector v yields a non-linear connection denoted $N(v)$ through relation $N(v) = \ker v$.*

An important remark here is that the last result admits a converse. Namely, if N is a non-linear connection let h_N, v_N the horizontal and vertical projection with respect to the decomposition (3.7). Then

Proposition 3.5. *v_N is a vertical projector with $N(v_N) = N$.*

Proof. From $\operatorname{im} v_N = V(M) = \ker J$ it follows (3.6_1). v_N being projector satisfy $v_N(V(M)) = V(M) = \operatorname{im} J$ and then we have (3.6_2). The second fact is immediately from the definition of $N(v_N)$. □

With respect to the identification non-linear connection = vertical projector let us point another equivalent choice inspired by [13]:

Definition 3.2. A $(1,1)$-tensor field Γ is called *non-linear connection of almost product type* if

$$\begin{cases} \Gamma \circ J = -J \\ J \circ \Gamma = J \end{cases} \tag{3.8}$$

Proposition 3.6. *If Γ is a non-linear connection of almost product type then*

(i) *$v_\Gamma = \frac{1}{2}(1_{\mathcal{X}(M)} - \Gamma)$ is a vertical projector,*
(ii) *$V(M)$ is the (-1)-eigenspace of Γ while $N(v_\Gamma)$ is the $(+1)$-eigenspace of Γ.*

It results that every vertical projector v yields a non-linear connection of almost product type: $\Gamma = 1_{\mathcal{X}(M)} - 2v$. From this last relation it results $\Gamma^2 = 1_{\mathcal{X}(M)}$, i.e. Γ is an almost product structure on M (hence the name).

Proof

(i) $J \circ v_\Gamma = \frac{1}{2}(J - J \circ \Gamma) \overset{(3.8_2)}{=} \frac{1}{2}(J - J) = 0$ and $v_\Gamma \circ J = \frac{1}{2}(J - \Gamma \circ J) \overset{(3.8_1)}{=} \frac{1}{2}(J + J) = J$.
(ii) $V(M) = \operatorname{im} v_\Gamma = \{X \in \mathcal{X}(M); \Gamma(X) = -X\}$ and $N(v_\Gamma) = \ker v_\Gamma = \{X \in \mathcal{X}(M); \Gamma(X) = X\}$. □

It follows a description in terms of Golden structures:

Proposition 3.7. *A non-linear connection N on M, given by the vertical projector v, can be also defined by a Golden structure $\Phi(= \Phi_\Gamma)$*

$$\Phi = \phi 1_{\mathcal{X}(M)} - \sqrt{5} v \tag{3.9}$$

with N the ϕ-eigenspace and $V(M)$ the $(1 - \phi)$-eigenspace.

4. Integrability and parallelism of Golden structures

Since in the previous section the integrability of a Golden structure was a main condition in some remarkable results let us treat in details this question. Recall the Nijenhuis tensor of Φ

$$N_\Phi(X,Y) = \Phi^2[X,Y] + [\Phi X, \Phi Y] - \Phi[\Phi X, Y] - \Phi[X, \Phi Y] \tag{4.1}$$

and denotes the complementary distributions R, S on M corresponding to ϕ and $1-\phi$, respectively. Let r, s be the corresponding projections; it results

$$\begin{cases} r^2 = r, \quad s^2 = s \\ rs = sr = 0, \quad r + s = I \end{cases} \tag{4.2}$$

A straightforward computation, using the expression of the corresponding almost product structure via (1.6) gives

$$\begin{cases} r = \frac{1}{\sqrt{5}}\Phi - \frac{1-\phi}{\sqrt{5}} I \\ s = -\frac{1}{\sqrt{5}}\Phi + \frac{\phi}{\sqrt{5}} I \end{cases} \tag{4.3}$$

Recall also:

(i) Φ is *integrable* if $N_\Phi = 0$; from (3.3) it results that Φ is integrable if and only if the associated (1.6) almost product structure is integrable.
(ii) The distribution R is integrable if $s[rX, rY] = 0$ and S is integrable if $r[sX, sY] = 0$ for every vector fields X, Y on M.

From

$$\begin{cases} \Phi r = r\Phi = \phi r = \frac{\phi}{\sqrt{5}}\Phi + \frac{1}{\sqrt{5}} I \\ \Phi s = s\Phi = (1-\phi)s = \frac{\phi-1}{\sqrt{5}}\Phi - \frac{1}{\sqrt{5}} I \end{cases} \tag{4.4}$$

we get

$$\begin{cases} s[rX, rY] = \frac{1}{5} s N_\Phi(rX, rY) \\ r[sX, sY] = \frac{1}{5} r N_\Phi(sX, sY) \end{cases} \tag{4.5}$$

Proposition 4.1. *R is integrable if and only if $sN_\Phi(rX, rY) = 0$ and S is integrable if and only if $rN_\Phi(sX, sY) = 0$. If Φ is integrable then both the distributions R and S are integrable.*

For the rest of this section, let ∇ be a fixed linear connection on M. To the pair (Φ, ∇) we associate two other linear connections [1,18]:

(i) *The Schouten connection*

$$\overset{Sc}{\nabla}_X Y = r(\nabla_X rY) + s(\nabla_X sY) \tag{4.6}$$

(ii) *The Vrănceanu connection*

$$\overset{V}{\nabla}_X Y = r(\nabla_{rX} rY) + s(\nabla_{sX} sY) + r[sX, rY] + s[rX, sY] \tag{4.7}$$

The following two propositions, generalizations of similar results from [5], show the importance of these connections from the point of view of the splitting (4.2).

Proposition 4.2. *The projectors r, s are parallels with respect to Schouten and Vrănceanu connection for every linear connection ∇ on M and Φ is parallel with respect to Schouten and Vrănceanu connections.*

Proof. For every $X, Y \in \mathscr{X}(M)$

$$\left(\overset{Sc}{\nabla}_X r\right) Y = \overset{Sc}{\nabla}_X rY - r\left(\overset{Sc}{\nabla}_X Y\right) = r(\nabla_X rY) - r(\nabla_X rY) = 0$$
$$\left(\overset{V}{\nabla}_X r\right) Y = \overset{V}{\nabla}_X rY - r\left(\overset{V}{\nabla}_X Y\right) r(\nabla_{rX} rY) + r[sX, rY] - r(\nabla_{rX} rY) - r[sX, rY] = 0.$$

Similar relations hold for s.

From (4.3) it results that Φ is parallel with respect to Schouten and Vrănceanu connections. □

Recall that a distribution D on M is called *parallel with respect to the linear connection* ∇ if $X \in \mathscr{X}(M)$ and $Y \in D$ implies $\nabla_X Y \in D$.

Proposition 4.3. *The distributions R, S are parallel with respect to Schouten and Vrănceanu connection for every linear connection ∇ on M.*

Proof. Let $X \in \mathscr{X}(M)$ and $Y \in R$. Since, $sY = 0$ and $rY = Y$, we get

$$\overset{Sc}{\nabla}_X Y = r(\nabla_X Y) \in R$$

$$\overset{V}{\nabla}_X Y = r(\nabla_{rX} Y) + r[sX, Y] \in R$$

Similar relations hold for S. □

5. Golden Riemannian metrics

Recall that a *Riemannian almost product structure* is a pair (g, P) with g a fixed Riemannian metric on M and P an almost product structure related by

$$g(PX, PY) = g(X, Y) \tag{5.1}$$

or equivalenty, P is a g-symmetric endomorphism

$$g(PX, Y) = g(X, PY) \tag{5.2}$$

Proposition 5.1. *The operator P is a g-symmetric endomorphism if and only if the associated (1.6) Golden structure is so.*

Definition 5.1. A *Golden Riemannian structure* is a pair (g, Φ) with

$$g(\Phi X, Y) = g(X, \Phi Y) \tag{5.3}$$

The triple (M, g, Φ) is a *Golden Riemannian manifold.*

Corollary 5.1. *On a Golden Riemannian manifold:*

(i) *The projectors r, s are g-symmetric*

$$\begin{cases} g(rX, Y) = g(X, rY) \\ g(sX, Y) = g(X, sY) \end{cases} \tag{5.4}$$

(ii) *The distributions R, S are g-orthogonal*

$$g(rX, sY) = 0 \tag{5.5}$$

(iii) *The Golden structure is N_Φ-symmetric*

$$N_\Phi(\Phi X, Y) = N_\Phi(X, \Phi Y) \tag{5.6}$$

Proposition 5.2. *A Riemannian almost product structure is a locally product structure if P is parallel with respect to the Levi–Civita connection $\overset{g}{\nabla}$ of g, i.e. $\overset{g}{\nabla} P = 0$ and if ∇ is a symmetric linear connection then the Nijenhuis tensor of P verifies*

$$N_P(X, Y) = (\nabla_{PX} P)(Y) - (\nabla_{PY} P)(X) - P(\nabla_X P)(Y) + P(\nabla_Y P)(X) \tag{5.7}$$

Proposition 5.3. *On a locally product Golden Riemannian manifold the Golden structure Φ is integrable.*

Inspired by this result let us search the linear connections making parallel the given Golden structure:

Theorem 5.1. *The set of linear connections ∇ for which $\nabla \Phi = 0$ is*

$$\nabla_X Y = \frac{1}{5}\left[3\tilde{\nabla}_X Y + 2\Phi(\tilde{\nabla}_X \Phi Y) - \Phi(\tilde{\nabla}_X Y) - \tilde{\nabla}_X \Phi Y\right] + O_P Q(X, Y) \tag{5.8}$$

where $\tilde{\nabla}$ is an arbitrary fixed linear connection and Q is an $(1,2)$-tensor field for which $O_P Q$ is an associated Obata operator

$$O_P Q(X, Y) = \frac{1}{2}[Q(X, Y) + PQ(X, PY)] \tag{5.9}$$

for the corresponding almost product structure (1.7).

As final remark let us mention the applications of the results of this paper to the real plane:

Example 5.6

$$\begin{cases} R = \mathrm{Span}\{x\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\} \\ S = \mathrm{Span}\{\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\} \end{cases} \tag{5.10}$$

A straightforward computation gives that R and S are globally defined complementary distributions orthogonal with respect to the Euclidean metric of $\mathbb{R}^2$. These distributions are associated to the Golden structure

$$\begin{cases} \Phi\left(\frac{\partial}{\partial x}\right) = \frac{\phi x^2 + (1-\phi)}{x^2+1}\frac{\partial}{\partial x} + \frac{\sqrt{5}x}{x^2+1}\frac{\partial}{\partial y} \\ \Phi\left(\frac{\partial}{\partial y}\right) = \frac{\sqrt{5}x}{x^2+1}\frac{\partial}{\partial x} + \frac{(1-\phi)x^2+\phi}{x^2+1}\frac{\partial}{\partial y} \end{cases} \tag{5.11}$$

which is integrable since $N_\Phi\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = 0$.

6. Conclusions and open problems

This paper has answered a number of questions on the application of the Golden proportion in differential geometry by studying the properties of a Golden structure defined on a manifold. A number of things were proved with respect to the Golden ratio.

It may be possible to extend these results to other if we consider a polynomial structure with the structure polynomial $Q(X) = X^{p+1} - X^p - I$, (given by the Golden algebraic equations $x^{p+1} = x^p + 1$ [28]), where p takes its values from the set $0, 1, 2, 3, \ldots$. For $p = 1$, the given equation is reduced to the well-known Golden Proportion equation $x^2 = x + 1$.

The investigation of Golden differential geometry is in initial stage. We hope that the current work contributes to motivate this research in both mathematics and physics according to [6,27].

References

[1] Bejancu A, Farran HR. Foliations and Geometric Structures. Mathematics and its applications, vol. 580. Springer; 2006.
[2] Cruceanu V. Sur la structure presque-produit associée à une connexion sur un espace fibré. An. St. Univ. Al. I. Cuza, Iaşi, Mat 1969;15(1):159–67 [MR 41 #950].
[3] Cruceanu V. On almost biproduct complex manifolds. An. St. Univ. Al. I. Cuza, Iaşi, Mat 2006;52(1):5–24 [MR 2007j:53033].
[4] Cushman RH, Bates LM. Global aspects of classical integrable systems. Basel: Birkhäuser; 1997 [MR 98a: 58083].
[5] Demetropoulou-Psomopoulou D. Linear connections on manifolds admitting an $f(2\nu+3,-1)$-structure or an $f(2\nu+3,1)$-structure. Tensor 1988;47(3):235–9 [MR 91b: 53037].
[6] El Nashie MS. Quantum mechanics and the possibility of a Cantorian spacetime. Chaos, Solitons & Fractals 1992;1(5):485–7.
[7] El Naschie MS. Kleinian groups in $E^{(\infty)}$ and their connection to particle physics and cosmology. Chaos, Solitons & Fractals 2003;16(4):637–49.
[8] El Naschie MS. The VAK of vacuum fluctuation: spontaneous self-organization and complexity theory interpretation of high energy particle physics and the mass spectrum. Chaos, Solitons & Fractals 2003;18(2):401–20.
[9] El Naschie MS. The concepts of E infinity: an elementary introduction to the Cantorian-fractal theory of quantum physics. Chaos, Solitons & Fractals 2004;22(2):495–511.
[10] Fischler RH. The shape of the Great Pyramid. Waterloo: Wilfrid Laurier University Press; 2000.
[11] Goldberg SI, Yano K. Polynomial structures on manifolds. Kodai Math Sem Rep 1970;22:199–218 [MR 42 #2380].
[12] Goldberg SI, Petridids NC. Differentiable solutions of algebraic equations on manifolds. Kodai Math Sem Rep 1973;25:111–28 [MR 47 #4176].
[13] Grifone J. Structure presque-tangente et connexions I, II. Ann Inst Fourier (Grenoble) 1972;22(13):287–334 [MR 0336636 (49 #1409)], p. 291–338 [MR 0341361 (49 #6112)].
[14] Harman CT. Complex Fibonacci numbers. Fibonacci Quart 1981;19:82–6 [MR 82h:10015].
[15] Herz-Fischler R. A mathematical history of division in extreme and mean ratio. Waterloo, Ontario, Canada: Wilfrid Laurier University Press; 1987.
[16] Heyrovska R. The Golden ratio ionic and atomic radii and bond lengths. Mol Phys 2005;103:877–82.
[17] Hreţcanu CE. Submanifolds in Riemannian manifold with Golden Structure. In: Workshop on Finsler geometry and its applications, Hungary; 2007.
[18] Ianuş S. Sur les structures presque produit des varietes a connection lineaire. C.R.A.S., Paris 1971;272:734–5 [MR 43 #3955].
[19] Livio M. The Golden Ratio: The Story of phi, the World's Most Astonishing Number, Broadway; 2002 [MR 2003k:11025].
[20] Marek-Crnjac L. On the mass spectrum of the elementary particles of the standard model using El Naschie's Golden field theory. Chaos, Solitons & Fractals 2003;15(4):611–8.
[21] Marek-Crnjac L. The Golden mean in the topology of four-manifolds in conformal field theory in the mathematical probability theory and in Cantorian spacetime. Chaos, Solitons & Fractals 2006;28(5):1113–8.
[22] Marek-Crnjac L. Periodic continued fraction representations of different quarks mass ratios. Chaos, Solitons & Fractals 2005;25:807–14.
[23] Miron R, Anastasiei M. The geometry of Lagrange spaces: theory and applications. Kluwer Academic Publishers, FTPH No. 59; 1994 [MR 95f: 53120].
[24] Procesi C. Lie groups: an approach through invariants and representations. Universitext, Springer; 2007 [MR 2007j:22016].
[25] Sigalotti LDiG, Mejias A. The Golden ratio in special relativity. Chaos, Solitons & Fractals 2006;30:521–4.
[26] Stakhov AP. The Golden section in the measurement theory. Comput Math Appl 1989;17(46):613–38.
[27] Stakhov AP. Fundamentals of a new kind of mathematics based on the Golden section. Chaos, Solitons & Fractals 2006;27(5):1124–46.
[28] Stakhov AP, Rozin B. The Golden algebraic equations. Chaos Solitons & Fractals 2006;27(5):1415–21.