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Applications of the Golden Ratio on Riemannian Manifolds

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Abstract

The Golden Ratio is a fascinating topic that continually generates new ideas. The main purpose of the present paper is to point out and find some applications of the Golden Ratio and of Fibonacci numbers in Differential Geometry. We study a structure defined on a class of Riemannian manifolds, called by us a Golden Structure. A Riemannian manifold endowed with a Golden Structure will be called a Golden Riemannian manifold. Precisely, we say that an (1,1)-tensor field $\widetilde{P}$ on a m-dimensional Riemannian manifold $(\widetilde{M}, \widetilde{g})$ is a Golden Structure if it satisfies the equation $\widetilde{P}^2 = \widetilde{P} + I$ (which is similar to that satisfied by the Golden Ratio ϕ) where I stands for the (1,1) identity tensor field. First, we establish several properties of the Golden Structure. Then we show that a Golden Structure induces on every invariant submanifold a Golden Structure, too. This fact is illustrated on a product of spheres in an Euclidean space.

Key Words: Riemannian manifold, Golden Structure, induced structures on submanifolds, Golden Ratio.

1. Introduction

The Golden Ratio (which sometimes is called "Golden Number", "Golden Section", "Golden Proportion" or "Golden Mean") has occupied an important place since antiquity in many parts of geometry, architecture, music, art and philosophies, being a symbol of great fascination to ancient and modern geometry. The Great Pyramid of Giza, built around 2560 BC, is one of the earliest examples of the use of this ratio([5]).

The Greeks usually attributed the discovery of the Golden Ratio to Pythagoras or his followers ([18]). In the Elements, Euclid of Alexandria (around 300 BC), provides the first known written definition of the Golden Ratio like a proportion derived from a division of a line into what he calls its "extreme and mean ratio". Euclid's definition states: "A straight line is said to have been cut in extreme and mean ratio when, as the whole line is to the greater segment, so is the greater to the lesser" ([16]).

The Fibonacci sequence (generated by the rule $f_{n+1} = f_n + f_{n-1}$ for every integer $n \geq 1$, with $f_0 = 0, f_1 = 1$) is well known in many different areas of mathematics and it is closely related to the Golden Ratio (in the sense that the ratio of successive pairs of the Fibonacci numbers tends to the Golden Ratio)([4]).

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Fibonacci $(1170-1250)$ mentioned this numerical sequence, now named after him, in his Liber Abaci. Fra Luca Pacioli $(1445-1517)$ published Divina Proportione (in Venice in 1509) and a detailed summary of the properties of the Golden Ratio is contained in the first of the three volume text. Golden Ratio was described by Johannes Kepler $(1571-1630)$ as "one of the two great treasures of geometry" (the other one is the Theorem of Pythagoras)([18]).

The Golden Ratio arises as a result of the solution regarding the division problem of the line segment AB with a point C (which belongs to the segment AB) in the ratio $\frac{AC}{CB}=\frac{AB}{AC}$. If we denote $\frac{AC}{CB}=x$, then the problem is reduced to the solution of the algebraic equation: $x^2=x+1$ ([6], [17]). The positive root of this equation is equal to the value of the Golden Ratio, denoted by ϕ, the first Greek letter in the name of Phidias, the Greek sculptor who lived around 450 BC ([18]).

The existence of the Golden Ratio in any place where life and beauty are present, has made us wonder how it can be used to approach new objects in Riemannian Geometry and what kinds of structures on manifolds can be obtained in this way. The idea to constructing a structure on a Riemannian manifold, called by us a Golden Structure, is based on several results from geometrical structures constructed on Riemannian manifolds ([1],[7],[8],[11],[13],[14]). Kentaro Yano introduced the notion of an f-structure ([15]). Extending this structure, Goldberg and Yano ([7]) introduced the notion of the polynomial structure on a manifold, as a C^{∞} tensor field f of type (1,1) defined on a differentiable manifold N, such that the algebraic equation is satisfied:

$$Q(x)=x^n+a_nx^{n-1}+...+a_2x+a_1I=0, \tag{1.1}$$

where I is the identity mapping and (for $x=f$) $f^{n-1}(p), f^{n-2}(p), ..., f(p), I$ are linearly independent in every point $p\in N$. The polynomial $Q(x)$ is called the structure polynomial.

For $Q(x)=x^2+I$ (or $Q(x)=x^2-I$) we obtain an almost complex structure (respectively, an almost product structure).

An almost product structure on a differentiable manifold N is determined by a system of differentiable distributions $T_1, T_2, ..., T_k$ so that the tangent space of N has the form ([8])

$$T(p)=T_1(p)+T_2(p)+...+T_k(p),\quad T_i(p)\bigcap T_j(p)=0,\ i\neq j \tag{1.2}$$

in every point $p\in N$. This structure is defined by a system of C^{∞} tensor fields of type $(1,1)$ on M, called projectors, given by the relation

$$\pi_i(p):T(p)\longrightarrow T_i(p),\quad \sum_{i=1}^{k}\pi_i=I,\quad \pi_i\pi_j=\delta_j^i\pi_i, \tag{1.3}$$

for every $i\in\{1,...,k\}$, where δ_j^i are the Kronecker symbols. The distributions T_i (for $i\in\{1,...,k\}$) are the basic distributions of the structure.

In this paper, we define a Golden Structure as a polynomial structure with the structure polynomial $Q(x)=x^2-x-I$. In Section 2 we establish several properties of the Golden Structure (also, we studied some properties of this structure in [3] and [10]). In Section 3, we give some properties of the induced structure on a submanifold in a Golden Riemannian manifold and we find a necessary and sufficient condition for this kind of submanifold to be a Golden Riemannian manifold. In Section 4 we give an example of Golden Structure on Euclidean manifold and we construct the induced structure on a product of spheres in Euclidean space.

2. Golden Riemannian Structure

In this section we define a polynomial structure on a m-dimensional Riemannian manifold $(\widetilde{M}, \widetilde{g})$, called by us a Golden Structure ([3], [10]), determined by an (1,1)-tensor field $\widetilde{P}$ which satisfies the equation:

$$\widetilde{P}^2 = \widetilde{P} + I, \tag{2.1}$$

where I is the identity operator on the Lie algebra $\chi(\widetilde{M})$ of vector fields on $\widetilde{M}$. We say that the metric $\widetilde{g}$ is $\widetilde{P}$-compatible if the equality

$$\widetilde{g}(\widetilde{P}(U), V) = \widetilde{g}(U, \widetilde{P}(V)) \tag{2.2}$$

is satisfied for every tangent vector fields $U, V \in \chi(\widetilde{M})$.

Remark 2.1 For a Golden Structure $\widetilde{P}$, defined on a Riemannian manifold $(\widetilde{M}, \widetilde{g})$, the condition (2.2) is equivalent with

$$\widetilde{g}(\widetilde{P}(U), \widetilde{P}(V)) = \widetilde{g}(\widetilde{P}(U), V) + \widetilde{g}(U, V), \tag{2.3}$$

for every tangent vector fields $U, V \in \chi(\widetilde{M})$.

Definition 2.1 *([10]) A Riemannian manifold $(\widetilde{M}, \widetilde{g})$, endowed with a Golden Structure $\widetilde{P}$ so that the Riemannian metric $\widetilde{g}$ is $\widetilde{P}$-compatible is named a Golden Riemannian manifold and $(\widetilde{g}, \widetilde{P})$ is named a Golden Riemannian structure on $\widetilde{M}$.*

Proposition 2.1 *([3]) A Golden Riemannian manifold $(\widetilde{M}, \widetilde{g}, \widetilde{P})$ has the property*

$$\widetilde{P}^n = f_n \widetilde{P} + f_{n-1} I \tag{2.4}$$

for every integer number $n > 0$, where $(f_n)_n$ is the Fibonacci sequence.

Proof. From (2.2) we obtain $\widetilde{P}^3 = 2\widetilde{P} + I$. Generally, if we suppose that $\widetilde{P}^k = f_k \widetilde{P} + f_{k-1} I$ $(k > 0)$, then we have

$$\widetilde{P}^{k+1} = f_k \widetilde{P}^2 + f_{k-1} \widetilde{P} = (f_k + f_{k-1}) \widetilde{P} + f_k I,$$

thus we obtain (2.4). □

Remark 2.2 Using an explicit expression for the Fibonacci sequence (Binet's formula [12]):

$$f_n = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}}, \tag{2.5}$$

we obtain a new form for the equality (2.4) the relation

$$\widetilde{P}^n = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}}\widetilde{P} + \frac{\phi^{n-1} - (1-\phi)^{n-1}}{\sqrt{5}}I, \tag{2.6}$$

for every natural number $n > 1$.

Proposition 2.2 *The Golden Structure* $\widetilde{P}$*, defined on a m-dimensional Riemannian manifold* $(\widetilde{M}, \widetilde{g})$*, is an isomorphism on the tangent space of the manifold* $\widetilde{M}$*,* $T_x\widetilde{M}$*, for every* $x \in \widetilde{M}$*.*

Proof. Denoting by $ker\widetilde{P} = \{V \in T_x\widetilde{M} \ : \ \widetilde{P}V = 0, (\forall)x \in \widetilde{M}\}$ we obtain $\widetilde{P}^2V = \widetilde{P}V + V$, thus $ker\widetilde{P} = \{0\}$ and from this we remark that $\widetilde{P}$ is an isomorphism on $T_x\widetilde{M}$, for every $x \in \widetilde{M}$. □

Proposition 2.3 *The eigenvalues of the Golden Structure* $\widetilde{P}$ *defined on a m-dimensional Riemannian manifold* $(\widetilde{M}, \widetilde{g})$ *are the Golden Ratio* ϕ *and* $(1-\phi)$*.*

Proof. If λ is an eigenvalue of the Golden Structure $\widetilde{P}$ on T_xM (for every $x \in \widetilde{M}$) then we have $\widetilde{P}V = \lambda V$ for every tangent vector fields $V \in T_x\widetilde{M}$ and every point $x \in \widetilde{M}$. From this we obtain $\lambda^2 = \lambda + 1$ and it follows that the eigenvalues of $\widetilde{P}$ are the Golden Ratio $\lambda_1 = \phi$ and $\lambda_2 = 1-\phi$. □

Proposition 2.4 *The Trace of the Golden Structure* $\widetilde{P}$ *defined on a m-dimensional Riemannian manifold* $(\widetilde{M}, \widetilde{g})$ *has the property*

$$trace(\widetilde{P}^2) = trace(\widetilde{P}) + m. \tag{2.7}$$

Proof. Denoting by $\{E_1, E_2, ..., E_m\}$ a local orthonormal basis of the tangent space $T_x\widetilde{M}$ in a point $x \in \widetilde{M}$, from (2.1) we obtain

$$\widetilde{g}(\widetilde{P}^2E_i, E_i) = \widetilde{g}(\widetilde{P}E_i, E_i) + \widetilde{g}(E_i, E_i).$$

and summing by i we obtain (2.7). □

Definition 2.2 *([15]) If a m-dimensional Riemannian manifold* $\widetilde{M}$*, endowed with a positive definite Riemannian metric* $\widetilde{g}$*, admits a non-trivial tensor field* F *of type* $(1,1)$ *such that* $F^2 = I$ *and* $\widetilde{g}(FX, FY) = \widetilde{g}(X, Y)$ *for every vector fields* $X, Y \in \chi(\widetilde{M})$*, then* F *is called an almost product structure and* $(\widetilde{M}, \widetilde{g}, F)$ *is called an almost product Riemannian manifold.*

It is well known ([8]) that a polynomial structure on a differentiable manifold M, defined by a C^∞ tensor field of type (1,1), induces an almost product structure on M. The number of distributions of the almost product structure is equal to the number of distinct irreducible factors over $\mathbb{R}$ of the structure polynomial and the projectors are expressed as polynomials in f.

Theorem 2.1 *([3]) Every almost product structure F on a m-dimensional Riemannian manifold $(\widetilde{M},\widetilde{g})$ induces two Golden Structures on $(\widetilde{M},\widetilde{g})$, given as follows:*

$$\widetilde{P}_1 = \frac{I+\sqrt{5}F}{2}, \quad \widetilde{P}_2 = \frac{I-\sqrt{5}F}{2} \tag{2.8}$$

Conversely, every Golden Structure $\widetilde{P}$ defined on a Riemannian manifold $(\widetilde{M},\widetilde{g})$ induces an almost product structure on this manifold.

Proof. We try to write the almost product structure F defined on a m-dimensional Riemannian manifold $(\widetilde{M},\widetilde{g})$, using a Golden Structure $\widetilde{P}$, in the form $F = a\widetilde{P}+bI$, where $a,b\in\mathbb{R}^*$. Thus $F^2 = a^2\widetilde{P}^2+2ab\widetilde{P}+b^2I$ and using that $F^2 = I$ and $\widetilde{P}^2 = \widetilde{P}+I$ we obtain the formulae (2.8). Moreover, we have

$$\widetilde{g}(\widetilde{P}_i(U),V) = \widetilde{g}(U,\widetilde{P}_i(V)) \Longleftrightarrow \widetilde{g}(\widetilde{P}(X),Y) = \widetilde{g}(X,\widetilde{P}(Y))$$

for every $i\in\{1,2\}$ and for every tangent vector fields $U,V\in\chi(\widetilde{M})$. □

On a Golden Riemannian manifold $(\widetilde{M},\widetilde{g},\widetilde{P})$, we can define two projection operators as

$$l = \frac{1}{\sqrt{5}}(\phi I-\widetilde{P}), \quad m = \frac{1}{\sqrt{5}}((\phi-1)I+\widetilde{P}). \tag{2.9}$$

We can find immediately that

Proposition 2.5 *([3]) On a Golden Riemannian manifold $(\widetilde{M},\widetilde{g},\widetilde{P})$, projector operators l and m defined in (2.9) verify that:*

$$l+m = I, \quad l^2 = l, \quad m^2 = m, \tag{2.10}$$

and

$$\widetilde{P}\circ l = l\circ\widetilde{P} = (1-\phi)l, \quad \widetilde{P}\circ m = m\circ\widetilde{P} = \phi m. \tag{2.11}$$

Remark 2.3 From (2.10) we obtain that there are two complementary distributions $\mathcal{D}_l$ and $\mathcal{D}_m$, corresponding to the projection operators l and m, respectively, on a Golden Riemannian manifold $(\widetilde{M},\widetilde{g},\widetilde{P})$.

3. Submanifolds in Golden Riemannian Manifolds

Let M be a n-dimensional submanifold of codimension r, isometrically immersed in a m-dimensional Golden Riemannian manifold $(\widetilde{M},\widetilde{g},\widetilde{P})$ (where $m,n,r\in\mathbb{N}$, $n+r=m\geq 2$). We denote by $T_x(M)$ the tangent space of M in a point $x\in M$ and by $T_x(M)^\perp$ the normal space of M in x, for every $x\in M$. Let i_* the differential of the immersion $i: M\to\widetilde{M}$. The induced Riemannian metric g of M is given by

$$g(X,Y) = \widetilde{g}(i_*X,i_*Y), \tag{3.1}$$

for every $X, Y \in \chi(M)$. We consider a local orthonormal basis $\{N_1, ..., N_r\}$ of the normal space $T_x(M)^{\perp}$ in a point $x \in M$. We suppose that the indices verify that: $\alpha, \beta, \gamma ... \in \{1, ..., r\}$ and $i, j, k ... \in \{1, ..., n\}$.

For every $X \in T_x(M)$, $\widetilde{P}(i_* X)$ and $\widetilde{P}(N_\alpha)$ can be decomposed in tangential and normal components at M as

$$\widetilde{P}(i_* X) = i_*(P(X)) + \sum_{\alpha=1}^{r} u_\alpha(X) N_\alpha, \ (\forall) X \in \chi(M) \tag{3.2}$$

and

$$\widetilde{P}(N_\alpha) = \varepsilon i_*(\xi_\alpha) + \sum_{\beta=1}^{r} a_{\alpha\beta} N_\beta, \quad (\varepsilon = \pm 1), \tag{3.3}$$

where P is an (1,1)-tensor field on M, ξ_α are tangent vector fields on M, u_α are 1-forms on M and $(a_{\alpha\beta})_r$ is a $r \times r$ matrix of real functions on M.

Theorem 3.1 *([10]) If M is a n-dimensional submanifold of codimension r, isometrically immersed in a Golden Riemannian manifold* $(\widetilde{M}, \widetilde{g}, \widetilde{P})$, *then the structure* $(P, g, u_\alpha, \varepsilon\xi_\alpha, (a_{\alpha\beta})_r)$, *induced on M by the Golden Structure* $\widetilde{P}$, *verifies these equalities:*

$$\begin{cases} (i) \quad P^2(X) = P(X) + X - \varepsilon \sum_\alpha u_\alpha(X)\xi_\alpha, \\ (ii) \quad u_\alpha(P(X)) = (1 - a_{\alpha\alpha}) u_\alpha(X), \\ (iii) \quad a_{\alpha\beta} = a_{\beta\alpha}, \\ (iv) \quad u_\beta(\xi_\alpha) = \varepsilon(\delta_{\alpha\beta} + a_{\alpha\beta} - \sum_\gamma a_{\alpha\gamma} a_{\gamma\beta}), \\ (v) \quad P(\xi_\alpha) = \xi_\alpha - \sum_\beta a_{\alpha\beta}\xi_\beta \end{cases} \tag{3.4}$$

and

$$\begin{cases} (i) \quad u_\alpha(X) = \varepsilon g(X, \xi_\alpha), \\ (ii) \quad g(P(X), Y) = g(X, P(Y)), \\ (iii) g(P(X), P(Y)) = g(P(X), Y) + g(X, Y) - \sum_\alpha u_\alpha(X) u_\alpha(Y) \end{cases} \tag{3.5}$$

for every $X, Y \in \chi(M)$.

Proof. Applying $\widetilde{P}$ in (3.2) we have

$$\widetilde{P}^2(i_* X) = \widetilde{P}(i_* P(X)) + \sum_{\alpha=1}^{r} u_\alpha(X) \widetilde{P}(N_\alpha),$$

for every $X \in \chi(M)$. Thus, we obtain

$$i_*(P(X)) + \sum_\alpha u_\alpha(X) N_\alpha + i_* X = i_* P^2(X) + \sum_\alpha u_\alpha(P(X)) N_\alpha + \sum_\alpha u_\alpha(X)(\varepsilon i_*(\xi_\alpha) + \sum_\beta a_{\alpha\beta} N_\beta),$$

for every $X \in \chi(M)$. Equalizing the tangential and normal parts, respectively, from the last equality, we obtain the relations (i) and (ii) from (3.4). The equality (i) has the equivalent form $P^2 = P+I-\varepsilon\sum_\alpha u_\alpha\otimes\xi_\alpha$. Applying the compatibility relation (2.2) for the normal vector fields N_α and N_β we have $\widetilde{g}(\varepsilon i_*(\xi_\alpha)+\sum_{\gamma=1} a_{\alpha\gamma}N_\gamma, N_\beta) = \widetilde{g}(N_\alpha, \varepsilon i_*(\xi_\beta)+\sum_{\gamma=1} a_{\beta\gamma}N_\gamma)$, and from this we obtain the relation (iii) from (3.4). Applying (2.1) to the normal vector field N_α we have $\widetilde{P}^2(N_\alpha) = \widetilde{P}(N_\alpha) + N_\alpha$ and from (3.3), it follows that

$$\widetilde{P}(\varepsilon i_*(\xi_\alpha) + \sum_\gamma a_{\alpha\gamma}N_\gamma) = \varepsilon i_*(\xi_\alpha) + \sum_\beta a_{\alpha\beta}N_\beta + N_\alpha.$$

So, we have

$$\varepsilon(i_*P(\xi_\alpha) + \sum_\beta u_\beta(\xi_\alpha)N_\beta) + \sum_\gamma a_{\alpha\gamma}(\varepsilon i_*(\xi_\gamma) + \sum_\beta a_{\gamma\beta}N_\beta) = \varepsilon i_*(\xi_\alpha) + \sum_\beta a_{\alpha\beta}N_\beta + N_\alpha.$$

Equalizing the tangential and normal parts respectively from the last equality we obtain the relations (v) and (iv) from (3.4). From (2.2) we have $\widetilde{g}(\widetilde{P}(X), N_\alpha) = \widetilde{g}(X, \widetilde{P}(N_\alpha))$ which follows that

$$\widetilde{g}(i_*P(X) + \sum_\beta u_\beta(X)N_\beta, N_\alpha) = \widetilde{g}(X, \varepsilon i_*\xi_\alpha + \sum_\beta a_{\alpha\beta}N_\beta)$$

for every $X \in \chi(M)$ and $N_\alpha \in T_x(M)^\perp$ (in every $x \in M$) and from the last equality we obtain the relation (i) from (3.5). From (3.1) and (3.2) we have

$$g(P(X), Y) - g(X, P(Y)) = \widetilde{g}(i_*P(X), i_*Y) - \widetilde{g}(i_*X, i_*P(Y)) =$$

$$= \widetilde{g}(\widetilde{P}(i_*X) - \sum_\alpha u_\alpha(X)N_\alpha, i_*Y) - \widetilde{g}(i_*X, \widetilde{P}(i_*Y) - \sum_\beta u_\beta(Y)N_\beta) =$$

$$= \widetilde{g}(\widetilde{P}(i_*X), i_*Y) - \widetilde{g}(i_*X, \widetilde{P}(i_*Y)) = 0,$$

and from this we obtain (ii) from (3.5). Applying (3.1) to the vector fields $X, P(Y) \in \chi(M)$, we have

$$g(P(X), P(Y)) = g(P^2(X), Y) = g(P(X), Y) + g(X, Y) - \varepsilon\sum_\alpha u_\alpha(X)g(\xi_\alpha, Y)$$

and from the equality (3.5)(i) we obtain (iii) from (3.5). □

Remark 3.1 If M is a n-dimensional invariant submanifold of codimension r (i.e. $\widetilde{P}(T_x(M)) \subseteq T_x(M)$), isometrically immersed in a Golden Riemannian manifold $(\widetilde{M}, \widetilde{g}, \widetilde{P})$, then ξ_α ($\alpha \in \{1, 2, ..., r\}$) are zero vector fields and the 1-forms u_α vanishes identically on M ($u_\alpha(X) = g(X, \xi_\alpha) = 0$). Consequently, (3.2) and (3.3) are respectively written as

$$\widetilde{P}(i_*X) = i_*(P(X)), \quad \widetilde{P}(N_\alpha) = \sum_\beta a_{\alpha\beta}N_\beta, \tag{3.6}$$

for every $X \in \chi(M)$ and $\alpha \in \{1,2,...,r\}$. In this situation the properties of the structure elements $P, g, u_\alpha, \varepsilon\xi_\alpha, (a_{\alpha\beta})_r$, verifies that ([10]):

$$\begin{cases} (i) \quad P^2(X) = P(X) + X, \\ (ii) \quad a_{\alpha\beta} = a_{\beta\alpha}, \\ (iii) \sum_\gamma a_{\alpha\gamma}a_{\gamma\beta} = a_{\alpha\beta} + \delta_{\alpha\beta}, \\ (iv) \quad g(P(X),Y) = g(X,P(Y)), \\ (v) \quad g(P(X),P(Y)) = g(P(X),Y) + g(X,Y), \end{cases} \tag{3.7}$$

for every $X, Y \in \chi(M)$ and $\alpha, \beta \in \{1,2,...,r\}$.

In the following considerations we suppose that the tangent vector fields $\xi_1, \xi_2, ..., \xi_r$ are linearly independent. In this situation, the 1-forms $u_1, ..., u_r$ are also, linearly independent.

In $T_x(M)$, we denote by $V(\xi_\alpha)$ a r-dimensional vector space spanned by ξ_α. When $r < n$, let η_A be the eigenvectors of P, which are perpendicular on $V(\xi_\alpha)$ and mutually orthogonal, for every $A \in \{r+1,...,n\}$. The eigenvalues of P corresponding to the eigenvalues η_A are $\lambda_A \in \{\phi; 1-\phi\}$, for every $A \in \{r+1,...,n\}$.

Next, if we take an eigenvector ξ of P in the vector space $V(\xi_\alpha)$, with the corresponding eigenvalue σ, then $P(\xi) = \sigma\xi$. Since $\xi = \sum_\alpha c_\alpha \xi_\alpha$ then $P(\xi) = \sigma \sum_\alpha c_\alpha \xi_\alpha$. From (3.4)(v) we have

$$P(\xi) = \sum_\beta c_\beta P(\xi_\beta) = \sum_{\alpha,\beta} c_\beta(\delta_{\alpha\beta} - a_{\alpha\beta})\xi_\alpha,$$

and from this we obtain $\sigma c_\alpha = \sum_\beta c_\beta(\delta_{\alpha\beta} - a_{\alpha\beta})$. Therefore, if σ is an eigenvalue of P then it is an eigenvalue of the matrix $(\delta_{\alpha\beta} - a_{\alpha\beta})_r$, too.

If $\{N_1,...,N_r\}$ and $\{N'_1,...,N'_r\}$ are two local orthonormal basis on a normal space $T_x^\perp M$, then the decomposition of N'_α in the basis $\{N_1,...,N_r\}$ is given by $N'_\alpha = \sum_{\gamma=1}^r k_\alpha^\gamma N_\gamma$, for every $\alpha \in \{1,...,r\}$, where (k_α^γ) is an $r \times r$ orthogonal matrix, and we have (from [2]): $u'_\alpha = \sum_\gamma k_\alpha^\gamma u_\gamma$, $\xi'_\alpha = \sum_\gamma k_\alpha^\gamma \xi_\gamma$ and $a'_{\alpha\beta} = \sum_\gamma k_\alpha^\gamma a_{\gamma\delta} k_\beta^\delta$. Thus, if $\xi_1,...,\xi_r$ are linearly independent vector fields, then $\xi'_1,...,\xi'_r$ are also linearly independent. Furthermore, because $a_{\alpha\beta}$ is symmetric in α and β, under a suitable transformation, we can find that $a_{\alpha\beta}$ can be reduced to $a'_{\alpha\beta} = \lambda_\alpha \delta_{\alpha\beta}$, where λ_α ($\alpha \in \{1,2,...,r\}$) are eigenvalues of the matrix $(a_{\alpha\beta})_r$. In this case we have $u'_\beta(\xi_\alpha) = \varepsilon\delta_{\alpha\beta}(1 + \lambda_\alpha - \lambda_\alpha\lambda_\beta)$ and from this we obtain $u'_\alpha(\xi_\alpha) = \varepsilon(1 + \lambda_\alpha - \lambda_\alpha^2)$.

We denoted by $\mathcal{A} := (a_{\alpha\beta})_r$. In the same manner like in [2], we obtain the following property:

Proposition 3.1 *Let M be a non-invariant n-dimensional submanifold of codimension r, immersed in a golden Riemannian manifold $(\widetilde{M}, \widetilde{g}, \widetilde{P})$ so that the tangent vector fields $\xi_1, \xi_2, ..., \xi_r$ are linearly independent. Then,*

$$trace(P) = \begin{cases} r - trace(\mathcal{A}) + \sum_{A=r+1}^n \lambda_A, & r < n \\ r - trace(\mathcal{A}), & r = n \end{cases} \tag{3.8}$$

with $\lambda_A \in \{\phi; (1-\phi)\}$ *for every* $A, B \in \{r+1, ..., n\}$.

Proof. From (3.4)(v) we have that the matrices (P) (of P), $U := (\xi_1\ \xi_2\ ...\ \xi_r)$ and $\mathcal{A} := (a_{\alpha\beta})_r$ verify that $(P)U = U(I_r - \mathcal{A})$, where $I_r = (\delta_{\alpha\beta})$ is the identity matrix of order r.

For $r = n$, from $\det U \neq 0$ we obtain $(P) = U(I_r - \mathcal{A})U^{-1}$, and from this we have

$$P_\alpha^\beta = \sum_{\mu,\nu} u_\mu^\beta (\delta_\nu^\mu - a_\nu^\mu) v_\alpha^\nu$$

for $\alpha, \beta, \mu, \nu \in \{1, 2, ..., r\}$, where $P_\alpha^\beta, u_\mu^\beta$, and v_α^ν are components of the matrices $(P), U$ and respectively U^{-1}. Thus, we have $\text{trace}(P) = r - \text{trace}(\mathcal{A})$.

For $r < n$, we define matrices $\overline{U}$ and L by: $\overline{U} = (\xi_1\ \xi_2\ ...\xi_r\ \eta_{r+1}...\eta_n)$ and $L = \begin{pmatrix} \delta_{\alpha\beta} - a_{\alpha\beta} & 0 \\ 0 & \lambda_A \delta_{AB} \end{pmatrix}$, where $\alpha, \beta \in \{1, 2, ..., r\}$, $A, B \in \{r+1, ..., n\}$, $\delta_{\alpha\alpha} = 1, \delta_{\alpha\beta} = 0$ for $\alpha \neq \beta$ and $\lambda_A \in \{\phi; (1-\phi)\}$ are solutions of the equation $\lambda^2 = \lambda + 1$ (for $A \in \{r+1, ..., n\}$).

Since $det(\overline{U}) \neq 0$, we have $(P) = \overline{U} L \overline{U}^{-1}$ and from this we obtain $P_\alpha^\beta = \sum_{\mu,\nu} \overline{u}_\mu^\beta l_\nu^\mu \overline{v}_\alpha^\nu$ $(\alpha, \beta, \mu, \nu \in \{1, 2, ..., r\})$, where $P_\alpha^\beta, \overline{u}_\mu^\beta$, l_ν^μ and $\overline{v}_\alpha^\nu$ are components of matrices $(P), \overline{U}, L$ and respectively $\overline{U}^{-1}$. Thus, we obtain $\text{trace}(P) = r - \text{trace}(\mathcal{A}) + \sum_{A=r+1}^n \lambda_A$. □

Theorem 3.2 *([10]) Let* M *be a n-dimensional submanifold of codimension r, isometrically immersed in a Golden Riemannian manifold* $(\widetilde{M}, \widetilde{g}, \widetilde{P})$ *and let* $(P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ *be the induced structure on* M *by structure* $(\widetilde{g}, \widetilde{P})$. *A necessary and sufficient condition for* M *to be invariant is that the induced structure* (P, g) *on* M *is a Golden Riemannian Structure, whenever* P *is non-trivial.*

Proof. From (3.7)(i) and (iv) it is obvious that, if M is an invariant submanifold in a Golden Riemannian manifold $(\widetilde{M}, \widetilde{g}, \widetilde{P})$, then (P, g) is a golden Riemannian structure.
Conversely, if we suppose that (M, g, P) is a golden Riemannian manifold, then $\sum_\alpha u_\alpha(X)\xi_\alpha = 0$ and we obtain

$$\sum_\alpha u_\alpha(X) g(X, \xi_\alpha) = \sum_\alpha (u_\alpha(X))^2 = 0$$

from which $u_\alpha(X) = 0$ for $\alpha \in \{1, 2, ..., r\}$. Therefore M is invariant. □

4. An Example of Golden Structure

We consider that the ambient space is a $(p+q)$-dimensional Euclidean space E^{p+q} $(p, q \in \mathbb{N}^*)$. Let $\widetilde{P} : E^{p+q} \to E^{p+q}$ be an (1,1) tensor field defined by

$$\widetilde{P}(x^1, ..., x^p, y^1, ..., y^q) = (\phi x^1, ..., \phi x^p, (1-\phi) y^1, ..., (1-\phi) y^q) \tag{4.1}$$

for every point $(x^1,...,x^p,y^1,...,y^q)\in E^{p+q}$, where $\phi=\frac{1+\sqrt{5}}{2}$ and $1-\phi=\frac{1-\sqrt{5}}{2}$ are the roots of the equation $x^2=x+1$.

On the other hand, for $(x^1,...,x^p,y^1,...,y^q),(z^1,...,z^p,t^1,...,t^q)\in E^{p+q}$, we have

$$\widetilde{P}^2(x^1,...,x^p,y^1,...,y^q)=(\phi^2x^1,...,\phi^2x^p,(1-\phi)^2y^1,...,(1-\phi)^2y^q)=$$

$$=(\phi x^1,...,\phi x^p,(1-\phi)y^1,...,(1-\phi)y^q)+(x^1,...,x^p,y^1,...,y^q).$$

Thus, we obtain $\widetilde{P}^2=\widetilde{P}+I$ and we have

$$<\widetilde{P}(x^1,...,x^p,y^1,...,y^q),(z^1,...,z^p,t^1,...,t^q)>=<(x^1,...,x^p,y^1,...,y^q),\widetilde{P}(z^1,...,z^p,t^1,...,t^q)>$$

for every $(x^1,...,x^p,y^1,...,y^q),(z^1,...,z^p,t^1,...,t^q)\in E^{p+q}$ so, the scalar product $<>$ on E^{p+q} is $\widetilde{P}$-compatible. Therefore, $\widetilde{P}$ is a Golden Structure defined on $(E^{p+q},<>)$ and $(E^{p+q},<>,\widetilde{P})$ is a Golden Riemannian manifold.

In the following issue, we identify i_*X with X (where $X\in\chi(E^{p+q})$). It is obvious that $E^{p+q}=E^p\times E^q$ and in each of spaces E^p and E^q respectively, we can get a hypersphere

$$S^{p-1}(r_1)=\{(x^1,...,x^p),\sum_{i=1}^{p}(x^i)^2=r_1^2\}$$

and

$$S^{q-1}(r_2)=\{(y^1,...,y^q),\sum_{j=1}^{q}(y^j)^2=r_2^2\}$$

respectively, where $r_1^2+r_2^2=r^2$.

We construct the product manifold $S^{p-1}(r_1)\times S^{q-1}(r_2)$ in the same manner like in [9]. Every point of $S^{p-1}(r_1)\times S^{q-1}(r_2)$ has the coordinates $(x^1,...,x^p,y^1,...,y^q):=(x^i,y^j)$ ($i\in\{1,...,p\},j\in\{1,...,q\}$) such that:

$$\sum_{i=1}^{p}(x^i)^2+\sum_{j=1}^{q}(y^j)^2=r^2. \tag{4.2}$$

Thus, $S^{p-1}(r_1)\times S^{q-1}(r_2)$ is a submanifold of codimension 2 in E^{p+q} and $S^{p-1}(r_1)\times S^{q-1}(r_2)$ is a submanifold of codimension 1 in $S^{p+q-1}(r)$. Therefore, we have:

$$S^{p-1}(r_1)\times S^{q-1}(r_2)\hookrightarrow S^{p+q-1}(r)\hookrightarrow E^{p+q}. \tag{4.3}$$

The tangent space in a point $(x^1,...,x^p,y^1,...,y^q):=(x^i,y^j)$ at the product of spheres $S^{p-1}(r_1)\times S^{q-1}(r_2)$ is

$$T_{(x^1,...,x^p,\underbrace{o,\,...,\,o}_{q})}S^{p-1}(r_1)\oplus T_{(\underbrace{o,\,...,\,o}_{p},y^1,...,y^q)}S^{q-1}(r_2).$$

A vector $(X^1,...,X^p)$ from $T_{(x^1,...,x^p)}E^p$ is tangent to $S^{p-1}(r_1)$ if and only if we have:

$$\sum_{i=1}^{p}x^iX^i=0 \tag{4.4}$$

and it can be identified by $(X^1,...,X^p,\underbrace{0,...,0}_{q})$ from E^{p+q}.

A vector $(Y^1,...,Y^q)$ from $T_{(y^1,...,y^q)}E^q$ is tangent to $S^{q-1}(r_2)$ if and only if we have:

$$\sum_{j=1}^{q} y^j Y^j = 0 \tag{4.5}$$

and it can be identified by $(\underbrace{0,...,0}_{p},Y^1,...,Y^q)$ from E^{p+q}.

Consequently, for every point $(x^i,y^j)\in S^{p-1}(r_1)\times S^{q-1}(r_2)$ we have

$$(X^1,...,X^p,Y^1,...,Y^q):=(X^i,Y^j)\in T_{(x^1,...,x^p,y^1,...,y^q)}(S^{p-1}(r_1)\times S^{q-1}(r_2))$$

if the relations (4.4) and (4.5) are satisfied. Furthermore, we remark that (X^i,Y^j) is a tangent vector field at $S^{p+q-1}(r)$ and from this it follows that

$$T_{(x^i,y^j)}(S^{p-1}(r_1)\times S^{q-1}(r_2))\subset T_{(x^i,y^j)}S^{p+q-1}(r),$$

for every point $(x^i,y^j)\in S^{p-1}(r_1)\times S^{q-1}(r_2)$.

We consider a local othonormal basis (N_1,N_2) of $T^{\perp}_{(x^i,y^j)}S^{p-1}(r_1)\times S^{q-1}(r_2)$ in every point $(x^i,y^j)\in S^{p-1}(r_1)\times S^{q-1}(r_2)$ given by

$$N_1=\frac{1}{r}(x^i,y^j);\quad N_2=\frac{1}{r}(\frac{r_2}{r_1}x^i,-\frac{r_1}{r_2}y^j). \tag{4.6}$$

From the decomposition of $\widetilde{P}(N_\alpha)$ $(\alpha\in\{1,2\})$ in tangential and normal components at $S^{p-1}(r_1)\times S^{q-1}(r_2)$, we obtain

$$\widetilde{P}(N_\alpha)=\xi_\alpha+a_{\alpha 1}N_1+a_{\alpha 2}N_2,\ \ \alpha\in\{1,2\}. \tag{4.7}$$

From $a_{\alpha\beta}=<\widetilde{P}(N_\alpha),N_\beta>$ $(\alpha,\beta\in\{1,2\})$, we obtain

$$a_{11}=\frac{\phi r_1^2+(1-\phi)r_2^2}{r^2},\quad a_{12}=a_{21}=\frac{r_1r_2(2\phi-1)}{r^2},\quad a_{22}=\frac{\phi r_2^2+(1-\phi)r_1^2}{r^2}.$$

Thus, the matrix $\mathcal{A}:=(a_{\alpha\beta})_2$ is given by

$$\mathcal{A}=\begin{pmatrix}\frac{\phi r_1^2+(1-\phi)r_2^2}{r^2} & \frac{r_1r_2(2\phi-1)}{r^2}\\ \frac{r_1r_2(2\phi-1)}{r^2} & \frac{\phi r_2^2+(1-\phi)r_1^2}{r^2}\end{pmatrix}. \tag{4.8}$$

From (4.7) we obtain

$$\xi_1=\xi_2=0_{p+q}. \tag{4.9}$$

From (4.8) and (4.9) we have

$$\widetilde{P}(N_\alpha)=a_{\alpha 1}N_1+a_{\alpha 2}N_2,\quad (\forall)\alpha\in\{1,2\}. \tag{4.10}$$

Therefore,

$$\widetilde{P}(T^{\perp}_{(x^i,y^j)}(S^{p-1}(r_1)\times S^{q-1}(r_2)))\subseteq T^{\perp}_{(x^i,y^j)}(S^{p-1}(r_1)\times S^{q-1}(r_2)). \tag{4.11}$$

From the decomposition of $\widetilde{P}(X^i,Y^j)$ in tangential and normal components at $S^{p-1}(r_1)\times S^{q-1}(r_2)$ (where $(X^1,...,X^p,Y^1,...,Y^q):=(X^i,Y^j)$ is a tangent vector field on $S^{p-1}(r_1)\times S^{q-1}(r_2)$), we obtain

$$\widetilde{P}(X^i,Y^j)=P(X^i,Y^j)+u_1(X^i,Y^j)N_1+u_2(X^i,Y^j)N_2. \tag{4.12}$$

From $u_\alpha(X^i,Y^j)=<(X^i,Y^j),\xi_\alpha>$ (with $\alpha\in\{1,2\}$) and (4.9) we obtain

$$u_1(X^i,Y^j)=u_2(X^i,Y^j)=0 \tag{4.13}$$

for every tangent vector (X^i,Y^j) on the product of spheres $S^{p-1}(r_1)\times S^{q-1}(r_2)$ in a point $(x^i,y^j)\in S^{p-1}(r_1)\times S^{q-1}(r_2)$. From (4.12) and (4.13), we obtain

$$P(X^i,Y^j)=\widetilde{P}(X^i,Y^j). \tag{4.14}$$

Thus, we have $\widetilde{P}(T_{(x^i,y^j)}(S^{p-1}(r_1)\times S^{q-1}(r_2)))\subseteq T_{(x^i,y^j)}(S^{p-1}(r_1)\times S^{q-1}(r_2))$ and $P^2=P+I$. From (4.8), (4.9), (4.13) and (4.14) we obtain the induced structure $(P,<>,\xi_\alpha=0_{p+q},u_\alpha=0,\mathcal{A})$ on the product of spheres $S^{p-1}(r_1)\times S^{q-1}(r_2)$ by the Golden Structure $(\widetilde{P},<>)$ on E^{p+q}, which is also, a Golden Riemannian Structure $(P,<>)$ on $S^{p-1}(r_1)\times S^{q-1}(r_2)$. Therefore, we find a Golden Structure P induced on the product of spheres $S^{p-1}(r_1)\times S^{q-1}(r_2)$ by the Golden Structure $\widetilde{P}$, defined in (4.1).

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