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KILLING POTENTIALS*

BY

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1. Introduction. Let (M, g) be a smooth, finite–dimensional and connected Riemannian manifold, ∇ its Levi–Civita connection, $C^\infty(M)$ the ring of real–valued functions and $\mathcal{X}(M)$ the Lie algebra of smooth vector fields on M. For $f \in C^\infty(M)$ we denote by ∇f the gradient of f([3, p. 83]) and by H_f the Hessian of f. Recall that $H_f : \mathcal{X}(M) \times \mathcal{X}(M) \longrightarrow C^\infty(M)$ is given by ([3, p. 141–142]):

$$H_f(X, Y) := g(\nabla_X(\nabla f), Y) \qquad \forall X, Y \in \mathcal{X}(M) \tag{1}$$

and $X \in \mathcal{X}(M)$ is a Killing vector field if ([3, p.81–82]):

$$g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0 \qquad \forall\ Y, Z \in \mathcal{X}(M). \tag{2}$$

We say $f \in C^\infty(M)$ is a *Killing potential* if ∇f is a Killing vector field. Then $X = \nabla f$ is called a *Killing gradient*. Killing potentials appear in the study of some fluids ([5, p. 9]).

2. Properties of Killing potentials. From $H_f(X, Y) = H_f(Y, X)$ we have:

$$2H_f(X, Y) = H_f(X, Y) + H_f(Y, X) = g(\nabla_X(\nabla f), Y) + g(X, \nabla_Y(\nabla f)) \tag{3}$$

which yields the following characterization:

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* Dedicated to Professor Dr. Vasile Crucianu on the occasion of his 65th birthday

Theorem. *$f \in C^\infty(M)$ is a Killing potential iff $H_f = 0$ $(*)$.*

Applications:

Proposition 1.

(i) *Every Killing potential is a harmonic map.*

(ii) *On a compact, orientable and without boundary Riemannian manifold the Killing potentials are only constant functions.*

Remark. For another formulation of (ii) see Proposition 3.1 of [11, p. 43].

Proof. (i) From $(*)$ and relation $\Delta f := -TrH_f$ ([3, p. 142]).

(ii) From (i) and a theorem of E. HOPF–BOCHNER ([3, p. 85]).

Proposition 2. *Let f be a Killing potential.*

(i) *If N is a regular level of f then N is totally geodesic and $|\nabla f|$ is constant along each component of N. Suppose that $C = \{p \in M; df(p) = 0\}$ is nonempty and let $p \in C$.*

(ii) *p is degenerate critical point of f.*

(iii) *There exists a neighborhood $V \subset M$ of p such that $V \cap C$ is a submanifold of M (this implies that every connected component of C is a submanifold of M).*

Proof. (i) From $(*)$ and Lemma 1 of [4].
(ii) From $(*)$ and a very known classification of critical points.
(iii) Ex. 12 b) from [3, p. 143].

Remark. For another formulation of the fact that if f is Killing potential then $|f|$ is constant see [1, Th.1.].

Proposition 3. *Denote by $K(M, g)$ the set of Killing potentials on (M, g) and by $Morse(M)$ the set of Morse functions on M. Then:*

(i) *$K(M, g)$ is submodule in $C^\infty(M)$*

(ii) *$K(M, g) \subset C^\infty(M) \setminus Morse(M)$*

(iii) *$K(M, g)$ is a rare set (i. e. int$(clK(M, g)) = \emptyset$ where cl=closure and int=interior) in Whitney $WO^\infty-$topology on $C^\infty(M)$.*

Proof (i) From $(*)$.
(ii) From Proposition 2, (ii).
(iii) From (i) and the fact that $Morse(M)$ is open and dense in $C^\infty(M)$ with respect to *Whitney $WO^\infty-$topology* ([6, p. 147] Th. 1.2.).

Proposition 4. *Let $f \in C^\infty(M)$.*

(i) *f is Killing potential if and only if ∇f is covariant constant field i.e.* $\nabla_X(\nabla f) = 0 \quad \forall X \in \mathcal{X}(M)$.

(ii) *If f is Killing potential then the orbits of ∇f are geodesics.*

Proof. (i) If f is Killing potential then $H_f(X,Y)=g(\nabla_X(\nabla f),Y)=0$, $\forall X,Y\in\mathcal{X}(M)$; let $Y=\nabla_X(\nabla f)$. If $\nabla_X(\nabla f)=0$ $\forall X\in\mathcal{X}(M)$ then $H_f=0$.

(ii) In (i) let $X = \nabla f$.

Let $f \in C^\infty(M)$ and $h_f : \mathcal{X}(M) \to \mathcal{X}(M)$ $h_f(X) := \nabla_X(\nabla f)$ $\forall X \in \mathcal{X}(M)$.

From $H_f(X, Y) = H_f(X, Y)$ we have that h_f is selfadjoint and then there exists a orthonormed base $(X_i)_{i=\overline{1,n}}$ in $\mathcal{X}(M)$ and $(d_i)_{i=\overline{1,n}} \in C^\infty(M)$ such that $h_f(X_i) = \nabla_{X_i}(\nabla f) = d_i X_i$ $i = \overline{1,n}$. After an ideea of [1] (Th.2, Th.3) we have:

Proposition 5. *$f \in C^\infty(M)$ is Killing potential if and only if $d_i = 0$, $i = \overline{1,n}$.*

Proof. Is consequence of Proposition 4, (i).

3. Killing potentials on two dimensional manifolds. By [2] a function $f \in C^\infty(M)$ is called *proper* if $\| \nabla f \|^2 = g^{ij}\dfrac{\partial f}{\partial x^i}\dfrac{\partial f}{\partial x^j} \neq 0$. For the rest of these paper we work only with proper functions.

In [9] it is proved (using the Brinkmann method [2, p.123]) that there exists a Killing potential on (M,g) if and only if there is a preferential atlas on M with respect to which the components of the metric tensor g are expressed by:

$$g_{11} = K^2 \quad g_{1\alpha} = 0 \quad g_{\alpha\beta} = Q_{\alpha\beta}(x^2,\dots,x^n) \qquad \alpha,\beta = \overline{2,n} \tag{4}$$

where K is a real number and n is the dimension of M. For improper functions see [2, p. 131–133].

In the following we obtain all Killing potentials for case $n = 2$. By (4) we have

$$g_{11} = K^2 \quad g_{12} = g_{21} = 0 \quad g_{22} = Q(x^2). \tag{4$'$}$$

The metric g is positive definite hence $Q(x^2) > 0$ for all x^2. The Christofell symbols are:

$$\Gamma^1_{ij} = 0 \quad \Gamma^2_{11} = \Gamma^2_{12} = 0 \quad \Gamma^2_{22} = \frac{Q'}{2Q} \quad i,j = 1,2 \tag{5}$$

where Q' denote the derivative of Q. Locally, the condition $(*)$ is expresssed as follows:

$$(6) \qquad \frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial f}{\partial x^k} = 0 \qquad i, j = 1, 2.$$

Using (5) and (6) we have:

$$\frac{\partial^2 f}{\partial (x^1)^2} = 0 \quad \frac{\partial^2 f}{\partial x^1 \partial x^2} = 0 \quad \frac{\partial^2 f}{\partial (x^2)^2} = \frac{Q'}{2Q} \frac{\partial f}{\partial x^2}.$$

From the first two equations it results $\frac{\partial f}{\partial x^1} = C_1$ and from the last $\frac{\partial f}{\partial x^2} = C_2 \sqrt{Q}$ where C_1, C_2 are real numbers. It follows:

Proposition 6. *On a two dimensional Riemannian manifold with respect to atlas who define* (4) *the Killing potentials are given by:*

$$f(x^1, x^2) = C_1 x^1 + C_2 \int_0^{x^2} \sqrt{Q(t)} dt + C_3$$

where C_1, C_2, C_3 *are real numbers. The asociate Killing gradients are:*

$$X = \frac{C_1}{K^2} \frac{\partial}{\partial x^1} + \frac{C_2}{\sqrt{Q(x^2)}} \frac{\partial}{\partial x^2}.$$

Remark. We observe that $\|X\|^2 = \frac{C_1^2}{K^2} + C_2^2 =$constant in agreement with Proposition 2, (i).

Let V_n be a Riemannian manifold with the components of metric tensor:

$$g_{ij} = a_{ij}(x^1, \ldots, x^m), g_{i\alpha} = 0, g_{\alpha\beta} = \psi(x^1, \ldots, x^m) b_{\alpha\beta}(x^{m+1}, \ldots, x^n)$$

$i, j = \overline{1, m}, \alpha, \beta = \overline{m+1, n}$ where a_{ij} are the components of the metric tensor for a V_m-Riemannian manifold and $b_{\alpha\beta}$ are the components of the metric tensor for a V_{n-m}-Riemannian manifold. If V_{n-m} is of constant curvature then V_n is called a *subprojectif manifold of (n-m-1)-order* ([10, p. 41]). For $n = 2, m = 1, a_{ij}(x^1) = K^2, \psi(x^1) = 1, b_{\alpha\beta}(x^2) = Q(x^2)$ we obtain the relations $(4')$ and the curvature tensor of $(V_{2-1}, b_{\alpha\beta} = Q)$ is zero. Then:

Proposition 7. *A two dimensional Riemannian manifold on which there exists Killing potentials is a subprojectif manifold of 0-order.*

Returning to general case (4), remark that $(Q_{\alpha\beta})$ is a Riemannian metric for a V_{n-1}-manifold. We obtain $\Gamma^1_{11}=\Gamma^1_{1\alpha}=\Gamma^1_{\alpha\beta}=\Gamma^\alpha_{11}=\Gamma^\alpha_{1\beta}=0$ and $\Gamma^\alpha_{\beta\rho}=\Gamma^\alpha_{\beta\rho}(x^2,\dots,x^n)$ are the Christoffel symbols for $(V_{n-1},Q_{\alpha\beta})$. From:

$$\frac{\partial^2 f}{\partial (x^1)^2}=0 \quad \frac{\partial^2 f}{\partial x^1 \partial x^\alpha}=0 \quad \frac{\partial^2 f}{\partial x^\alpha \partial x^\beta}=\Gamma^\rho_{\alpha\beta}\frac{\partial f}{\partial x^\rho} \tag{7}$$

it follows that $\dfrac{\partial f}{\partial x^1}=C_1$ and for every x^1 the function

$$(x^2,...,x^n)\longrightarrow f(x^1,x^2,...,x^n)$$

is Killing potential for $(V_{n-1},Q_{\alpha\beta})$.

4. Killing potentials with respect to a pair of Riemannian metrics. Two Riemannian metrics $g,\tilde g$ on M is called *in g-subgeodesic correspondence* ([7]) if there exists $\xi=\left(\xi^i\right)_{i=\overline{1,n}}\in T^1_0(M)$ and $\psi=(\psi_i)_{i=\overline{1,n}}\in$ $\in T^0_1(M)$ such that:

$$\tilde\Gamma^i_{jk}=\Gamma^i_{jk}+\delta^i_j\psi_k+\delta^i_k\psi_j+g_{jk}\xi^i \tag{8}$$

Proposition 8. *Let $g,\tilde g$ two Riemannian metrics on M in g-subgeodesic correspondence and $f\in C^\infty(M)$ proper. If f is Killing potential with respect to both $g,\tilde g$ then $\psi\equiv 0$ and there is a preferential atlas on M with respect to which we have*

$$\xi^1=0,\tilde\Gamma^\alpha_{11}=K^2\xi^\alpha,\tilde\Gamma^\alpha_{1\beta}=0,\ \alpha,\beta=\overline{2,n} \tag{9}$$

with $K\neq 0$ a real number.

Proof. By [2 p.123] there is a preferential atlas on M with respect to which $f=x^1, g_{1\alpha}=g^{1\alpha}=0,\alpha=\overline{2,n}$. Then g is given by (4) and then it result (7). We have:

$$\frac{\partial^2 f}{\partial\left(x^1\right)^2}=0\Rightarrow\tilde\Gamma^1_{11}=\Gamma^1_{11}=0\Rightarrow(a)\ 2\psi_1+K^2\xi^1=0$$
$$\frac{\partial^2 f}{\partial x^1\partial x^\alpha}=0\Rightarrow\tilde\Gamma^1_{1\alpha}=\Gamma^1_{1\alpha}=0\Rightarrow\text{(b) }\psi_\alpha=0$$
$$\frac{\partial^2 f}{\partial x^\alpha\partial x^\beta}=0\Rightarrow\tilde\Gamma^1_{\alpha\beta}=\Gamma^1_{\alpha\beta}=0\Rightarrow g_{\alpha\beta}\xi^1=0\Rightarrow\text{(c) }\xi^1=0$$

(a)+(c) $\Rightarrow \psi_1 = 0$ and then $\psi = 0$.
$\Gamma^{\alpha}_{11} = 0 \Rightarrow$ (d) $\tilde{\Gamma}^{\alpha}_{11} = K^2 \xi^{\alpha}$
$\Gamma^{\alpha}_{1\beta} = 0 \Rightarrow$ (e) $\tilde{\Gamma}^{\alpha}_{1\beta} = 0$.

Particular cases:

1. If $\xi = 0$ in (8) the metrics $g, \tilde{g}$ are called in *geodesic correspondence* ([10, p. 322]) Then:

Proposition 9. *Let $g, \tilde{g}$ two Riemannian metrics on M in geodesic correspondence and $f \in C^{\infty}(M)$ proper. If f is Killing potential with respect to both $g, \tilde{g}$ then $\psi = 0$ $\left(i.e. \tilde{\Gamma} = \Gamma \right)$ and there is a preferential atlas on M with respect to which we have* (7).

2. Let $g, \tilde{g} = e^{2u} g$ with $u \in C^{\infty}(M)$ two conformal metrics on M. Then:

$$\tilde{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j \frac{\partial u}{\partial x^k} + \delta^i_k \frac{\partial u}{\partial x^j} - g_{jk} g^{im} \frac{\partial u}{\partial x^m}$$

i.e. (8) with $\psi = du, \xi = -\nabla u$ where gradient is with respect to g. Then:

Proposition 10. *Let g and $\tilde{g} = e^u g, u \in C^{\infty}(M)$ two conformal Riemannian metrics on M and $f \in C^{\infty}(M)$ proper. If f is Killing potential with respect to both g and $\tilde{g}$ then $\psi = du = 0$ i.e. u is constant.*

5. Lifts on TM. Let G be the Sasaki lift of g to TM. By $(*)$ a function $\tilde{f} \in C^{\infty}(TM)$ is Killing potential on (TM, G) if and only if:

$$A_{ij}(\tilde{f}) := \frac{\partial^2 \tilde{f}}{\partial x^i \partial x^j} - \tilde{\Gamma}^k_{ij} \frac{\partial \tilde{f}}{\partial x^k} - \tilde{\Gamma}^{n+k}_{ij} \frac{\partial \tilde{f}}{\partial y^k} = 0$$

$$B_{ij}(\tilde{f}) := \frac{\partial^2 \tilde{f}}{\partial x^i \partial y^j} - \tilde{\Gamma}^k_{in+j} \frac{\partial \tilde{f}}{\partial x^k} - \tilde{\Gamma}^{n+k}_{in+j} \frac{\partial \tilde{f}}{\partial y^k} = 0$$

$$C_{ij}(\tilde{f}) := \frac{\partial^2 \tilde{f}}{\partial y^i \partial y^j} - \tilde{\Gamma}^k_{n+in+j} \frac{\partial \tilde{f}}{\partial x^k} - \tilde{\Gamma}^{n+k}_{n+in+j} \frac{\partial \tilde{f}}{\partial y^k} = 0$$

where $(\tilde{\Gamma})$ are the Christoffel symbols for G ([8, p. 194]).

Let $f \in C^{\infty}(M)$ be a Killing potential on the base manifold M and the lifts on TM:

(i) the vertical lift: $f^V := f \circ \pi$ where $\pi : TM \longrightarrow M$ is the natural projection

(ii) the complet lift: $f^C = \dfrac{\partial f}{\partial x^i} y^i$

(iii) the horizontal lift: $f^H = 0$.

In the following the Greek indices $\lambda, \mu, \dots$ range over $1, 2, \dots, n$ and $(R = R^i_{jkl})$ is the curvature tensor of M. After a straightforward computation we obtain for f^V :

$$A^V_{ij} = -\frac{1}{2}(R^k_{jh\mu}\Gamma^h_{\lambda i} + R^k_{ih\mu}\Gamma^h_{\lambda j})y^\lambda y^\mu \frac{\partial f}{\partial x^k}$$

$$B^V_{ij} = -\frac{1}{2}R^k_{ij\lambda}y^\lambda \frac{\partial f}{\partial x^k}$$

$$C^V_{ij} = 0.$$

Remark that $A^V_{ij} = (B^V_{jh}\Gamma^h_{\lambda i} + B^V_{ih}\Gamma^h_{\lambda j})y^\lambda$ which yields:

Proposition 11. *If f is a Killing potential on M then f^V is Killing potential on (TM, G) if and only if* $R^k_{ijl}\dfrac{\partial f}{\partial x^k} = 0 \quad i, j, l = \overline{1, n}$ *or globally $df \circ R = 0$ where df is the differential of f.*

For f^C we obtain:

$$A^C_{ij} = -\frac{1}{2}(R^k_{i\lambda j} + R^k_{j\lambda i})y^\lambda \frac{\partial f}{\partial x^k}$$

$$B^C_{ij} = C^C_{ij} = 0$$

and then:

Proposition 12. *If f is a Killing potential on M then f^C is Killing potential on (TM, G) if and only if* $(R^k_{ilj} + R^k_{jli})\dfrac{\partial f}{\partial x^k} = 0 \quad i, j, l = \overline{1, n}$ *or globally $df(R(X,Y)Z + R(X,Z)Y) = 0 \quad \forall X, Y, Z \in \mathcal{X}(M)$.*

6. Classification of Killing potentials. In this section we provide a definitive solution to the following problem.

Describe the complete connected Riemann manifolds (M, g) which admit nontrivial Killing potentials.

We will prove the following result.

Classification Theorem. *A complete, connected Riemann manifold (M, g) which admits a nontrivial Killing potential is isometric to a product $\mathbf{R}\times N$. The Killing potential in this case is the natural projection $\mathbf{R}\times N \to \mathbf{R}$.*

Proof. If f is a nontrivial Killing potential then ∇f is not identically 0. Condition (i) by Proposition 4 above implies (since M is connected) that ∇f never vanishes. Since the flow lines of ∇f are geodesics and M is complete the flow is defined for all moments of time. Then we have the following isometry (since the flow generated by ∇f is a flow of isometries)

$$M \cong \mathbf{R} \times f^{-1}(0).$$

The theorem is proved.

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