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Last multipliers on Lie algebroids

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Abstract. In this paper we extend the theory of last multipliers as solutions of the Liouville's transport equation to Lie algebroids with their top exterior power as trivial line bundle (previously developed for vector fields and multivectors). We define the notion of exact section and the Liouville equation on Lie algebroids. The aim of the present work is to develop the theory of this extension from the tangent bundle algebroid to a general Lie algebroid (e.g. the set of sections with a prescribed last multiplier is still a Gerstenhaber subalgebra). We present some characterizations of this extension in terms of Witten and Marsden differentials.

Keywords. Liouville equation; volume form; last multiplier; Lie algebroid; Gerstenhaber algebra; Schouten bracket; exact section; Casimir function; Witten differential; Marsden differential.

0. Introduction

In 1838, Joseph Liouville [18] published a note on the time-dependence of the Jacobian of the 'transformation' exerted by the solution of an ODE on its initial condition. If $A = A(x)$ is the vector field corresponding to the given ODE and $m = m(t, x)$ is a smooth function, then the main equation of the cited paper [18] is:

$$\frac{\mathrm{d}m}{\mathrm{d}t} + m \cdot \operatorname{div} A = 0 \tag{LE}$$

called the *Liouville equation*. Some authors use the name *generalized Liouville equation* [9] but we prefer to name it the *Liouville equation of transport* (or *of continuity*). This equation is a main tool in statistical mechanics where a solution is called a *probability density function* [28]. Recently, a very interesting theory called *Liouville dynamics* in [11] and [12] was derived around volume-preserving vector fields.

The notion of *last multiplier* (also named *Jacobi multiplier*), introduced by Carl Gustav Jacob Jacobi around 1844, was treated in detail in *Vorlesugen über Dynamik*, edited by R F A Clebsch in Berlin in 1866. For understanding ODE's, this tool was intensively studied in the usual Euclidean space $\mathbb{R}^n$ by mathematicians (as can be seen in the bibliography of [3], [23]–[26]). For all those interested in historical aspects an excellent survey can be found in [1].

Several geometrical aspects of last multipliers viewed as autonomous (i.e. time-independent) solutions of LE are derived in two papers [3], [4] of the first author. Our study has been inspired by the results presented in [27], using the calculus on manifolds, especially the Lie derivative, a well-known tool for the geometry of vector fields. In [5] the previous theory is extended to general multivectors by means of the *curl operator* (i.e. a conjugate of the usual exterior derivative with respect to contraction of a given volume form). This operator was introduced by Koszul [17] in Poisson geometry and is detailed in Chapter 2 of [8], §2 of [32] and [30]. A version of the curl operator for Lie algebroids appears in [14].

The aim of the present paper is to describe a new step of generalization changing from vector fields or multivectors, seen as sections of the tangent bundle TM of a given manifold M, to sections of a general Lie algebroid E over M. This passing is possible since two important structures of our previous theory, namely the Schouten (particularly Lie) bracket and the exterior derivative are still in the general framework of Lie algebroids. It follows that this extension preserves a series of remarkable characterizations and results e.g. the set of sections with a prescribed last multiplier is again a Gerstenhaber subalgebra like in tangent bundle case.

The paper is structured as follows. The first section recalls the definition of last multipliers both for vector fields and multivectors and several previous results of [5].

The next section is intended to the announced extension to sections of a Lie algebroid with trivial top exterior power and some of the previous results are re-obtained in this extended framework. A notion which, to our knowledge, does not appear in the bibliography is introduced here, namely *exact section* with respect to a top degree element in E although the notion of exact Poisson structure is known ([6], [7], [32]). Consequences with respect to the Schouten bracket on sections, Witten and Marsden differentials are still present on Lie algebroids. The term *Liouville equation on Lie algebroids* is introduced.

The paper ends with a list of conclusions completed with an open problem regarding the cohomology of a Lie algebroid with respect to the Marsden differential.

1. Last multipliers for vector fields and multivectors

1.1 *The case of vector fields*

Let M be a real, smooth, n-dimensional manifold, $C^\infty(M)$ the algebra of smooth real functions on M, $\mathcal{X}(M)$ the Lie algebra of vector fields and $\Lambda^k(M)$ the $C^\infty(M)$-module of k-differential forms, $0 \leq k \leq n$. Assume that M is orientable with the fixed volume form $V \in \Lambda^n(M)$.

Let

$$\dot{x}^i(t) = A^i(x^1(t), \ldots, x^n(t)), \quad 1 \leq i \leq n$$

be an ODE system on M defined by the vector field $A \in \mathcal{X}(M)$, with $A = (A^i)_{1 \leq i \leq n}$ and let us consider the $(n-1)$-form $\Omega_A = i_A V \in \Lambda^{n-1}(M)$.

DEFINITION 1.1 (p. 107 of [10], p. 428 of [27])

The function $m \in C^\infty(M)$ is called a *last multiplier* of the ODE system generated by A, (*last multiplier* of A, for short) if $m\Omega_A$ is closed:

$$d(m\Omega_A) := (dm) \wedge \Omega_A + md\Omega_A = 0. \tag{1.1}$$

For example, in dimension 2, the notions of last multiplier and integrating factor are identical and Sophus Lie suggested a method to associate a last multiplier to every symmetry vector field of A (Theorem 1.1 in p. 752 of [15]). Lie's method is extended to any dimension in [27].

Characterizations of last multipliers can be obtained in terms of Witten's differential [31] and Marsden's differential (p. 220 of [19]). If $f \in C^{\infty}(M)$ and $t \geq 0$, Witten deformation of the usual differential is defined by

$$d_{tf}\colon \Lambda^{*}(M) \to \Lambda^{*+1}(M), \quad d_{tf} = \mathrm{e}^{-tf} d \mathrm{e}^{tf}$$

which means [31]:

$$d_{tf}(\omega) = t df \wedge \omega + d\omega.$$

Hence, m is a last multiplier if and only if we have

$$d_m \Omega_A = (1 - m) d\Omega_A$$

i.e. Ω_A belongs to the kernel of the differential operator

$$d_m + (m-1)d\colon \Lambda^{n-1}(M) \to \Lambda^{n}(M).$$

Marsden differential is defined by

$$d^{f}\colon \Lambda^{*}(M) \to \Lambda^{*+1}(M), \quad d^{f}(\omega) = \frac{1}{f} d(f\omega)$$

and m is a last multiplier if and only if Ω_A is d^m-closed.

The following characterization of last multipliers will be useful.

Lemma 1.1 (p. 428 of [27])

(i) *$m \in C^{\infty}(M)$ is a last multiplier for A if and only if*

$$A(m) + m \cdot \mathrm{div}_V A = 0, \tag{1.2}$$

where $\mathrm{div}_V A$ *is the divergence of A with respect to volume form V.*

(ii) *Let $0 \neq h \in C^{\infty}(M)$ such that*

$$L_A h := A(h) = (\mathrm{div}_V A) \cdot h. \tag{1.3}$$

Then $m = h^{-1}$ is a last multiplier for A.

Remark 1.1.

(i) Equation (1.2) is exactly the time-independent version of LE from the Introduction. An important feature of eq. (1.2) is that it does not always admit solutions (p. 269 of [13]).
(ii) In the terminology of p. 89 of [1], a function h satisfying (1.3) is called an *inverse multiplier*.

(iii) A first result given by (1.2) is the characterization of last multipliers for divergence-free vector fields: *$m \in C^{\infty}(M)$ is a last multiplier for the divergenceless vector field A if and only if m is a first integral of A*. The importance of this result is shown by the fact that three remarkable classes of divergence-free vector fields are provided by Killing vector fields in Riemannian geometry, Hamiltonian vector fields in symplectic geometry and Reeb vector fields in contact geometry. Also, there are many equations of mathematical physics corresponding to the vector fields without divergence.

(iv) For the general case, namely A is not divergenceless, there is a strong connection between first integrals and last multipliers as well. Namely, from properties of Lie derivative, the ratio of two last multipliers is a first integral and conversely, the product between a first integral and a last multiplier is a last multiplier. So, denoting $F\mathrm{Int}(A)$ the set of first integrals of A, since $F\mathrm{Int}(A)$ is a subalgebra in $C^{\infty}(M)$ it results that the set of last multipliers for A is a $F\mathrm{Int}(A)$-module.

(v) Recalling formula

$$\mathrm{div}_V(fX) = X(f) + f\mathrm{div}_V X \tag{1.4}$$

it follows that m is a last multiplier for A if and only if the vector field mA is with null divergence i.e. $\mathrm{div}_V(mA) = 0$. Thus, the set of last multipliers is a 'measure of how far away' is A from being divergence-free.

An important structure generated by a last multiplier is given by the following.

PROPOSITION 1.1

Let $m \in C^{\infty}(M)$ be fixed. The set of vector fields admitting m as last multiplier is a Lie subalgebra in $\mathcal{X}(M)$.

Proof. Let X and Y be vector fields with the required property. From p. 123 of [20], we have

$$\mathrm{div}_V[X, Y] = X(\mathrm{div}_V Y) - Y(\mathrm{div}_V X)$$

out of which we obtain

$$\begin{aligned}[X, Y](m) + m\mathrm{div}_V[X, Y] &= (X(Y(m)) + mX(\mathrm{div}_V Y)) \\ &\quad - (Y(X(m)) + mY(\mathrm{div}_V X)) \\ &= (-\mathrm{div}_V Y \cdot X(m)) - (-\mathrm{div}_V X \cdot Y(m)) \\ &= \mathrm{div}_V Y \cdot m\mathrm{div}_V X - \mathrm{div}_V X \cdot m\mathrm{div}_V Y = 0.\end{aligned}$$

1.2 *The case of multivectors*

Let $A \in \mathcal{X}^k(M)$ be a multivector (where $\mathcal{X}^k(M)$ is the $C^{\infty}(M)$-module of k-vector fields, $1 \leq k \leq n$). Thus, A defines a map $i_A\colon \Lambda^p(M) \to \Lambda^{p-k}(M)$ given by

- if $p \geq k$, then $\langle i_A\omega, B\rangle = \langle \omega, A \wedge B\rangle$ for every $B \in \mathcal{X}^{p-k}(M)$ and
- if $p < k$, then $i_A\omega = 0$,

where $\langle , \rangle$ is the natural duality between forms and multivectors and $\wedge$ is the Grassmann wedge product on $\bigoplus_{k=1}^{n} \mathcal{X}^k(M)$.

Thus, we have a map on (M, V) given by

$$V^\flat \colon \mathcal{X}^k(M) \to \Lambda^{n-k}(M), \qquad V^\flat(A) = i_A V. \tag{1.5}$$

This map is a $C^\infty(M)$-isomorphism between $\mathcal{X}^k(M)$ and $\Lambda^{n-k}(M)$, for $0 \leq k \leq n$. The inverse map of $V^\flat$ is denoted by $V^\natural \colon \Lambda^{n-k}(M) \to \mathcal{X}^k(M)$.

DEFINITION 1.2 (p. 70 of [8])

The map $D_V \colon \mathcal{X}^k(M) \to \mathcal{X}^{k-1}(M)$:

$$D_V = V^\natural \circ d \circ V^\flat, \tag{1.6}$$

is called *the curl operator* with respect to the volume form V. Thus, if $A \in \mathcal{X}^k(M)$, then $D_V A$ is called *the curl of* A.

Example 1.1 (p. 70 of [8]). If $k = 1$, then $D_V = \mathrm{div}_V$. Indeed, if $A \in \mathcal{X}(M)$ then

$$(D_V A)V = V^\flat \circ D_V(A) = d \circ V^\flat(A) = d \circ i_A(V) = L_A V = (\mathrm{div}_V A)V.$$

Inspired by this example and relation (1.4) we introduce the following.

DEFINITION 1.3

The function $m \in C^\infty(M)$ is called a *last multiplier* of $A \in \mathcal{X}^k(M)$ if

$$D_V(mA) = 0 \tag{1.7}$$

and this relation can be called *Liouville equation for the multivector* A.

Since $V^\natural$ is a $C^\infty(M)$-isomorphism between $\Lambda^{n-k}(M)$ and $\mathcal{X}^k(M)$ it results that (1.7) means $d(V^\flat(mA)) = 0$ i.e.

$$d(mV^\flat(A)) = 0 \tag{1.8}$$

which is the natural extension of condition (1.1) from Definition 1.1. With the same computation as in the previous section we derive the following equivalent characterizations of last multipliers for $A \in \mathcal{X}^k(M)$:

- *In terms of Witten differential.* $V^\flat(A) = i_A V$ belongs to the kernel of the differential operator $d_m + (m-1)d \colon \Lambda^{n-k}(M) \to \Lambda^{n-k+1}(M)$.
- *In terms of Marsden differential.* $V^\flat(A) = i_A V$ is d^m-closed with $d^m \colon \Lambda^k(M) \to \Lambda^{k+1}(M)$ as in §§1.

From the $C^\infty(M)$-linearity of $V^\flat$ we have $V^\flat(mA) = mV^\flat(A) = (mV)^\flat(A)$ and then $(mV)^\natural = \frac{1}{m} V^\natural$ (we suppose $m > 0$ everywhere). It follows that

$$mD_{mV}(A) = V^\natural \circ d \circ V^\flat(mA) = D_V(mA) \tag{1.9}$$

which yields the following.

PROPOSITION 1.2

A positive function $m \in C^{\infty}(M)$ *is a last multiplier of* $A \in \mathcal{X}^k(M)$ *if and only if*

$$D_{mV}(A) = 0. \tag{1.10}$$

The last formula has some important consequences, all in terms of an operation on $\bigoplus_{k=1}^{n} \mathcal{X}^k(M)$ called *Schouten bracket* [,] which is a natural generalization of Lie bracket from $\mathcal{X}(M)$ and generates a *Gersternhaber algebra* structure on the set of multivectors [16]. For details regarding this bracket see [8], [29]. The first corollary of (1.10) is a formula for the curl.

PROPOSITION 1.3

If $m \in C^{\infty}(M)$ *is a non-vanishing last multiplier of* $A \in \mathcal{X}^k(M)$ *then the curl of* A *can be expressed in terms of the Schouten bracket*

$$D_V A = -[A, \ln |m|]. \tag{1.11}$$

Proof. This is a direct consequence of formula (2.90) from p. 71 of [8]:

$$D_{mV} A = D_V A + [A, \ln |m|].$$

□

A second formula relates the Schouten bracket to the product $\wedge$ of $\bigoplus_{k=1}^{n} \mathcal{X}^k(M)$. After Theorem 2.6.7, p. 71 of [8] if A is an a-multivector and B is a b-multivector, then

$$[A, B] = (-1)^b D_V(A \wedge B) - (D_V A) \wedge B - (-1)^b A \wedge (D_V B). \tag{1.12}$$

COROLLARY 1.1

Let $m \in C^{\infty}(M)$ *be a last multiplier for both* A *and* B. *Then* m *is a last multiplier for* $A \wedge B$ *if and only if* A *and* B *Schouten-commutes i.e. their Schouten bracket vanishes*: $[A, B] = 0$.

Another consequence of (1.10) is a straightforward generalization of Proposition 1.1.

Theorem 1.1. *Let* $m \in C^{\infty}(M)$ *be fixed. The set of multivectors admitting* m *as last multiplier is a Gerstenhaber subalgebra in* $\bigoplus_{k=1}^{n} \mathcal{X}^k(M)$.

Proof. The curl operator is, up to a sign, a derivation of the Schouten bracket, namely p. 71 of [8]:

$$D_V[A, B] = [A, D_V B] + (-1)^{b-1}[D_V A, B]. \tag{1.13}$$

This relation gives the conclusion. □

DEFINITION 1.4 [32]

The multivector A is called *exact* with respect to the volume form V if $D_V(A) = 0$.

Remark 1.2.

(i) It follows from (1.7) that the set of last multipliers of A is a 'measure of how far away' is A from being exact.

(ii) Equation (1.12) gives that if A and B are exact multivectors then $A \wedge B$ is exact if and only if they Schouten-commute.
(iii) Using again (1.13) it results that the set of exact multivectors is a Gerstenhaber subalgebra in $\bigoplus_{k=1}^{n} \mathcal{X}^k(M)$.

Example 1.2. From [21] the volume form V yields a Nambu multivector, p. 160 of [8], $A_V \in \mathcal{X}^n(M)$; if $(x^1, \ldots, x^n)$ is a local chart on M such that $V = f dx^1 \wedge \cdots \wedge dx^n$ with $f > 0$ then $A_V = \frac{1}{f} \frac{\partial}{\partial x^1} \wedge \cdots \wedge \frac{\partial}{\partial x^n}$. A straightforward computation gives that A_V is exact with respect to V.

Remark 1.3. Let $f \in C^\infty(M)$ and A an a-multivector. From (1.12) and $D_V(f) = 0$ we get

$$[A, f] = D_V(fA) - f D_V(A)$$

and then $D_V(fA) = f D_V(A)$ if and only if f is a *Casimir* of A, i.e. $[A, f] = 0$.

PROPOSITION 1.4

If A is exact then fA is exact if and only if f is a Casimir function of A.

2. Last multipliers for Lie algebroids

Let us take a Lie algebroid $(E, \rho, [\,,\,])$ over M; we refer the reader to the basics of Lie algebroids calculus, for instance [2] and [8]. It is a well-known fact that, two structures are associated to E.

(i) The algebra of sections of ΛE, $\Gamma(\Lambda E)$, equipped with the exterior product $\wedge$ together with the generalized Schouten bracket $[\,,\,]$, forms a Gerstenhaber algebra (p. 132 of [2]).
(ii) The algebra of sections of ΛE^*, $\Gamma(\Lambda E^*)$ is endowed with a derivation of square 0 of its graded commutative associative structure, denoted by d_E (p. 131 of [2]).

Suppose that E is of rank r and assume that there is a nowhere vanishing section μ in $\Gamma(\Lambda^r E^*)$. Such a section defines an isomorphism $*_\mu$ of ΛE onto ΛE^* [16], so that, for each degree k ($0 \leq k \leq r$), this isomorphism is defined by

$$*_\mu \colon \Lambda^k E \to \Lambda^{r-k} E^*,$$

(i) $*_\mu Q = i_Q \mu, \quad (\forall) Q \in \Gamma(\Lambda^k E), k > 0,$
(ii) $*_\mu f = f\mu, \quad (\forall) f \in \Gamma(\Lambda^0 E) = C^\infty(M).$

Inspired by §5 of [16] let us introduce on $\Gamma(\Lambda E)$ the operator:

$$\partial_\mu = - *_\mu^{-1} d_E *_\mu \,. \tag{2.1}$$

PROPOSITION 2.1 (Proposition 3 of [16])

The operator ∂_μ has the following properties:

(i) *It is of degree -1 and square* 0.
(ii) *It generates the Schouten bracket on* $\Gamma(\Lambda E)$.
(iii) *It is a derivation of the Schouten bracket.*

The fact that ∂_μ generates the Schouten bracket means that a similar relation to (1.12) holds, [16]:

$$[A, B] = (-1)^{|A|}(\partial_\mu(A \wedge B) - \partial_\mu A \wedge B - (-1)^{|A|} A \wedge \partial_\mu B), \tag{2.2}$$

where, as is usual, $|A|$ denotes the degree of A.

DEFINITION 2.1

Given the triple (M, E, μ), a section $A \in \Gamma(\Lambda E)$ is called *exact* if $\partial_\mu(A)$ is identically zero. The function $m \in C^\infty(M)$ is called a *last multiplier* of $A \in \Gamma(\Lambda E)$ if we have:

$$\partial_\mu(mA) = 0. \tag{2.3}$$

The relation (2.3) is called *the Liouville equation on the Lie algebroid E*.

It results that the set of last multipliers of A is a 'measure of how far away' is A from being exact. Using the same arguments like in the previous section we get the following.

PROPOSITION 2.2

(i) *If A and $B \in \Gamma(\Lambda E)$ are exact sections then $A \wedge B$ is exact if and only if they Schouten-commute i.e. their Schouten bracket vanishes*: $[A, B] = 0$.

(ii) *A positive function $m \in C^\infty(M)$ is a last multiplier of A if and only if, similar to* (1.8):

$$d_E(m *_\mu A) = 0 \tag{2.4}$$

or, similar to (1.10):

$$\partial_{m\mu}(A) = 0. \tag{2.5}$$

(iii) *Let $m \in C^\infty(M)$ be a last multiplier for both A and B. Then m is a last multiplier for $A \wedge B$ if and only if A and B Schouten-commute.*

(iv) *Let $m \in C^\infty(M)$ be fixed. The set of multivectors admitting m as last multiplier is a Gerstenhaber subalgebra in $\Gamma(\Lambda E)$ since applying ∂_μ to relation* (2.2) *we get*:

$$\partial_\mu([A, B]) = (-1)^{|A|+1}\partial_\mu(\partial_\mu A \wedge B) - \partial_\mu(A \wedge \partial_\mu B). \tag{2.6}$$

Remark 2.1. Let $f \in C^\infty(M)$ and $A \in \Gamma(\Lambda E)$. From (2.2) and $\partial_\mu(f) = 0$ we derive

$$[A, f] = (-1)^{|A|}(\partial_\mu(fA) - f\partial_\mu(A))$$

and then $\partial_\mu(fA) = f\partial_\mu(A)$ if and only if f is a *Casimir* of A, i.e. $[A, f] = 0$.

Connecting this with Proposition 2.2(i) we obtain the following.

PROPOSITION 2.3

If $A \in \Gamma(\Lambda E)$ is exact, then fA is exact if and only if f is a Casimir function of A.

The tools of Witten and Marsden differential can be extended to Lie algebroids by replacing d with d_E. Therefore, the characterizations of last multipliers in terms of these differentials admit straightforward generalization: a positive function $m \in C^\infty(M)$ is a last multiplier for $A \in \Gamma(\Lambda^{|A|}E)$:

- *In terms of Witten differential* $d_{E,tf} := \mathrm{e}^{-tf} d_E \mathrm{e}^{tf}$. If $*_\mu A$ belongs to the kernel of the differential operator $d_{E,m} + (m-1)d_E \colon \Gamma(\Lambda^{r-|A|}E) \to \Gamma(\Lambda^{r-|A|+1}E)$.
- *In terms of Marsden differential* $d_E^f(\cdot) := \frac{1}{f} d_E(f\cdot)$. If $*_\mu A$ is d_E^m -closed with $d_E^m \colon \Gamma(\Lambda^{|A|}E) \to \Gamma(\Lambda^{|A|+1}E)$ as in §§1.2.

3. Conclusions

1) The last multipliers constitute a measure to count the 'perturbation' from exactness. So, this notion can be thought in the framework of [22].
2) The theory of last multipliers can be extended from vector fields and general multivectors on a manifold M to sections of a Lie algebroid over M preserving a series of remarkable characterizations and results.
3) An important structure generated by a last multiplier is of algebraic nature: the set of sections with a prescribed last multiplier is a Gerstenhaber subalgebra.
4) The Liouville equation is extended to Lie algebroids and the notion of exact section is introduced into this framework.
5) A natural open problem from the last sentence of §2 is to study the Marsden cohomology of a Lie algebroid.

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