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LAST MULTIPLIERS FOR MULTIVECTORS WITH APPLICATIONS TO POISSON GEOMETRY

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Abstract. The theory of the last multipliers as solutions of the Liouville's transport equation, previously developed for vector fields, is extended here to general multivectors. Characterizations in terms of Witten and Marsden differentials are reobtained as well as the algebraic structure of the set of multivectors with a common last multiplier, namely Gerstenhaber algebra. Applications to Poisson bivectors are presented by obtaining that last multipliers count for "how far away" is a Poisson structure from being exact with respect to a given volume form. The notion of exact Poisson cohomology for an unimodular Poisson structure on $I\!R^n$ is introduced.

0. Introduction

In January 1838, Joseph Liouville(1809-1882) published a note, [16], on the time-dependence of the Jacobian of the "transformation" exerted by the solution of an ODE on its initial condition. In modern language, if $A = A(x)$ is the vector field corresponding to the given ODE and $m = m(t, x)$ is a smooth function (depending also on time t), then the main equation of the cited paper is:

$$(LE) \qquad \frac{dm}{dt} + m \cdot divA = 0$$

called, by then, the *Liouville equation*. Some authors use the name *generalized Liouville equation*, [9], but we prefer to name it the *Liouville equation of transport* (or *of continuity*). This equation is a main tool in statistical mechanics where a solution is called a *probability density function*, [26].

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The notion of the *last multiplier*, introduced by Carl Gustav Jacob Jacobi (1804-1851) around 1844, was treated in details in *Vorlesugen über Dynamik*, edited by R. F. A. Clebsch in Berlin in 1866. Thus, sometimes it has been used under the name of *Jacobi multiplier*. Since then, this tool for understanding ODE's was intensively studied by mathematicians in the usual Euclidean space $I\!R^n$, as can be seen in the bibliography of [2], [21-24]. For all those interested in historical aspects, an excellent survey can be found in [1].

Several geometrical aspects of the last multipliers viewed as autonomous, i.e. time-independent, solutions of LE are derived in two papers by the same author: [2], [3]. Our study has been inspired by the results presented in [25] using the calculus on manifolds especially the Lie derivative, a well-known tool for the geometry of vector fields.

The aim of the present paper is to extend this theory of the last multipliers from vector fields to general multivectors by means of the *curl operator*. This operator, a conjugate of usual exterior derivative with respect to contraction of a given volume form, was introduced by J.-L. Koszul in Poisson geometry [15] and is detailed in Chapter 2 of [8] and Section 2 of [30].

Since the Poisson multivectors are most frequently used, a Poisson bracket is added to our study and we show that the last multipliers are a measure of "how far away" is a Poisson structure from being exact regarding the given volume form. Exact Poisson structures are the theme of papers [6] and [30] and form a remarkable class of Poisson structures closed to symplectic structures as it is pointed out in [28] and the second paper cited above and proved in our Section 3. There are other two important features of these Poisson structures:

(a) in [8, p. 149] the problem of classification of quadratic Poisson structures is reduced to the problem of classification of exact quadratic Poisson structures and linear vector fields which preserve them,

(b) [30, Remark 3.2.]: *in dimension 3 any Hamiltonian vector field associated to an exact Poisson structure is completely integrable.*
Let us remark that previously, in [5], the same notion was called *locally exact*.

The paper is structured as follows. The first section recalls the definition of last multipliers and some previous results. Characterizations in terms of other types of differentials than the usual exterior derivative, namely Witten and Marsden, are recalled from [3]. For a fixed smooth function m, the set of vector fields admitting m as last multiplier is shown to be a Lie subalgebra of the Lie algebra of vector fields.

The next section is devoted to the announced extension to multivectors and the previous results regarding Marsden and Witten differentials are reobtained in this extended framework. Several consequences with respect to the Schouten bracket on multivectors are derived including the extension of final result from last paragraph.

In the following section the Poisson case is discussed and local expressions for the main results of this section are provided in terms of the bivector π defining the Poisson bracket. Again, last multipliers count for the "deformation" from exactness of a given Poisson structure. Two concrete examples (two-dimensional Poisson structures and Lie-Poisson structures) are discussed and some results of [30] are reobtained in this way.

The last section is dedicated to a new notion namely *exact Poisson cohomology for an unimodular Poisson structure* in $\mathbb{R}^n$. It is an open problem both the computation of this cohomology and the relation with classical Poisson cohomology. For this last theory details appear in [8] and [27].

1. Last Multipliers for Vector Fields

Let M be a real, smooth, n-dimensional manifold, $C^{\infty}(M)$ the algebra of smooth real functions on M, $\mathcal{X}(M)$ the Lie algebra of vector fields and $\Lambda^k(M)$ the $C^{\infty}(M)$-module of k-differential forms, $0 \leq k \leq n$. Assume that M is orientable with the fixed volume form $V \in \Lambda^n(M)$.

Let:

$$\dot{x}^i(t) = A^i\left(x^1(t), \ldots, x^n(t)\right), 1 \leq i \leq n$$

be an ODE system on M defined by the vector field $A \in \mathcal{X}(M)$, $A = \left(A^i\right)_{1\leq i\leq n}$ and let us consider the $(n-1)$-form $\Omega_A = i_A V \in \Lambda^{n-1}(M)$.

Definition 1.1. ([10, p. 107], [25, p. 428]) The function $m \in C^{\infty}(M)$ is called a *last multiplier* of the ODE system generated by A, (*last multiplier* of A, for short) if $m\Omega_A$ is closed:

$$d(m\Omega_A) := (dm) \wedge \Omega_A + md\Omega_A = 0. \tag{1.1}$$

For example, in dimension 2, the notions of the last multiplier and integrating factor are identical and Sophus Lie suggested a method to associate a last multiplier to every symmetry vector field of A (Theorem 1.1 in [13, p. 752]). Lie's method is extended to any dimension in [25].

Characterizations of last multipliers can be obtained in terms of Witten's differential [29] and Marsden's differential [17, p. 220]. If $f \in C^{\infty}(M)$ and $t \geq 0$, Witten deformation of the usual differential $d_{tf} : \Lambda^*(M) \to \Lambda^{*+1}(M)$ is defined by:

$$d_{tf} = e^{-tf} d e^{tf}$$

which means [29]:

$$d_{tf}(\omega) = tdf \wedge \omega + d\omega.$$

Hence, m is a last multiplier if and only if:

$$d_m\Omega_A = (1-m)\, d\Omega_A$$

i.e. Ω_A belongs to the kernel of the differential operator $d_m + (m-1)\, d : \Lambda^{n-1}(M) \to \Lambda^n(M)$. Marsden differential is $d^f : \Lambda^*(M) \to \Lambda^{*+1}(M)$ defined by:

$$d^f(\omega) = \frac{1}{f} d(f\omega)$$

and m is a last multiplier if and only if Ω_A is d^m-closed.

The following characterization of the last multipliers will be useful:

Lemma 1.2. ([25, p. 428]).

(i) *$m \in C^\infty(M)$ is a last multiplier for A if and only if:*

$$A(m) + m \cdot div_V A = 0 \tag{1.2}$$

where $div_V A$ is the divergence of A with respect to volume form V.

(ii) *Let $0 \neq h \in C^\infty(M)$ such that:*

$$L_A h := A(h) = (div_V A) \cdot h \tag{1.3}$$

Then $m = h^{-1}$ is a last multiplier for A.

Remark 1.3.

(i) Equation (1.2) is exactly the time-independent version of LE from the Introduction. An important feature of equation (1.2) is that it does not always admit solutions [11, p. 269].

(ii) In the terminology of [1, p. 89], a function *h* satisfying (1.3) is called an *inverse multiplier*.

(iii) A first result given by (1.2) is the characterization of last multipliers for divergence-free vector fields: *$m \in C^\infty(M)$ is a last multiplier for the divergenceless vector field A if and only if m is a first integral of A.* The importance of this result is shown by the fact that three remarkable classes of divergence-free vector fields are provided by: Killing vector fields in Riemannian geometry, Hamiltonian vector fields in symplectic geometry and Reeb vector fields in contact geometry. Also, there are many equations of mathematical physics corresponding to the vector fields without divergence.

(iv) For the general case, namely A is not divergenceless, there is a strong connection between the first integrals and the last multipliers as well. Namely, from properties of Lie derivative, the ratio of two last multipliers is a first integral and conversely, the product between a first integral and a last multiplier is a last multiplier. So, denoting $FInt(A)$ the set of first integrals of A, since $FInt(A)$ is a subalgebra in $C^\infty(M)$ it results that the set of last multipliers for A is a $FInt(A)$-module.

(v) Recalling formula:

$$div_V(fX) = X(f) + f div_V X \tag{1.4}$$

it follows that m is a last multiplier for A if and only if the vector field mA is with null divergence i.e. $div_V(mA) = 0$. Thus, the set of last multipliers is a "measure of how far away" is A from being divergence-free.

An important structure generated by a last multiplier is given by:

Proposition 1.4. *Let $m \in C^{\infty}(M)$ be fixed. The set of vector fields admitting m as last multiplier is a Lie subalgebra in $\mathcal{X}(M)$.*

Proof. Let X and Y be vector fields with the required property. Since [18, p. 123]:

$$div_V[X, Y] = X(div_V Y) - Y(div_V X)$$

one has:

$$\begin{aligned} & [X, Y](m) + m div_V [X, Y] \\ = & (X(Y(m)) + mX(div_V Y)) - (Y(X(m)) + mY(div_V X)) \\ = & (-div_V Y \cdot X(m)) - (-div_V X \cdot Y(m)) \\ = & div_V Y \cdot m div_V X - div_V X \cdot m div_V Y = 0. \end{aligned}$$ ■

2. Last Multipliers for Multivectors

Denote by $\mathcal{X}^k(M)$ the $C^{\infty}(M)$-module of k-vector fields, $1 \leq k \leq n$ and fix $A \in \mathcal{X}^k(M)$. The multivector A defines the map $i_A : \Lambda^p(M) \to \Lambda^{p-k}(M)$ given by:
· $< i_A\omega, B > = < \omega, A \wedge B >$ for every $B \in \mathcal{X}^{p-k}(M)$ with $<,>$ the natural duality between forms and multivectors and $\wedge$ the Grassmann wedge product on $\bigoplus_{k=1}^{n} \mathcal{X}^k(M)$, if $p \geq k$,
· $i_A\omega = 0$ if $p < k$.

It follows that on (M, V) lives the map:

$$V^{\flat} : \mathcal{X}^k(M) \to \Lambda^{n-k}(M), \qquad V^{\flat}(A) = i_A V, \tag{2.1}$$

which is a $C^{\infty}(M)$-isomorphism between $\mathcal{X}^k(M)$ and $\Lambda^{n-k}(M)$, for $0 \leq k \leq n$. The inverse map of $V^{\flat}$ is denoted $V^{\natural} : \Lambda^{n-k}(M) \to \mathcal{X}^k(M)$.

Definition 2.1. ([8, p. 70]). The map $D_V : \mathcal{X}^k(M) \to \mathcal{X}^{k-1}(M)$:

$$D_V = V^{\natural} \circ d \circ V^{\flat}, \tag{2.2}$$

is called *the curl operator* with respect to the volume form V. So, if $A \in \mathcal{X}^k(M)$ then $D_V A$ is called *the curl of* A.

Example 2.2. ([8, p. 70]). If $k = 1$ then $D_V = div_V$. Indeed, if $A \in \mathcal{X}(M)$ then:

$$(D_V A) V = V^\flat \circ D_V (A) = d \circ V^\flat (A) = d \circ i_A (V) = L_A V = (div_V A) V.$$

Inspired by this example and relation (1.4) we introduce here the main notion of this paper:

Definition 2.3. The function $m \in C^\infty(M)$ is called a *last multiplier* of $A \in \mathcal{X}^k(M)$ if:

$$D_V(mA) = 0. \tag{2.3}$$

Since $V^\natural$ is a $C^\infty(M)$-isomorphism between $\Lambda^{n-k}(M)$ and $\mathcal{X}^k(M)$ it results that (2.3) means $d\left(V^\flat(mA)\right) = 0$ i.e.:

$$d\left(mV^\flat(A)\right) = 0 \tag{2.4}$$

which is the natural extension of condition (1.1) from Definition 1.1. With the same computation as in the previous section we derive the following equivalent characterizations of last multipliers for $A \in \mathcal{X}^k(M)$:
· in terms of Witten differential: $V^\flat(A) = i_A V$ belongs to the kernel of the differential operator $d_m + (m-1)\, d : \Lambda^{n-k}(M) \to \Lambda^{n-k+1}(M)$,
· in terms of Marsden differential: $V^\flat(A) = i_A V$ is d^m-closed with $d^m : \Lambda^k(M) \to \Lambda^{k+1}(M)$ as in Section 1.

From the $C^\infty(M)$-linearity of $V^\flat$ we have $V^\flat(mA) = mV^\flat(A) = (mV)^\flat(A)$ and then $(mV)^\natural = \frac{1}{m} V^\natural$ (we suppose $m > 0$ everywhere). It follows:

$$mD_{mV}(A) = V^\natural \circ d \circ V^\flat(mA) = D_V(mA) \tag{2.5}$$

which yields:

Proposition 2.4. *$m \in C^\infty(M)$ is a last multiplier of $A \in \mathcal{X}^k(M)$ if and only if*:

$$D_{mV}(A) = 0. \tag{2.6}$$

The last formula has some important consequences, all in terms of an operation on $\bigoplus_{k=1}^{n} \mathcal{X}^k(M)$ called *Schouten bracket* $[,]$ which is a natural generalization of Lie bracket from $\mathcal{X}(M)$ and generates a *Gersternhaber algebra* structure on the set of multivectors, [14]. For details regarding this bracket see [8], [27]. The first corollary of (2.6) is a formula for the curl:

Proposition 2.5. *If $m \in C^{\infty}(M)$ is a non-vanishing last multiplier of $A \in \mathcal{X}^k(M)$ then the curl of A can be expressed in terms of the Schouten bracket*:

$$D_V A = -[A, \ln|m|]\,. \tag{2.7}$$

Proof. Is a direct consequence of formula (2.90) from [8, p. 71]:

$$D_{mV} A = D_V A + [A, \ln|m|]\,.$$ ∎

A second formula relates the Schouten bracket with the product $\wedge$ of $\bigoplus_{k=1}^{n} \mathcal{X}^k(M)$. After [8, Th. 2.6.7 p. 71] if A is an a-multivector and B is a b-multivector then:

$$[A, B] = (-1)^b D_V(A \wedge B) - (D_V A) \wedge B - (-1)^b A \wedge (D_V B)\,. \tag{2.8}$$

Corollary 2.6. *Let $m \in C^{\infty}(M)$ be a last multiplier for both A and B. Then m is a last multiplier for $A \wedge B$ if and only if A and B Schouten-commutes i.e. their Schouten bracket vanishes*: $[A, B] = 0$.

Another consequence of (2.6) is a straightforward generalization of Proposition 1.4:

Theorem 2.7. *Let $m \in C^{\infty}(M)$ be fixed. The set of multivectors admitting m as last multiplier is a Gerstenhaber subalgebra in $\bigoplus_{k=1}^{n} \mathcal{X}^k(M)$.*

Proof. The curl operator is, up to a sign, a derivation of the Schouten bracket, namely [8, p. 71]:

$$D_V[A, B] = [A, D_V B] + (-1)^{b-1}[D_V A, B]\,. \tag{2.9}$$

This relation combined with (2.6) gives the conclusion. ∎

Definition 2.8. ([30]). The multivector A is called *exact* with respect to the volume form V if $D_V(A) = 0$.

Remark 2.9.

(i) It follows from (2.3) that the set of last multipliers of A is a "measure of how far away" is A from being exact.

(ii) Equation (2.8) gives that if A and B are exact multivectors then $A \wedge B$ is exact if and only if they Schouten-commutes.

(iii) Using again (2.9) it results that the set of exact multivectors is a Schouten subalgebra in $\bigoplus_{k=1}^{n} \mathcal{X}^k(M)$.

Example 2.10. From [19] the volume form V yields a Nambu multivector, [8, p. 160], $A_V \in \mathcal{X}^n(M)$; if $\left(x^1, \ldots, x^n\right)$ is a local chart on M such that $V = f dx^1 \wedge \ldots \wedge dx^n$ then $A_V = \frac{1}{f}\frac{\partial}{\partial x^1} \wedge \ldots \wedge \frac{\partial}{\partial x^n}$. A straightforward computation gives that A_V is exact with respect to V.

Remark 2.11. Let $f \in C^\infty(M)$ and A an a-multivector. From (2.8) and $D_V(f) = 0$ we get:

$$[A, f] = D_V(fA) - fD_V(A)$$

and then $D_V(fA) = fD_V(A)$ if and only if f is a *Casimir* of A i.e. $[A, f] = 0$. Connecting this with Remarks 2.9. (ii) we derive:

Proposition 2.12. *If A is exact then fA is exact if and only if f is a Casimir function of A.*

3. Last Multipliers for Poisson Bivectors

Let us assume that M is endowed with a Poisson bracket $\{,\}$ induced by the Poisson bivector $\pi \in \mathcal{X}^2(M)$. Let $f \in C^\infty(M)$ and $A_f \in (M)$ be the associated *Hamiltonian vector field* of the *Hamiltonian* f, [18].

Given the volume form V there exists a unique vector field $X_{\pi,V}$, called the *modular vector field*, so that [15], [28]:

$$div_V A_f = X_{\pi,V}(f). \tag{3.1}$$

From Proposition 1 of [7, p. 4] we have:

$$X_{\pi,V} = D_V(\pi). \tag{3.2}$$

Definition 3.1. The triple (M, π, V) is called [28] *unimodular* if $X_{\pi,V}$ is a Hamiltonian vector field, A_ρ of $\rho \in C^\infty(M)$. The triple (M, π, V) is called [6], [30] *exact* if $X_{\pi,V}$ is identically zero.

Let us introduce:

Definition 3.2. The function $m \in C^\infty(M)$ is called a *last multiplier* of (M, π, V) if:

$$D_V(m\pi) = 0 \tag{3.3}$$

equivalently:

$$D_{mV}(\pi) = 0. \tag{3.4}$$

It results that the set of the last multipliers of (M, π, V) is a "measure of how far away" is (M, π, V) from being exact and the characterization:

Proposition 3.3. $m \in C^{\infty}(M)$ *is a last multiplier* of (M, π, V) *if and only if*:

$$X_{\pi, mV} = 0. \tag{3.5}$$

Example 3.4.

(i) Poisson structures induced by symplectic structures are exact. This statement appears in the introduction of [30] and we provide here a proof using [28](or item 1 of Remark 2.3. from [30]): a Poisson structure is exact with respect to V if and only if V is invariant of any Hamiltonian vector field A_f. But in symplectic geometry this is a well-known fact.

(ii) A condition for a quadratic Poisson structure on $I\!R^3$ to be exact is given in Example 5.6.8. from [8, p. 149].

The two notions of Definition 3.1 are equivalent as it is pointed out in [6]. Moreover, in the MR review of [30] it is put in evidence that at local level there is no problem about the dependence of volume form V. So, in the following we work in local coordinates. Let $\left(x^1, \ldots, x^n\right)$ be a local chart on M such that $V = dx^1 \wedge \ldots \wedge dx^n$ and the bivector π of $(M, \{,\})$ is: $\pi = \sum\limits_{i<j} \pi^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$.

Denoting $\pi^i = \sum\limits_{j=1}^{n} \frac{\partial \pi^{ij}}{\partial x^j}$ we have [7, Proposition 1, p. 4], [6]:

$$X_{\pi, V} = \sum_{i=1}^{n} \pi^i \frac{\partial}{\partial x^i} \tag{3.6}$$

and then, Proposition 3.3 becomes:

Proposition 3.5. $m \in C^{\infty}(M)$ *is a last multiplier for* (M, π, V) *if and only if*:

$$\pi^i_m := \sum_{j=1}^{n} \frac{\partial \left(m \pi^{ij}\right)}{\partial x^j} = 0, \qquad 1 \leq i \leq n. \tag{3.7}$$

Examples 3.6.

3.6.1.

After [27, p. 31] the bivector $\pi = h(x, y) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ defines a Poisson structure on $I\!R^2$. So, $\pi^{12} = -\pi^{21} = h$ and then (3.7) becomes:

$$\frac{\partial (mh)}{\partial y} = -\frac{\partial (mh)}{\partial x} = 0$$

with the obvious solution $m_\pi = \frac{C}{h}$ (if we suppose $h > 0$ everywhere), where C is a real constant. Therefore, on the Poisson manifold $\left(I\!R^2, \pi\right)$ above, the function C/h is a last multiplier.

In this way we reobtain part (a) of Theorem 3.2. from [30] that any smooth 2-dimensional Poisson structure is exact if and only if it is constant; indeed the exact Poisson $m_\pi \cdot \pi = C\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ is constant. Also, the second phrase of Remark 3.2. item 3): *the set of exact 2-dimensional Poisson structures is a 1-dimensional space isomorphic with* $I\!R$ is also verified.

3.6.2. Lie-Poisson structures

The interest for this example is pointed out in [30]: *Lie-Poisson structures play important roles in studying normal forms for a class of Poisson structures.*

Let $\mathcal{G}$ be an n-dimensional Lie algebra with a fixed basis $B = \{e_i\}_{1 \leq i \leq n}$ and let $B^* = \{e^i\}$ be the dual basis on the dual $\mathcal{G}^*$. Recall the definition of *structure constants* of $\mathcal{G}$:

$$[e_i, e_j] = c^k_{ij} e_k.$$

Then, on $\mathcal{G}^*$ we have the so-called *Lie-Poisson structure* given by [27, p. 31]:

$$\pi^{ij}\left(x_u e^u\right) = c^k_{ij} x_k. \tag{3.8}$$

We get:

$$\pi^i_m = \sum_{j=1}^{n} c^k_{ij} \frac{\partial (m x_k)}{\partial x_j} \tag{3.9}$$

Particular case: n=2

Although from the previous example we know all about the 2-dimensional case it is interesting to reobtain the conclusion within this example. The structure relations $[e_1, e_1] = [e_2, e_2] = 0, [e_1, e_2] = c^1_{12} e_1 + c^2_{12} e_2$ yield:

$$\begin{cases} \pi^1_m = c^2_{12} m + \dfrac{\partial m}{\partial y}\left(c^1_{12} x + c^2_{12} y\right) \\ \pi^2_m = -c^1_{12} m - \dfrac{\partial m}{\partial x}\left(c^1_{12} x + c^2_{12} y\right) \end{cases}. \tag{3.10}$$

Supposing $\mathcal{G}$ nontrivial (i.e. $\left(c^1_{12}\right)^2 + \left(c^2_{12}\right)^2 > 0$) there result three cases:
I) $c^1_{12} \cdot c^2_{12} \neq 0$ i.e. $h = c^1_{12} x + c^2_{12} y$. From the system (3.7) $\pi^1_m = \pi^2_m = 0$ we have:

$$c^2_{12} \frac{\partial m}{\partial x} - c^1_{12} \frac{\partial m}{\partial y} = 0 \tag{3.11}$$

with solution $m = A\left(\frac{x}{c_{12}^2} + \frac{y}{c_{12}^1}\right) + B$ which replaced in (3.10) yields $A = B = 0$. In conclusion, the last multiplier of π for this case is zero and the associated Poisson structure is trivial (hence exact).
II) $c_{12}^2 = 0$ (i.e. $h = c_{12}^1 x$) with solution $m = m(x)$ of (3.11). Inserting this function in (3.10_2) we get $m + x \cdot m' = 0$ with solution $m_\pi = \frac{C}{x}$.
III) $c_{12}^1 = 0$ (i.e. $h = c_{12}^2 y$) with solution $m = m(y)$ of (3.12). With the same computations as above it results $m_\pi = \frac{C}{y}$.

4. Exact Poisson Cohomology of Unimodular Poisson Structures

Returning to the general case of Poisson structures in $\mathbb{R}^n$ let us point out an interesting consequence of (2.8) and (2.9) respectively:

Proposition 3.7.

(i) *Let $X, Y \in \mathcal{X}(\mathbb{R}^n)$ be such that:*

- (a) *their wedge product $\pi = X \wedge Y$ is a Poisson structure,*
- (b) *they Lie-commutes: $[X, Y] = 0$.*
- (c) *they are divergence-free. Then π is an unimodular Poisson bivector.*

(ii) *Let π be a Poisson structure and $X \in \mathcal{X}(\mathbb{R}^n)$ such that their Schouten bracket $[\pi, X]$ is again a Poisson structure. If π is unimodular and X is divergence-free then $[\pi, X]$ is unimodular.*

(iii) *Let π be an unimodular Poisson structure and A an exact multivector. Then their Schouten bracket $[\pi, A]$ is an exact multivector.*

In the following suppose $(\mathbb{R}^n, \pi)$ is an unimodular Poisson manifold. Let us consider, after [8, p. 39], the map $\delta_\pi : \bigoplus_{k=1}^{n} \mathcal{X}^k(\mathbb{R}^n) \to \bigoplus_{k=1}^{n} \mathcal{X}^k(\mathbb{R}^n), \delta_\pi(A) = [\pi, A]$ (for a local expression see [27, Formula (4.8), p. 43]) and let us denote $\mathcal{X}_e^k(\mathbb{R}^n)$ the set of exact k-multivectors. From the last item of the previous result and the fact that $\left(\bigoplus_{k=1}^{n} \mathcal{X}^k(\mathbb{R}^n), \delta_\pi\right)$ is a complex [8, p. 39], it results a new differential complex:

$$\ldots \to \mathcal{X}_e^{k-1}(\mathbb{R}^n) \xrightarrow{\delta_\pi} \mathcal{X}_e^k(\mathbb{R}^n) \xrightarrow{\delta_\pi} \mathcal{X}_e^{k+1}(\mathbb{R}^n) \to \ldots \tag{4.1}$$

which will be called *the exact Lichnerowicz complex*. Let us call the cohomology of this complex *exact Poisson cohomology*. Obviously, the exact Poisson cohomology is included in the usual Poisson cohomology treated in detail in [8] and [27].

Therefore we set the exact Poisson groups:

$$H^k_e(I\!R^n, \pi) = \frac{\ker\{\delta_\pi : \mathcal{X}^k_e(I\!R^n) \to \mathcal{X}^{k+1}_e(I\!R^n)\}}{Im\{\delta_\pi : \mathcal{X}^{k-1}_e(I\!R^n) \to \mathcal{X}^k_e(I\!R^n)\}}. \tag{4.2}$$

$H^k_e(I\!R^n, \pi)$ is a subgroup of the group $H^k(I\!R^n, \pi)$ of Poisson cohomology. For example $H^0_e(I\!R^n, \pi) = H^0(I\!R^n, \pi)$ which is the group of Casimir functions of π, [8, p. 40].

5. Conclusions

(0) The last multipliers constitute a measure to count the ”perturbation” from exactness. So, this notion can be thought in the framework of [20].

(1) The theory of the last multipliers can be extended from vector fields to general multivectors preserving a series of remarkable characterizations and results.

(2) An important structure generated by a last multiplier is of algebraic nature: the set of multivectors with a prescribed last multiplier is a Gerstenhaber subalgebra.

(3) From the two previous remarks it results that a natural extension of our theory seems to work on Lie algebroids using the tools of [12] and [14]. Hence, a sequel paper [4] is forthcoming.

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