

NEW ASPECTS IN KILLING TENSORS OF RANK TWO

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First integrals quadratic in moments corresponding to the natural Lagrangian systems are discussed with a special view towards the two-dimensional case. Since these first integrals are provided by Killing tensors of rank two we pointed out some geometrical features of this class of tensor fields.

Key words: Natural Lagrangian system, First integral, Killing tensors.

INTRODUCTION

In four-dimensional space times like the Kerr metric [1], the existence of conserved quantities for geodesics (constants of motion) [2] and the tensorial structures that generate them Killing vectors, Killing tensors [3, 18], and Killing-Yano tensors [4–13] have been very important, not only elucidating particle motion in these space times, but also leading to the separation of the Klein-Gordon [2], massless neutrino [14, 15], massive Dirac [16, 17], electromagnetic [14], and gravitational wave [14] equations.

Several dynamical systems modelling physical or biological processes are of Lagrangian type, namely are described by a Lagrangian $L(x, \dot{x})$ where x is the dependent variables and the dot means the derivative with respect to the independent variable, usually considered as proper time, denoted t . If the given system has n degrees of freedom and the Lagrangian has the form

$$L = \frac{1}{2} \delta_{ij} \dot{x}^i \dot{x}^j - V(x)$$

where $x = (x^i)_{1 \leq i \leq n}$, $\dot{x} = (\dot{x}^i)_{1 \leq i \leq n}$ which is called *natural Lagrangian system* and V represents *the potential*. The evolution paths of the system are solutions of Euler-Lagrange equations:

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$$E_i(L): \frac{d^2 x}{dt^2}(t) + \frac{\partial V}{\partial x^i}(x(t)) = 0, \quad 1 \leq i \leq n \quad (1)$$

which is a system of ordinary differential equations with special features for the *natural case*.

The present paper is devoted to a main notion in the study of (1), with great importance from a dynamical point of view, namely *first integrals*. These are functions $\mathbf{F}(x, \dot{x})$ such that for every solution $x(t)$ of (1) we have: $\mathbf{F}\left(x(t), \frac{dx}{dt}(t)\right) =$ constant, namely it does not depend of t .

More precisely, we search for quadratic in moments first integrals, namely:

$$\mathbf{F} = A_{ab}(x) \dot{x}^a \dot{x}^b + \Phi(x). \quad (2)$$

2. HAMILTONIAN AND FIRST INTEGRALS OF MOTION

As it is known (1) admits the first integral:

$$H = \frac{1}{2} \delta_{ab} \dot{x}^a \dot{x}^b + V \quad (3)$$

called *Hamiltonian* of (1) which is quadratic.

In [21, p. 193] there are given the equations in order to obtain \mathbf{F} from (2). We present on short the arguments of cited paper. Namely, set $p_i = \frac{\partial L}{\partial \dot{x}^i} = \delta_{ij} \dot{x}^j = \dot{x}^i$, then $F = A^{ab}(x) p_a p_b + \Phi(x)$ where $A^{ab} = \delta^{ai} \delta^{bj} A_{ij}$ is the contravariant version of the covariant tensor A . The Hamiltonian becomes: $H = \frac{1}{2} \delta^{ab} p_a p_b + V$ and then \mathbf{F} is a first integral if and only if the usual Poisson bracket $\{\mathbf{F}, H\}$ vanishes. In [21, p. 193] is obtained:

$$\{\mathbf{F}, H\} = \delta^{ab} p_b (A^{cd}{}_{,a} p_c p_d + \Phi_{,a}) - 2A^{ab} p_a V_{,b} \quad (4)$$

where $T_{\dots, a}$ means the derivative with respect to x^a i.e. $T_{\dots, a} = \frac{\partial T_{\dots}}{\partial x^a}$. It follows:

Proposition ([21]). \mathbf{F} is first integral if and only if

$$\begin{cases} A_{(ab,c)} = 0 \\ 2A^b{}_a V_{,b} = \Phi_{,a} \end{cases} \quad (5)$$

3. FIRST INTEGRALS AND SUFFICIENT CONDITIONS

Our main remark is that (5) can be obtained exactly in the Lagrangian framework and the Hamiltonian setting (moments, Poisson bracket) is not necessary. The starting point for our approach is given by the following Lemma

Lemma. *The function $f(q)$ vanishes on the solutions of the system $\phi_\alpha(q) = 0, 1 \leq \alpha \leq n$ if there exist functions $\mu^\alpha(q), 1 \leq \alpha \leq n$ such that $f = \mu^\alpha \phi_\alpha$.*

Therefore, \mathcal{F} is first integral of (1) if there exist functions $\mu^i(x, \dot{x}), 1 \leq i \leq n$ such that $\frac{d\mathcal{F}}{dt} = \mu^i E_i(L)$ and this relation yields exactly (5). We give here a proof only for the case $n = 2$.

For this case we have

$$\begin{cases} E_1(L) = \ddot{x}^1 + V_{,1} \\ E_2(L) = \ddot{x}^2 + V_{,2} \\ \mathcal{F} = A(\dot{x}^1)^2 + 2B\dot{x}^1\dot{x}^2 + C(\dot{x}^2)^2 + \Phi \end{cases} \quad (6)$$

or denoting $A^{(2)} = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$ it results that $\mathcal{F} = (\dot{x}^1, \dot{x}^2) \cdot A^{(2)} \cdot \begin{pmatrix} \dot{x}^1 \\ \dot{x}^2 \end{pmatrix} + \Phi$ which yields

$$\begin{aligned} \frac{d\mathcal{F}}{dt} &= (\dot{x}^1, \dot{x}^2) A^{(2)} \begin{pmatrix} \ddot{x}^1 \\ \ddot{x}^2 \end{pmatrix} + (\dot{x}^1, \dot{x}^2) \frac{d}{dt} A^{(2)} \begin{pmatrix} \dot{x}^1 \\ \dot{x}^2 \end{pmatrix} + (\dot{x}^1, \dot{x}^2) A^{(2)} \begin{pmatrix} \dot{x}^1 \\ \dot{x}^2 \end{pmatrix} + \frac{d\Phi}{dt} = \\ &= \mu^1 (\ddot{x}^1 + V_{,1}) + \mu^2 (\ddot{x}^2 + V_{,2}). \end{aligned} \quad (7)$$

Then,

$$\begin{cases} \mu^1 = 2(A\dot{x}^1 + B\dot{x}^2) \\ \mu^2 = 2(B\dot{x}^1 + C\dot{x}^2) \end{cases} \text{ or } \begin{pmatrix} \mu^1 \\ \mu^2 \end{pmatrix} = 2A^{(2)} \cdot \begin{pmatrix} \dot{x}^1 \\ \dot{x}^2 \end{pmatrix}. \quad (8)$$

Returning to (7) it follows

$$\begin{cases} A_{,1} = 0 \\ A_{,2} + 2B_{,1} = 0 \\ 2B_{,2} + C_{,1} = 0 \\ C_{,2} = 0 \end{cases}$$

which is exactly (5) and

$$\begin{cases} \Phi_{,1} = 2(AV_{,1} + BV_{,2}) \\ \Phi_{,2} = 2(BV_{,1} + CV_{,2}) \end{cases}$$

which is exactly (5). The last relation can be written as

$$\begin{pmatrix} \frac{\partial}{\partial x^1} \\ \frac{\partial}{\partial x^2} \end{pmatrix} \Phi = 2A^{(2)} \cdot \begin{pmatrix} \frac{\partial}{\partial x^1} \\ \frac{\partial}{\partial x^2} \end{pmatrix} V. \quad (9)$$

An important observation is that (5) does not depend of the potential V i.e. does not depend of system. In [21, p. 195] is given the general solution of (5):

$$A^{(2)} = aM + bL_1 + cL_2 + eE_1 + dE_2 + gE_3 \quad (10)$$

with a, b, c, d, e, g arbitrary real numbers and

$$M = \begin{pmatrix} y^2 & -xy \\ -xy & x^2 \end{pmatrix}, \quad L_1 = \begin{pmatrix} 0 & -y \\ -y & 2x \end{pmatrix}, \quad L_2 = \begin{pmatrix} 2y & -x \\ -x & 0 \end{pmatrix} \quad (11)$$

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (12)$$

Let us remark that in the cited paper the matrices M, L_1, L_2, E_3 appear times $\frac{1}{2}$ but this factor can be included in the parameters a, b, c, g .

Let us discuss in details the last three solutions within a simpler notation $(x^1, x^2) = (x, y)$. Firstly, let us remark that from a geometrical point of view the matrices E_1 and E_2 are the projections on the x -axis respectively the y -axis while $2E_3$ is the basis of the Lie algebra $lor(1, 1)$ ([19, p. 162, Remark 6.5]) associated to the Lorentzian Lie group [19, p. 161, Example 6.4]:

$$Lor(1,1) = \left\{ \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}; t \in \mathbb{R} \right\}. \quad (13)$$

Secondly, let us give some examples of potentials corresponding to these first integrals

A) For E_1 the potential $V = cx$ with c a real constant yields the Euler-Lagrange equations:

$$\begin{cases} \ddot{x} + c = 0 \\ \ddot{y} = 0 \end{cases} \quad (14)$$

Then $\Phi = 2cx$ and:

$$\mathcal{F} = \dot{x}^2 + 2cx. \quad (15)$$

For E_2 a similar example holds with the twist $x \leftrightarrow y$.

B) For E_3 let us presents two examples

B.1)

The linear potential $\Phi = \alpha x + \beta y$ yields the Euler-Lagrange equations:

$$\begin{cases} \ddot{x} + \alpha = 0 \\ \ddot{y} + \beta = 0 \end{cases} \quad (16)$$

Then $\Phi = \beta x + \alpha y$ and:

$$\mathcal{F} = \dot{x}\dot{y} + \beta x + \alpha y. \quad (17)$$

B.2)

The product potential $\Phi = xy$ gives the Euler-Lagrange equations:

$$\begin{cases} \ddot{x} + y = 0 \\ \ddot{y} + x = 0 \end{cases} \quad (18)$$

Then $\Phi = \frac{1}{2}(x^2 + y^2)$ and:

$$\mathcal{F} = \dot{x}\dot{y} + \frac{1}{2}(x^2 + y^2). \quad (19)$$

Let us end this note with the form of equation (5) in a general Riemannian metric which appears in [22, p. 141, Eq. (4.51)]:

$$\begin{cases} \nabla_1 A = 0 \\ \nabla_2 A + 2\nabla_1 B = 0 \\ 2\nabla_2 B + \nabla_1 C = 0 \\ \nabla_2 C = 0 \end{cases} \quad (20)$$

4. CONCLUSIONS

Finding the first integrals of motion for a given Hamiltonian system is an important task especially for a two-dimensional dynamical systems. Killing tensors play an important role in finding the integral of motions. In this study a lemma is presented and sufficient conditions for finding the first integrals of motion are

given in two dimensional case. Illustrative examples show the validity of the new proposed approach.

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REFERENCES

1. R.P. Kerr, Phys. Rev. Lett. **11**, 237 (1963).
2. B. Carter, Commun. Math. Phys. **10**, 280 (1968).
3. P. Stackel, C.R. Acad. Sci. Paris, **121**, 489 (1895).
4. K. Yano, Ann. Math. **55**, 328 (1952).
5. R. Penrose, Ann. N.Y. Acad. Sci. **224**, 125 (1973).
6. R. Floyd, The Dynamics of Kerr Fields, Ph.D. Thesis, London (1973).
7. I.I. Cotaescu, M. Visinescu, Fortschr. Phys. **56**, 400 (2008).
8. M. Visinescu, Class. Quant. Grav. **11**, 1867 (1994).
9. M. Visinescu, Phys. Lett. B, **339**, 28 (1994).
10. I.I. Cotaescu, M. Visinescu, Mod. Phys. Lett. **15**, 145 (2000).
11. D. Baleanu, S. Baskal, Nuovo Cim. B, **117**(5), 501 (2002).
12. D. Baleanu, S. Baskal, Int. J. Mod. Phys. A, **17**(26), 3737 (2002).
13. V.P. Frolov, Progr. Theor. Phys. Suppl. **172**, 210 (2008).
14. S.A. Teukolsky, Astrophys. J. **185**, 635(1973).
15. W. Unruh, Phys. Rev. Lett. **31**, 1265 (1973).
16. S. Chandrasekhar, Proc. Roy. Soc. London, A **349**, 571 (1976).
17. D.N. Page, Phys. Rev. D **14**, 1509(1976).
18. M. Walkerand, R. Penrose, Commun. Math. Phys. **18**, 265 (1970).
19. Andrew Baker, *Matrix Groups. An Introduction to Lie Group Theory*, Springer Undergraduate Mathematics Series, 2nd printing, Springer-Verlag London, Ltd. (2003).
20. I.M. Benn, J. Math. Phys. **47**(2), 022903 (2006).
21. D. Boccaletti, G. Pucacco, Nuovo Cim., B **112**(2-3), 181 (1997).
22. Marco Pettini, *Geometry and Topology to Hamiltonian Dynamics and Statistical Mechanics*, Interdisciplinary Applied Mathematics 33, Springer, New York (2007).