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From the Eisenhart Problem to Ricci Solitons in f-Kenmotsu Manifolds

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Abstract. The Eisenhart problem of finding parallel tensors is solved for the symmetric case in the regular f-Kenmotsu framework. In this way, the Olszack-Rosca example of Einstein manifolds provided by f-Kenmotsu manifolds via locally symmetric Ricci tensors is recovered as well as a case of Killing vector fields. Some other classes of Einstein-Kenmotsu manifolds are presented. Our result is interpreted in terms of Ricci solitons and special quadratic first integrals.

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Dedicated to the memory of Neculai Papaghiuc 1947–2008

1. Introduction

In 1923, Eisenhart [9] proved that if a positive definite Riemannian manifold (M, g) admits a second order parallel symmetric covariant tensor other than a constant multiple of the metric tensor, then it is reducible. In 1926, Levy [18] proved that a second order parallel symmetric non-degenerated tensor α in a space form is proportional to the metric tensor. Note that this question can be considered as the dual to the the problem of finding linear connections making parallel a given tensor field; a problem which was considered by Wong in [35]. Also, the former question implies topological restrictions, namely, if the (pseudo) Riemannian manifold M admits a parallel symmetric $(0, 2)$ tensor field, then M is locally the direct product of a number of (pseudo) Riemannian manifolds [36] (cited by [37]). Another situation where the parallelism of α is involved appears in the theory of totally geodesic maps, namely, as is point out in [22, p. 114], $\nabla\alpha = 0$ is equivalent with the fact that $1 : (M, g) \to (M, \alpha)$ is a totally geodesic map.

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While both Eisenhart and Levy work locally, Ramesh Sharma gives in [26] a global approach based on Ricci identities. In addition to space-forms, Sharma considered this *Eisenhart problem* in contact geometry [27, 28, 29], for example, for K-contact manifolds in [28]. Since then, several other studies appeared in various contact manifolds: Nearly-Sasakian [33], (para) P-Sasakian [6, 19, 32], α-Sasakian [5]. Another framework was that of quasi-constant curvature in [13]. Also, contact metrics with nonvanishing ξ-sectional curvature are studied in [10].

Returning to contact geometry, an important class of manifolds are introduced by Kenmotsu in [15] and generalized by Olszack and Rosca in [21]. Recently, there is an increasing flow of papers in this direction, e.g., that of our Professor N. Papaghiuc [23, 24] to whom we dedicate this short note. Motivated by this fact, we studied the case of f-Kenmotsu manifolds satisfying a special condition called *regular* and show that a symmetric parallel tensor field of second order must be a constant multiple of the Riemannian metric. There are three remarks regarding our result:

(i) It is in agreement with what happens in all previously recalled contact geometries for the symmetric case,
(ii) it is obtained in the same manner as in Sharma's paper [26], and
(iii) yields a class of Einstein manifolds already indicated by Olszack and Rosca but with a more complicated proof.

Let us point out also that the anti-symmetric case appears without proof in [20].

Our main result is connected with the recent theory of Ricci solitons, a subject included in the Hamilton-Perelman approach (and proof) of Poincaré conjecture. Ricci solitons in contact geometry were first studied by Sharma in [11] and [30]; the preprint [34] is also available in arxiv. In these papers the K-contact and (k,μ)-contact (including Sasakian) cases are treated; thus our treatment for the Kenmotsu variant of almost contact geometry seems to be new.

Our work is structured as follows. The first section is a very brief review of Kenmotsu geometry and Ricci solitons. The next section is devoted to the (symmetric case of) Eisenhart problem in a f-Kenmotsu manifold and several situations yielding Einstein manifolds are derived. Also, the relationship with the Ricci solitons is pointed out. The last section offers a dynamical picture of the subject via Killing vector fields and quadratic first integrals of a special type.

2. f-Kenmotsu manifolds. Ricci solitons

Let M be a real $2n+1$-dimensional differentiable manifold endowed with an almost contact metric structure (φ, ξ, η, g):

$$\begin{aligned} &(a) \quad \varphi^2 = -I + \eta \otimes \xi, \quad (b) \quad \eta(\xi) = 1, \quad (c) \quad \eta \circ \varphi = 0, \\ &(d) \quad \varphi(\xi) = 0, \quad (e) \quad \eta(X) = g(X, \xi), \\ &(f) \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \end{aligned} \tag{2.1}$$

for any vector fields $X, Y \in \mathcal{X}(M)$ where I is the identity of the tangent bundle TM, φ is a tensor field of $(1,1)$-type, η is a 1-form, ξ is a vector field and g is a metric tensor field. Throughout the paper all objects are differentiable of class C^∞.

We say that $(M, \varphi, \xi, \eta, g)$ is an *f-Kenmotsu manifold* if the Levi-Civita connection of g satisfy [20]:

$$(\nabla_X \varphi)(Y) = f(g(\varphi X, Y)\xi - \varphi(X)\eta(Y)) \tag{2.2}$$

where $f \in C^\infty(M)$ is strictly positive and $df \wedge \eta = 0$ holds. A $f = constant \equiv \beta > 0$ is called *β-Kenmotsu manifold* with the particular case $f \equiv 1$-Kenmotsu manifold which is a usual *Kenmotsu manifold* [15].

In a general f-Kenmotsu manifold we have, [21]:

$$\nabla_X \xi = f(X - \eta(X)\xi) \tag{2.3}$$

and the curvature tensor field:

$$R(X, Y)\xi = f^2(\eta(X)Y - \eta(Y)X) + Y(f)\varphi^2 X - X(f)\varphi^2 Y \tag{2.4}$$

while the Ricci curvature and Ricci tensor are, [16]:

$$S(\xi, \xi) = -2n(f^2 + \xi(f)) \tag{2.5}$$

$$Q(\xi) = -2nf^2\xi - \xi(f)\xi - (2n-1)gradf. \tag{2.6}$$

In the last part of this section we recall the notion of Ricci solitons according to [30, p. 139]. On the manifold M, a *Ricci soliton* is a triple (g, V, λ) with g a Riemannian metric, V a vector field and λ a real scalar such that:

$$\mathcal{L}_V g + 2S + 2\lambda g = 0. \tag{2.7}$$

The Ricci soliton is said to be *shrinking, steady* or *expanding* according as λ is negative, zero or positive.

3. Parallel symmetric second order tensors and Ricci solitons in f-Kenmotsu manifolds

Fix α a symmetric tensor field of $(0,2)$-type which we suppose to be parallel with respect to ∇ i.e. $\nabla\alpha = 0$. Applying the Ricci identity

$$\nabla^2\alpha(X, Y; Z, W) - \nabla^2(X, Y; W, Z) = 0$$

we obtain the relation (1.1) of [26, p. 787]:

$$\alpha(R(X, Y)Z, W) + \alpha(Z, R(X, Y)W) = 0, \tag{3.1}$$

which is fundamental in all papers treating this subject. Replacing $Z = W = \xi$ and using (2.4) results in

$$f^2[\eta(X)\alpha(Y, \xi) - \eta(Y)\alpha(X, \xi)] + Y(f)\alpha(\varphi^2 X, \xi) - X(f)\alpha(\varphi^2 Y, \xi) = 0, \tag{3.2}$$

by the symmetry of α. With $X = \xi$ we derive

$$[f^2 + \xi(f)][\alpha(Y, \xi) - \eta(Y)\alpha(\xi, \xi)] = 0$$

and supposing $f^2 + \xi(f) \neq 0$ it results

$$\alpha(Y, \xi) = \eta(Y)\alpha(\xi, \xi). \tag{3.3}$$

Let us call a *regular f-Kenmotsu manifold* a f-Kenmotsu manifold with $f^2 + \xi(f) \neq 0$ and remark that β-Kenmotsu manifolds are regular.

Differentiating the last equation covariantly with respect to X we have

$$\alpha(\nabla_X Y, \xi) + f[\alpha(X, Y) - \eta(X)\eta(Y)\alpha(\xi, \xi)] = X(\eta(Y))\alpha(\xi, \xi), \tag{3.4}$$

which means via (3.3) with $Y \to \nabla_X Y$:

$$f[\alpha(X,Y) - \eta(X)\eta(Y)\alpha(\xi,\xi)] = [X(g(Y,\xi)) - g(\nabla_X Y,\xi)]\alpha(\xi,\xi)$$
$$= g(Y,\nabla_X \xi)\alpha(\xi,\xi) = f[g(X,Y) - \eta(X)\eta(Y)]\alpha(\xi,\xi). \tag{3.5}$$

From the positiveness of f we deduce that

$$\alpha(X,Y) = \alpha(\xi,\xi)g(X,Y) \tag{3.6}$$

which together with the standard fact that the parallelism of α implies the $\alpha(\xi,\xi)$ is a constant, via (2.3) yields:

Theorem 3.1. *A symmetric parallel second order covariant tensor in a regular f-Kenmotsu manifold is a constant multiple of the metric tensor. In other words, a regular f-Kenmotsu metric is irreducible which means that the tangent bundle does not admits a decomposition $TM = E_1 \oplus E_2$ parallel with respect of the Levi-Civita connection of g.*

Corollary 3.1. *A locally Ricci symmetric ($\nabla S \equiv 0$) regular f-Kenmotsu manifold is an Einstein manifold.*

Remark 3.1.

(1) The particular case of dimension three and β-Kenmotsu of our theorem appears in Theorem 3.1 from [7, p. 2689]. The above corollary has been proved by Olszack and Rosca in another way.

(2) In [2] it is shown the equivalence of the following statements for an Kenmotsu manifold:
 (i) Is Einstein,
 (ii) is locally Ricci symmetric,
 (iii) is Ricci semi-symmetric i.e. $R \cdot S = 0$ where

 $$(R(X,Y)\cdot S)(X_1,X_2) = -S(R(X,Y)X_1,X_2) - S(X_1,R(X,Y)X_2).$$

 The same implication (iii) $\to$ (i) for Kenmotsu manifolds is Theorem 1 from [14, p. 438]. But we have the implication (iii) $\to$ (i) in the more general framework of regular f-Kenmotsu manifols since $R \cdot S = 0$ means exactly (3.1) with α replaced by S. Every semisymmetric manifold, i.e. $R \cdot R = 0$, is Ricci-semisymmetric but the converse statement is not true.

In conclusion:

Proposition 3.1. *A Ricci-semisymmetric, particularly semisymmetric, regular f-Kenmotsu manifold is Einstein.*

Another class of spaces related to the Ricci tensor was introduced in [31]; namely a Riemannian manifold is a *special weakly Ricci symmetric space* if there exists a 1-form ρ such that

$$(\nabla_X S)(Y,Z) = 2\rho(X)S(Y,Z) + \rho(Y)S(Z,X) + \rho(Z)S(X,Y). \tag{3.7}$$

The same condition was sometimes called *generalized pseudo-Ricci symmetric manifold* [12] or simply *pseudo-Ricci symmetric manifold* [3]. By taking $X = Y = Z = \xi$ yields

$$\xi(S(\xi,\xi)) = 4\rho(\xi)S(\xi,\xi) \tag{3.8}$$

and then for a β-Kenmotsu manifold we get $\rho(\xi) = 0$. Returning to (3.7) with $Y = Z = \xi$ will result in $\rho(X) = 0$ for every vector field X and thus lead to a generalization of Theorem 3.3. in [1, p. 96].

Proposition 3.2. *A β-Kenmotsu manifold which is special weakly Ricci symmetric is an Einstein space.*

We close this section with applications of our Theorem to Ricci solitons:

Corollary 3.2. *Suppose that on a regular f-Kenmotsu manifold the $(0,2)$-type field $\mathcal{L}_V g+2S$ is parallel where V is a given vector field. Then (g, V) yield a Ricci soliton. In particular, if the given regular f- Kenmotsu manifold is Ricci-semisymmetric or semisymmetric with $\mathcal{L}_V g$ parallel, we have the same conclusion.*

Naturally, two situations appear regarding the vector field V: $V \in span\xi$ and $V \perp \xi$ but the second class seems far too complex to analyse in practice. For this reason it is appropriate to investigate only the case $V = \xi$.

We are interested in expressions for $\mathcal{L}_\xi g + 2S$. A straightforward computation gives

$$\mathcal{L}_\xi g(X,Y) = 2f(g(X,Y) - \eta(X)\eta(Y)) = 2fg(\varphi X, \varphi Y). \tag{3.9}$$

A general expression of S is known by us only for the the 3-dimensional case and η-Einstein Kenmotsu manifolds. Let us treat these situations in the following manner

(I) [8, p. 251]:

$$\begin{aligned} S(X,Y) = &\left(\frac{r}{2} + \xi(f) + f^2\right) g(X,Y) - \left(\frac{r}{2} + \xi(f) + 3f^2\right) \eta(X)\eta(Y) \\ &- Y(f)\eta(X) - X(f)\eta(Y) \end{aligned} \tag{3.10}$$

where r is the scalar curvature. Then, for a 3-dimensional f-Kenmotsu manifold we obtain

$$\begin{aligned} \alpha &:= (\mathcal{L}_\xi g + 2S)(X,Y) \\ &= (r + 2\xi(f) + 2f + 2f^2)g(X,Y) - (r + 2\xi(f) + 2f + 6f^2)\eta(X)\eta(Y) \\ &\quad - 2Y(f)\eta(X) - 2X(f)\eta(Y) \end{aligned} \tag{3.11}$$

while, for β-Kenmotsu

$$\alpha(X,Y) = (r + 2\beta + 2\beta^2)g(\varphi X, \varphi Y) - 4\beta^2\eta(X)\eta(Y), \tag{3.12}$$

$$\begin{aligned} (\nabla_Z \alpha)(X,Y) = Z(r)g(\varphi X, \varphi Y) &- \beta(r + 2\beta + 6\beta^2)[\eta(X)g(\varphi Y, \varphi Z) \\ &+ \eta(Y)g(\varphi X, \varphi Z)]. \end{aligned} \tag{3.13}$$

Substituting $Z = \xi, X = Y \in (span\xi)^\perp$, and respectively $X = Y = Z \in (span\xi)^\perp$ in (3.13), we derive that r is a constant, provided α is parallel. Thus, we can state the following.

Proposition 3.3. *A 3-dimensional β-Kenmotsu Ricci soliton (g, ξ, λ) is expanding and with constant scalar curvature.*

Proof. $\lambda = -\alpha(\xi, \xi)/2 = 2\beta^2$. ∎

At this point we remark that the Ricci solitons of almost contact geometry studied in [30] and [34] in relationship with the Sasakian case are shrinking and this observation is in accordance with the diagram of Chinea from [4] that Sasakian and Kenmotsu are opposite sides of the trans-Sasakian moon. Also, the expanding character may be considered as a manifestation of the fact that a β-Kenmotsu manifold can not be compact.

(II) Recall that the metric g is called *η-Einstein* if there exists two real functions a, b such that the Ricci tensor of g is

$$S = ag + b\eta \otimes \eta.$$

For an η-Einstein Kenmotsu manifold we have, [14, p. 441]:

$$S(X,Y) = \left(\frac{r}{2n} + 1\right) g(X,Y) - \left(\frac{r}{2n} + 2n + 1\right) \eta(X)\eta(Y) \tag{3.14}$$

and then

$$\alpha(X,Y) = \left(\frac{r}{n} + 4\right) g(X,Y) - \left(\frac{r}{n} + 4 + 4n\right) \eta(X)\eta(Y) \tag{3.15}$$

$$\begin{aligned}(\nabla_Z \alpha)(X,Y) = \frac{1}{n} Z(r) g(\varphi X, \varphi Y) - \left(\frac{r}{n} + 4n + 4\right) [\eta(Y) g(\varphi X, \varphi Z) \\ + \eta(X) g(\varphi Y, \varphi Z)].\end{aligned} \tag{3.16}$$

Proposition 3.4. *An η-Einstein Kenmotsu Ricci soliton (g, ξ, λ) is expanding and with constant scalar curvature, thus Einstein.*

Proof. $\lambda = -\alpha(\xi, \xi)/2 = 2n$. The same computation as in Proposition 3.3 implies constant scalar curvature. ∎

4. The dynamical point of view

We begin this section with a straightforward consequence of the main theorem, which also appears in the Olzack-Rosca paper, and is related to the last part of Corollary 3.2.

Corollary 4.1. *An affine Killing vector field in a β-Kenmotsu manifold is Killing. As consequence, that scalar provided by the Ricci soliton (g, V) of a Ricci-semi-symmetric β-Kenmotsu manifold is $\lambda = -S(V, V)$.*

Proof. (Inspired by [10, p. 504]), fix $X \in \mathcal{X}(M)$ an affine Killing vector field: $\nabla \mathcal{L}_X g = 0$. From Theorem 3.1 it follows that X is *conformal Killing* i.e. $\mathcal{L}_X g = cg$; more precisely X is *homothetic* since c is a constant. Lie differentiating the identity (2.5) along X and using $\mathcal{L}_X S = 0$ (since X is homothetic) and equation (2.6) we obtain $g(\mathcal{L}_X \xi, \xi) = 0$. Hence $c = (\mathcal{L}_X g)(\xi, \xi) = -2g(\mathcal{L}_X \xi, \xi) = 0$. Thus X is Killing. ∎

Let us present another dynamical picture of our results. Let (M, ∇) be a m-dimensional manifold endowed with a symmetric linear connection. A *quadratic first integral* (QFI in short) for the geodesics of ∇ is defined by $\mathcal{F} = a_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}$ with a symmetric 2-tensor field $a = (a_{ij})$ satisfying the *Killing-type equations*

$$a_{ij:k} + a_{jk:i} + a_{ki:j} = 0, \tag{4.1}$$

where, as usual, the double dot means the covariant derivative with respect to ∇.

The QFI defined by a is called *special* (SQFI) if $a_{ij:k} = 0$ and the maximum number of linearly independent SQFI a pair (M, ∇) can admit is $\frac{m(m+1)}{2}$; a flat space will admit this number. In [17, p. 117] it is shown that a non-flat Riemannian manifold may admit as many as $M_S(m) = 1+\frac{(m-2)(m-1)}{2}$ linearly independent SQFI. Therefore, for an almost contact manifold ($m = 2n + 1$) the maximum number of SQFI is $M_S(2n + 1) = 1 + n(2n - 1) > 1$.

Our main result implies that for a regular f-Kenmotsu manifold the number of SQFI is exactly 1 and the only SQFI is the *kinetic energy* $\mathcal{F} = g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}$. So,

Proposition 4.1. *There exist almost contact manifolds which does not admit M_S $(2n + 1)$ SQFI.*

It remains as an open problem to find examples of almost contact metrics with exactly $M_S(2n + 1)$ SQFI.

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References

[1] N. Aktan, A. Görgülü and E. Özüsağlam, On special weakly Ricci-symmetric Kenmotsu manifolds, *Sarajevo J. Math.* **3(15)** (2007), no. 1, 93–97.
[2] T. Q. Binh, L. Tamássy, U. C. De and M. Tarafdar, Some remarks on almost Kenmotsu manifolds, *Math. Pannon.* **13** (2002), no. 1, 31–39.
[3] M. C. Chaki, On pseudo Ricci symmetric manifolds, *Bulgar. J. Phys.* **15** (1988), no. 6, 526–531.
[4] D. Chinea, On horizontally conformal (ϕ, ϕ')-holomorphic submersions, *Houston J. Math.* **34** (2008), no. 3, 721–737.
[5] L. Das, Second order parallel tensors on α-Sasakian manifold, *Acta Math. Acad. Paedagog. Nyházi.* (*N.S.*) **23** (2007), no. 1, 65–69 (electronic).
[6] U. C. De, Second order parallel tensors on P-Sasakian manifolds, *Publ. Math. Debrecen* **49** (1996), no. 1-2, 33–37.
[7] U. C. De and A. K. Mondal, On 3-dimensional normal almost contact metric manifolds satisfying certain curvature conditions, *Commun. Korean Math. Soc.* **24** (2009), no. 2, 265–275.
[8] U. C. De and M. M. Tripathi, Ricci tensor in 3-dimensional trans-Sasakian manifolds, *Kyungpook Math. J.* **43** (2003), no. 2, 247–255.
[9] L. P. Eisenhart, Symmetric tensors of the second order whose first covariant derivatives are zero, *Trans. Amer. Math. Soc.* **25** (1923), no. 2, 297–306.
[10] A. Ghosh and R. Sharma, Some results on contact metric manifolds, *Ann. Global Anal. Geom.* **15** (1997), no. 6, 497–507.
[11] A. Ghosh, R. Sharma and J. T. Cho, Contact metric manifolds with η-parallel torsion tensor, *Ann. Global Anal. Geom.* **34** (2008), no. 3, 287–299.
[12] S. K. Jana and A. A. Shaikh, On quasi-conformally flat weakly Ricci symmetric manifolds, *Acta Math. Hungar.* **115** (2007), no. 3, 197–214.
[13] X. Jia, Second order parallel tensors on quasi-constant curvature manifolds, *Chinese Quart. J. Math.* **17** (2002), no. 2, 101–105.
[14] J.-B. Jun, U. C. De and G. Pathak, On Kenmotsu manifolds, *J. Korean Math. Soc.* **42** (2005), no. 3, 435–445.
[15] K. Kenmotsu, A class of almost contact Riemannian manifolds, *Tôhoku Math. J.* (*2*) **24** (1972), 93–103.

[16] J.-S. Kim, R. Prasad and M. M. Tripathi, On generalized Ricci-recurrent trans-Sasakian manifolds, *J. Korean Math. Soc.* **39** (2002), no. 6, 953–961.

[17] J. Levine and G. H. Katzin, On the number of special quadratic first integrals in affinely connected and Riemannian spaces, *Tensor* (*N.S.*) **19** (1968), 113–118.

[18] H. Levy, Symmetric tensors of the second order whose covariant derivatives vanish, *Ann. of Math.* (*2*) **27** (1925), no. 2, 91–98.

[19] Z. Li, Second order parallel tensors on P-Sasakian manifolds with a coefficient k, *Soochow J. Math.* **23** (1997), no. 1, 97–102.

[20] V. Mangione, Harmonic maps and stability on f-Kenmotsu manifolds, *Int. J. Math. Math. Sci.* **2008**, Art. ID 798317, 7 pp.

[21] Z. Olszak and R. Roşca, Normal locally conformal almost cosymplectic manifolds, *Publ. Math. Debrecen* **39** (1991), no. 3-4, 315–323.

[22] C. Oniciuc, Nonlinear connections on tangent bundle and harmonicity, *Ital. J. Pure Appl. Math.* **6** (1999), 109–122 (2000).

[23] N. Papaghiuc, Semi-invariant submanifolds in a Kenmotsu manifold, *Rend. Mat.* (*7*) **3** (1983), no. 4, 607–622.

[24] N. Papaghiuc, On the geometry of leaves on a semi-invariant $\xi^{\perp}$-submanifold in a Kenmotsu manifold, *An. Ştiinţ. Univ. Al. I. Cuza Iaşi Secţ. I a Mat.* **38** (1992), no. 1, 111–119.

[25] G. Pitiş, *Geometry of Kenmotsu Manifolds*, Publishing House of Transilvania University of Braşov, Braşov, 2007.

[26] R. Sharma, Second order parallel tensor in real and complex space forms, *Internat. J. Math. Math. Sci.* **12** (1989), no. 4, 787–790.

[27] R. Sharma, Second order parallel tensors on contact manifolds, *Algebras Groups Geom.* **7** (1990), no. 2, 145–152.

[28] R. Sharma, Second order parallel tensors on contact manifolds. II, *C. R. Math. Rep. Acad. Sci. Canada* **13** (1991), no. 6, 259–264.

[29] R. Sharma, On the curvature of contact metric manifolds, *J. Geom.* **53** (1995), no. 1-2, 179–190.

[30] R. Sharma, Certain results on K-contact and (k, μ)-contact manifolds, *J. Geom.* **89** (2008), no. 1-2, 138–147.

[31] H. Singh and Q. Khan, On special weakly symmetric Riemannian manifolds, *Publ. Math. Debrecen* **58** (2001), no. 3, 523–536.

[32] D. Tarafdar and U. C. De, Second order parallel tensors on P-Sasakian manifolds, *Northeast. Math. J.* **11** (1995), no. 3, 260–262.

[33] M. Tarafdar and A. Mayra, On nearly Sasakian manifold, *An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat.* (*N.S.*) **45** (1999), no. 2, 291–294 (2000).

[34] M. M. Tripathi, *Ricci solitons in contact metric manifolds*, arXiv:0801.4222.

[35] Y. Wong, Existence of linear connections with respect to which given tensor fields are parallel or recurrent, *Nagoya Math. J.* **24** (1964), 67–108.

[36] H. Wu, Holonomy groups of indefinite metrics, *Pacific J. Math.* **20** (1967), 351–392.

[37] G. Zhao, *Symmetric covariant tensor fields of order 2 on pseudo-Riemannian manifolds*, Viena Preprint ESI 479 (1997). Available at http://www.esi.ac.at/preprints/esi479.ps