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# EUCLIDEAN GEOMETRY OF FINSLER WAVEFRONTS THROUGH GAUSSIAN CURVATURE

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*Lucrarea studiază curbura Gauss $K$ a indicatoarei spaţiilor Finsler. Pentru $(\alpha, \beta)$-metrici, se obţine o expresie a curburii similară celei determinate de M. Hashiguchi ([2], [3]), care extinde cazurile Randers, Kropina, Matsumoto şi Riemann. Suprafeţele Randers în context Funk 2-dimensional şi metrica Randers-Funk pentru discul unitate sunt de asemenea studiate. Se pune în evidenţă relaţia curburii cu metricile Finsler fundamentală şi angulară, utilizând reperul Berwald pentru cazul $n = 2$. Se demonstrează că curbura Gauss a indicatoarei este h-covariant constantă relativ la conexiunile Cartan şi Chern-Rund. În încheiere, este descris cazul structurilor pseudo-Finsler de tip Berwald-Moor.*

*The present paper deals with the Gaussian curvature $K$ of a particular indicatrix in a general Finsler manifold. A formula for $K$ similar to the one obtained by M. Hashiguchi ([2], [3]), is derived for $(\alpha, \beta)$-metrics and then specialized to Randers, Kropina, Matsumoto and Riemann-type metrics. For the Funk two-dimensional case the class of Randers surfaces is detailed, and the Randers-Funk metric on the unit disk is investigated. Alternative expressions for $K$ are provided, in terms of Finslerian metric and angular metric, and in terms of Berwald frame for $n = 2$. It is shown that $K$ is h-covariant constant with respect to the Cartan and Chern-Rund connections. The last section describes the pseudo-Finsler locally-Minkowski Berwald-Moor case.*

**Keywords:** Finsler spaces, Gauss curvature, indicatrix, hypersurface, $(\alpha, \beta)$-metric, Randers metric, Funk metric, Finsler connection.
**MSC2000:** 53C60, 53B40.

## 1. Introduction

It is well-known the major rôle played by the Gaussian curvature in Differential Geometry. The aim of this paper is investigate the Gaussian curvature in Finsler spaces by means of the *indicatrix* (or *unit tangent space, unit tangent sphere*) associated to the Finsler structure. The importance of indicatrices in the Finslerian setting has been pointed out by Okubo (1, p.13]), which shows that, not only the fundamental Finsler function $F$ defines the indicatrices, but conversely, the indicatrices determine $F$ as well. One of the major differences between Riemannian and Finsler geometry is that the latter provides *several* remarkable connections - and not only one, as the Levi-Civita connection of the Riemannian case. Hence, in the Finslerian framework there are several curvatures with different geometric importance.

The first formula for Gaussian curvature $K_x$ of a fixed indicatrix in Lagrange and Finsler geometry was given by Masao Hashiguchi in [2] and [3] and was further generalized by the second author of the present paper to Generalized Lagrange Geometry in [4].

The present study extends M. Hashiguchi's formula developed in the Randers-Kropina cases to the more general $(\alpha, \beta)$ case. Much attention is given to the $n = 2$ case, where the laborious computations are partially simplified by using in a special way the homogeneity of the Finsler fundamental function through the *associated function* $\lambda = \lambda(z)$ recently introduced by P. Funk ([5]). This reduction of number of variables from two to one becomes essential in replacing PDEs associated to conditions imposed on $K$ to ODEs in $\lambda(z)$.

The paper is organized as follows: the first section recalls the Hashiguchi formula for Lagrange and Finsler manifolds and provides the formula for $(\alpha, \beta)$-metrics. This naturally extends the Randers and Kropina cases studied in [2] and [3], and the Matsumoto and "Riemann"-type metrics are obtained as particular cases. As well, a new expression for $K_x$ in terms of the Finsler and angular metrics, is presented.

The second section focuses on the dimension two - which has been intensively studied in recent works (e.g., [6] and Chapter 4 from [1]). The formula (both in the general and in the previously mentioned particular cases) is computed in terms of the Funk associated fundamental function $\lambda$. A new expression for $K_x$ is obtained in terms of the Berwald frame, specific to Finslerian surfaces.

The next section provides concrete examples, supplementing thus the works [2] and [3]. The Randers metric on the unit disk considered by T. Okada in [7] appears as an example of Finsler metric of Funk type having constant Finslerian-Gaussian (or flag) negative curvature.

The last section is devoted to the behavior of the total function $K$ in terms of its horizontal and vertical covariant derivatives with respect to certain Finsler connections. It is proved that $K$ is horizontal covariant constant with respect to the Cartan and Chern-Rund connections.

## 2. **Gaussian curvature for wavefronts in Lagrange and Finsler geometry**

In the $n$-dimensional Euclidean space $\mathbf{R}^n$, consider a hypersurface

$$S = \{x \in \mathbf{R}^n \mid f(x) = 0, \; \nabla f(x) \neq 0\},$$

where $f \in C^\infty(\mathbf{R}^n)$, and $\nabla f$ denotes the gradient of $f$ ($\nabla f = (f_i)_{i=\overline{1,n}}$, $f_i = \frac{\partial f}{\partial x^i}$). We further choose the inward unit normal vector field of $S$, $N = -\frac{\nabla f}{||\nabla f||}$. Then we have the following classical result ([3, p. 37], [2, p. 23]):

**Proposition 1.1** *The Gaussian curvature of pair $(S, N)$ is:*

$$K = -\begin{vmatrix} f_{ij} & f_i \\ f_j & 0 \end{vmatrix} \cdot \| \nabla f \|^{-(n+1)} .$$

This formula has been used in the above cited papers in order to obtain the Gaussian curvature for the indicatrix of a *Lagrange space* $L^n = (M, L)$ on an open domain $M$ of $\mathbf{R}^n$. More precisely, let $L : TM \to \mathbf{R}$ be a regular Lagrangian ([8]) on $M$ considered as a smooth $n$-dimensional manifold. This means that the halved $y$-Hessian matrix of $L$, $g_{ij} = \frac{1}{2}\,\dot{\partial}_i\dot{\partial}_j L$ is of rank $n$ (i.e., $\det(g_{ij}) \neq 0$), where $\dot{\partial}_i = \frac{\partial}{\partial y^i}$ using the local coordinates $x = (x^i)_{i=\overline{1,n}}$ on $M$ and $(x, y) = (x^i, y^i)$ on the tangent space $TM$. Associated to this Lagrangian we have *the indicatrix of* $L$ at $x \in M$: $I_x = \{y \in T_xM; L(x, y) = 1\}$. This appears as a hypersurface in $T_xM \simeq \mathbf{R}^n$ implicitly defined by the function: $f(x, y) = L(x, y) - 1$.

**Remark.** Throughout this paper we shall alternatively call $I_x$ *the wavefront at* $x$, following the paper [9].

**Proposition 1.2** ([3, p. 39]) *Let $L^n$ be a Lagrange space. Then at each point $x \in M$, the Gaussian curvature $K_x$ of the wavefront $I_x$ oriented in the direction $N_x = -\frac{\dot{\nabla}L}{||\dot{\nabla}L||}$ is:*

$$K_x = -\begin{vmatrix} \dot{\partial}_i\dot{\partial}_j L & \dot{\partial}_i L \\ \dot{\partial}_j L & 0 \end{vmatrix} \cdot \left[\sum_{i=1}^{n}(\dot{\partial}_i L)^2\right]^{-\frac{n+1}{2}}. \tag{1}$$

In this proposition $\dot{\nabla}L$ denotes the gradient of $L$ with respect to $y$ i.e. $\dot{\nabla}L = (\dot{\partial}_i L)_{i=\overline{1,n}}$.

**Important remark.** The formula (1) and its offsprings belong to the Euclidean geometry of the family of (Euclidean) hypersurfaces $\{I_x\}_{x\in M}$, and not to the Lagrange (particularly Finsler) geometry of the pair $(M, L)$.

If there exists a function $F : TM \to \mathbf{R}$, smooth on $TM$ minus the null section and positively homogeneous of degree 1 with respect to $y$ such that $L = F^2$ is a regular Lagrangian then the pair $F^n = (M, F)$ is a *Finsler space* and $F$ is called *the fundamental function* of $F^n$ ([8]). For Finsler spaces, it has been proved that:

**Proposition 1.3** ([3, p. 40]) *Let $(M, F)$ be a Finsler space and let $g = \det(g_{ij}) = \det(\frac{1}{2}\dot{\partial}_i\dot{\partial}_j F^2)$. Then at each point $x \in M$, the Gaussian curvature $K_x$ of the wavefront $I_x$ oriented in the direction opposite to $\dot{\nabla}L$ is:*

$$K_x = g \cdot \left(\sum_{i=1}^{n} l_i^2\right)^{-\frac{n+1}{2}} \tag{2}$$

*where $l_i = \dot{\partial}_i F$.*

In the following we shall provide alternative expressions for the Gaussian curvature $K_x$. The homogeneity of the fundamental Finsler function $F$ yields $l_i = F^{-1}g_{ij}y^j$, where $l = (l_i)$ is *the covariant normalized supporting element*; then using the equation of $I_x :\ F^2 = 1$, we get:

**Proposition 1.4** *The Gaussian curvature of the Finslerian wavefront $I_x$ is:*

$$K_x = g \cdot \left[ \sum_{i=1}^{n} (g_{ij} y^j)^2 \right]^{-\frac{n+1}{2}}. \tag{3}$$

*This shows that the Gaussian curvature is a metric object, i.e., it depends only on the Finsler metric $g_{ij}$ (and not directly on the Finsler function $F$).*

Using the notation $y_i = g_{ik} y^k$, the relation (3) rewrites in the simpler form

$$K_x = g \cdot \left[ \sum_{i=1}^{n} y_i^2 \right]^{-\frac{n+1}{2}}.$$

As well, using the tensor field $h = (h_{ij} = F \frac{\partial^2 F}{\partial y^i \partial y^j})$ called *the angular metric* related to $g_{ij}$ via ([6, p. 296]):

$$g_{ij} = h_{ij} + l_i l_j,$$

we get $l_i^2 = g_{ii} - h_{ii}$ and then we yield a formula for $K$ in terms of both Finsler and angular metrics:

$$K_x = g \cdot \left[ \sum_{i=1}^{n} (g_{ii} - h_{ii}) \right]^{-\frac{n+1}{2}}.$$

**Example.** The case of $(\alpha, \beta)$-**metrics**. Let us consider a Riemannian metric $a = (a_{ij}(x))_{1 \leq i,j \leq n}$ and an 1-form: $b = (b_i(x))_{1 \leq i \leq n}$, both living globally on $M$, and let us associate to these objects the following functions on $TM$:

- $\alpha(x, y) = \sqrt{a_{ij} y^i y^j}$;
- $\beta(x, y) = b_i y^i$;
- $F(x, y) = \alpha \phi(\frac{\beta}{\alpha})$, with $\phi$ a $C^\infty$ positive function on some interval $[-r, r]$, big enough such that $r \geq \frac{\beta}{\alpha}$ for all $(x, y) \in TM$.

Then $F$ is a Finsler fundamental function if the following conditions are satisfied ([10, p. 307, eq. (2-11)])

$$\phi(s) > 0, \phi(s) - s\phi'(s) > 0, (\phi(s) - s\phi'(s)) + (b^2 - s^2)\phi''(s) > 0,$$

for all $|s| \leq b \leq r$. Then using the results from ([10, p. 307]) we obtain:

$$g = \phi^{n+1}(s)(\phi(s) - s\phi'(s))^{n-2}(\phi(s) - s\phi'(s) + (\|\beta\|_x^2 - s^2)\phi''(s)) \det a,$$

where $s = \frac{\beta}{\alpha}$ and $\|\beta\|_x = \sup\{\frac{\beta(x,y)}{\alpha(x,y)}; y \in T_x M\}$. Also:

$$l_i = \frac{\tilde{y}_i}{\alpha}(\phi - s\phi') + \phi' b_i,$$

where $(\tilde{y}_i)$ is the $a$-covariant version of $(y^i)$: $\tilde{y}_i = a_{ij} y^j$. In conclusion, for an $(\alpha, \beta)$-metric, we have:

$$K_x = \frac{\phi^{n+1}(\phi - s\phi')^{n-2}[(\phi - s\phi') + (\|\beta\|_x^2 - s^2)\phi''] \det a}{[\sum_{i=1}^{n} (\frac{\tilde{y}_i}{\alpha}(\phi - s\phi') + b_i \phi')^2]^{\frac{n+1}{2}}}.$$

Then, using the equation of $I_x : \phi = \frac{1}{\alpha}$, the final formula is:

$$K_x = \frac{(\phi - s\phi')^{n-2}[(\phi - s\phi') + (\|\beta\|_x^2 - s^2)\phi''] \det a}{[\sum\limits_{i=1}^{n} (\tilde{y}_i(\phi - s\phi') + \alpha b_i \phi')^2]^{\frac{n+1}{2}}}.$$

**Particular cases.**

**I. Randers metrics ($\phi(s) = 1 + s$).** From $\phi' = \phi - s\phi' = 1, \phi'' = 0$, it results ([3, Th. 3.3., p. 41], [2, Th. 3.2., p. 25]):

$$K_x = (\det a) \cdot \left[ \sum_{i=1}^{n} (\tilde{y}_i + \alpha b_i)^2 \right]^{-\frac{n+1}{2}}.$$

**II. Kropina metrics ($\phi(s) = \frac{1}{s}$).** From $\phi' = -\frac{1}{s^2}, \phi - s\phi' = \frac{2}{s}, \phi'' = \frac{2}{s^3}$, it follows:

$$K_x = 2^{n-1} \|\beta\|_x^2 \cdot (\det a) \cdot \left[ \sum_{i=1}^{n} (2\tilde{y}_i - \frac{\alpha}{s} b_i)^2 \right]^{-\frac{n+1}{2}},$$

and using the equation of $I_x : \frac{\alpha^2}{\beta} = \frac{\alpha}{s} = 1$, we infer ([2, Th. 3.3., p. 26]):

$$K_x = 2^{n-1} \|\beta\|_x^2 \cdot (\det a) \cdot \left[ \sum_{i=1}^{n} (2\tilde{y}_i - b_i)^2 \right]^{-\frac{n+1}{2}}.$$

**III. Matsumoto metrics ([11, p. 553], $\phi(s) = \frac{1}{1-s}$).** From $\phi' = \frac{1}{(1-s)^2}$, $\phi - s\phi' = \frac{1-2s}{(1-s)^2}, \phi'' = \frac{2}{(1-s)^3}$, it results:

$$K_x = \frac{(1-s)^6 (1-2s)^{n-2} (1 - 3s + 2\|\beta\|_x^2) \det a}{[\sum\limits_{i=1}^{n} (\tilde{y}_i(1-2s) + \alpha b_i)^2]^{\frac{n+1}{2}}}$$

or, using the equation of $I_x : 1 - s = \alpha$:

$$K_x = \frac{\alpha^6 (2\alpha - 1)^{n-2} (3\alpha - 2 + 2\|\beta\|_x^2) \det a}{[\sum\limits_{i=1}^{n} (\tilde{y}_i(2\alpha - 1) + \alpha b_i)^2]^{\frac{n+1}{2}}}.$$

**IV. Riemann-type $(\alpha, \beta)$-metrics ([11, p. 553], $\phi(s) = \sqrt{1+s^2}$).** From $\phi' = \frac{2}{\sqrt{1+s^2}}, \phi - s\phi' = \frac{1}{\sqrt{1+s^2}}, \phi'' = \frac{1}{(1+s^2)^{3/2}}$, it follows:

$$K_x = (1 + \|\beta\|_x^2) \cdot (\det a) \cdot \left[ \sum_{i=1}^{n} (\tilde{y}_i + \alpha b_i s)^2 \right]^{-\frac{n+1}{2}}.$$

But $I_x : \sqrt{1+s^2} = \frac{1}{\alpha}$ gives $s = \frac{\sqrt{1-\alpha^2}}{\alpha}$, and then:

$$K_x = (1 + \|\beta\|_x^2) \cdot (\det a) \cdot \left[ \sum_{i=1}^{n} (\tilde{y}_i + b_i \sqrt{1-\alpha^2})^2 \right]^{-\frac{n+1}{2}}.$$

3. **The two-dimensional case**

For $n = 2$, $I_x$ becomes a curve with a unique curvature - hence we shall omit the word "Gaussian". This case considerably simplifies by means of the Funk associated fundamental function ([5]). Namely, $F = F(x^1, x^2; y^1, y^2)$ being positively 1-homogeneous in $y = (y^1, y^2)$, we get for positive $y^1$: $F = y^1 f(x^1, x^2; 1, z)$ with $z = y^2/y^1$.

**Definition 2.1 (**[5]**)** The function $\lambda(x; z) = \lambda(x^1, x^2; z) := f(x^1, x^2; 1, z)$ is called *the associated fundamental function* of the Finsler surface $(M, F^2)$.

In the following, we shall make use of the following formulae ([12, (3.4), p.307; (3.6), p.308; (3.7), p.308])

$$\begin{cases} (l_1, l_2) = (\lambda - \lambda' z, \lambda') \\ (g_{11}, g_{12}, g_{22}) = ((\lambda\lambda')' z^2 - 2\lambda\lambda' z + \lambda^2, \lambda\lambda' - (\lambda\lambda')' z, (\lambda\lambda')') \\ g = \lambda^3 \lambda'', \end{cases}$$

where the prime stands for differentiation by $z$. Hence we infer:

**Proposition 2.2.** *For a Finsler surface, the curvature of the wavefront $I_x$ is:*

$$K_x(z) = \lambda^3 \lambda'' \cdot \left[ (\lambda - \lambda' z)^2 + (\lambda')^2 \right]^{-3/2}.$$

**Example.** $(\alpha, \beta)$**-metrics**.

$$\lambda = \sqrt{a_{11} + 2a_{12} z + a_{22} z^2} \cdot \phi \left( \frac{b_1 + b_2 z}{\sqrt{a_{11} + 2a_{12} z + a_{22} z^2}} \right).$$

Using the notations: $\alpha_\lambda = \sqrt{a_{11} + 2a_{12} z + a_{22} z^2}, \beta_\lambda = b_1 + b_2 z$, we have:

$$\begin{cases} \lambda' = b_2 \phi' + \frac{a_{12} + a_{22} z}{\alpha_\lambda} (\phi - s\phi') \\ \lambda'' = \frac{\det a}{\alpha_\lambda^3} (\phi - s\phi') + \phi'' \left[ \frac{b_2^2}{\alpha_\lambda} - \frac{\beta_\lambda (a_{12} + a_{22} z)(b_2 + a_\lambda)}{\alpha_\lambda^3} \right], \end{cases}$$

and then:

$$K_x(z) = \frac{(\alpha_\lambda \phi)^3 [\det a(\phi - s\phi') + \phi''(b_2^2 \alpha_\lambda^2 - \beta_\lambda (a_{12} + a_{22} z)(b_2 + \beta_\lambda))]}{[(b_1 \alpha_\lambda \phi' + (a_{11} + a_{12} z)(\phi - s\phi'))^2 + (b_2 \alpha_\lambda \phi' + (a_{12} + a_{22} z)(\phi - s\phi'))^2]^{\frac{3}{2}}}.$$

In the particular case of a Randers metric, one easily infers

$$K_x(z) = \frac{(\alpha_\lambda + \beta_\lambda)^3 \det a}{[(b_1 \alpha_\lambda + a_{11} + a_{12} z)^2 + (b_2 \alpha_\lambda + a_{12} + a_{22} z)^2]^{\frac{3}{2}}}.$$

For example, if the Riemannian metric is diagonal, i.e., $a_{12} = 0$, then the last formula becomes:

$$K_x(z) = \frac{(\sqrt{a_{11} + a_{22} z^2} + b_1 + b_2 z)^3 a_{11} a_{22}}{[(b_1 \sqrt{a_{11} + a_{22} z^2} + a_{11})^2 + (b_2 \sqrt{a_{11} + a_{22} z^2} + a_{22} z)^2]^{\frac{3}{2}}}.$$

Returning to the usual coordinates $(x, y)$ on $TM$, we shall futher use the remarkable *Berwald* frame (specific to the 2-dimensional case) $(l, m)$, with $m = (m^i)$, $m^i = \frac{1}{C} C^i)$, where we consider the following Finsler tensors:

- $C_{ijk} = \frac{1}{4} \dot\partial_i \dot\partial_j \dot\partial_k F^2$ (*the Cartan tensor*, [6, p. 291]),

- $C^i_{jk} = g^{ia}C_{ajk}$, $C^i = C^i_{jk}g^{jk}$ (*the Cartan vector*), and
- $C = g_{ij}C^iC^j$ (*the length of the Cartan vector*).

But from ([6, p. 297]), one has $g_{ij} = l_il_j + m_im_j$, whence a straightforward computation gives ([1, p. 95])

$$g = (l_1m_2 - l_2m_1)^2.$$

This infers the curvature in terms of Berwald frame

$$K_x = (l_1m_2 - l_2m_1)^2 \cdot (l_1^2 + l_2^2)^{-3/2}.$$

## 4. **A concrete example**

We shall further consider the Randers metric on the unit disk $D^1$ determined by ([7, p. 123]):

$$\begin{cases} a_{11} = \frac{1-(x^2)^2}{\varphi^2(x)},\ a_{12} = a_{21} = \frac{x^1x^2}{\varphi^2(x)},\ a_{22} = \frac{1-(x^1)^2}{\varphi^2(x)} \\ b_1 = \frac{x^1}{\varphi(x)},\ b_2 = \frac{x^2}{\varphi(x)}, \end{cases}$$

where $\varphi$ defines the boundary $\partial D^1 = S^1 : \varphi(x) = 1 - (x^1)^2 - (x^2)^2$. It is known that the associated Randers-Funk metric is of constant negative flag curvature $(-1/4)$ ([7, p. 123]; see also [1, p. 20]). We have $\det a = \frac{1}{\varphi^3}$, and from the equation of $I_x : 1 = \alpha + \beta = \alpha + \frac{x^1y^1+x^2y^2}{\varphi}$, it results that on $I_x$ we have:

$$\alpha\varphi = \varphi - x^1y^1 - x^2y^2. \tag{4}$$

Also,

$$\tilde{y}_1 + \alpha b_1 = a_{11}y^1 + a_{12}y^2 + \frac{\alpha x^1}{\varphi} = \frac{\alpha\varphi x^1 + (1-(x^2)^2)y^1 + x^1x^2y^2}{\varphi^2}$$

and (4) implies $\tilde{y}_1 + \alpha b_1 = (x^1 + y^1) \cdot \varphi^{-1}$. Concluding, we get

$$K_x(y) = [(x^1 + y^1)^2 + (x^2 + y^2)]^{-3/2}. \tag{5}$$

Moreover, a straightforward computation gives the wavefront

$$I_x :\ \ (x^1 + y^1)^2 + (x^2 + y^2)^2 = 1,$$

and then the above result yields $K_x = 1$ for every $x \in D^1$.

## 5. *K* **is covariant constant with respect to certain Finsler connections**

An important topic is the change of different geometrical structures with respect to certain changes of main objects of the theory. So, if we make a *conformal change* of the Finsler function, $\widetilde{F} = e^{\varphi(x)}F$, then the Finsler metrics are $M$-conformally equivalent $\widetilde{g}_{ij} = e^{2\varphi(x)}g_{ij}$ and $\widetilde{l}_i = e^{\varphi(x)}l_i$. Then the relation (2) infers that the Gauss curvature $\widetilde{K}_x$ of $\widetilde{I}_x : e^{\varphi(x)}F = 1$ is related to the curvature $K_x$ of $I_x : F = 1$ through $\widetilde{K}_x = e^{(n-1)\varphi(x)}K_x$.

As well, the Finslerian Gauss curvature exhibits interesting properties in terms of its covariant derivative with respect to linear connections. We recall ([6], [8]) that a linear connection on $TM$,

$$D : (X, Y) \in \mathcal{X}(TM) \times \mathcal{X}(TM) \to D_X Y \in \mathcal{X}(TM)$$

is a *Finsler connection* if it preserves by parallelism the horizontal distribution generated by $F$ and the almost tangent structure of $TM$.

A Finsler connection yields two covariant derivatives: a horizontal one denoted $_{|k}$ and a vertical one denoted $|_k$. The covariant derivatives of the Liouville vector field $\Gamma = y^i \frac{\partial}{\partial y^i}$ are usually denoted $D^i_j$ =*the horizontal* (briefly $h-$) *deflection tensor* respectively $d^i_j$=*the vertical* (briefly $v-$) *deflection tensor*. $D$ is called:

- *h-metrical* if $g_{ij|k} = 0$ and *v-metrical* if $g_{ij}|_k = 0$,
- *metrical* if it is $h-$ and $v-$metric,
- *h-deflection free* if $D^i_j = 0$, and *v-deflection free* if $d^i_j = 0$.

Denoting $I(TM) = \{(x, y) \in TM; y \in I_x M\}$ the set of all wavefronts of $M$ and considering now the Gaussian curvature as a global function $K : I(TM) \subset \mathbf{R}^{2n} \to \mathbf{R}$, from (3) we get

**Proposition 4.1** *If the Finsler connection $D$ is h-metrical (v-metrical) and h- (v-)deflection free, then $K : I(TM) \to \mathbf{R}$ is a horizontal (vertical) covariant constant function. In particular, if $D$ is metrical and $\Gamma$ is $*$-covariant constant with respect to $D$, then $K$ is $*$-covariant constant with respect to $D$ where $* \in \{h, v\}$.*

**Examples of Finsler connections.**

- *the Berwald connection*: neither $h$-metrical nor $v$-metrical;
- *the Cartan connection*: metrical and $d^i_j = \delta^i_j$;
- *the Chern-Rund connection*: $h$-metrical, but not $v$-metrical and $d^i_j = \delta^i_j$;
- *the Hashiguchi connection*: $v$-metrical and not $h$-metrical.

All these four connections are $h$-deflection free and then we infer:

**Corollary 4.2** *The function $K$ is h-covariant constant with respect to the Cartan and Chern-Rund connections.*

We note that results on closely related topics have been published in [13], [14], [15], and [16] (the latter, for the case of non-compact Finsler indicatrix).

## 6. Pseudo-Finsler notable structures

Recently, much attention has been paid to the pseudo-Finsler framework ([17]), and especially to $m-$root structures, mainly due to their applications to Relativity, Ecology and Diffusion Imaging ([18],[19],[20]). Primarily studied and developed by M.Matsumoto and H.Shimada ([21],[22],[23]), these structures have the generic form:

$$F(x, y) = \sqrt[m]{a_{i_1 i_2 \ldots i_m}(x) y^{i_1} \ldots y^{i_m}} \ ,$$

where $a_{i_1 i_2 \ldots i_m}$ are the coefficients of an $m-$covariant tensor field on the manifold $M$. The remarkable particular flat locally-Minkowski $n-$root pseudo-Finsler space $(M = \mathbb{R}^n, F)$, with

$$F(y) = \sqrt[n]{y^1 \ldots y^n},$$

has the property that its indicatrix $I_x: \ y^1 \ldots y^n = c$, $(c = 1)$ is the Tzitzeica hypersurface ([24],[25],[26],[27]) whose Gauss curvature at some point $y \in I_x \subset T_x M$ is proportional to the $(n+1)$ - th power of the distance $d_y$ from origin to the tangent hyperplane $T_y I_x$. This infers

$$K(y) = d_y^{n+1} \cdot \frac{(-1)^n}{n^{n+1} c^2} = \frac{n \cdot (-1)^{n+1}}{c^2} \left( \sum_{i=1}^{n} (y^i)^{-2} \right)^{-(n+1)/2}, \quad (\text{for } c = 1).$$

In particular, for $n = 4$, one obtains the Gauss curvature of the $H_4$ Berwald-Moor indicatrix ([18]), and for $n = 6$ the curvature of the uniform $m-$root metric used in diffusion imaging ([20]).

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## REFERENCES

[1] *D. Bao, S.S. Chern, Z. Shen*, "An Introduction to Riemann-Finsler Geometry", Graduate Texts in Mathematics 200, Springer-Verlag, 2000.

[2] *M. Hashiguchi*, "On a Finsler-geometrical expression of the Gaussian Curvature of a hypersurface in an Euclidian space", Rep. Fac. Sci. Kagoshima Univ. **25** (1992), pp. 21-27.

[3] *S.-I. Nishimura, M. Hashiguchi*, "On the Gaussian curvature of the indicatrix of a Lagrange space", Rep. Fac. Sci. Kagoshima Univ., **24** (1991), pp. 33-41.

[4] *M. Crasmareanu*, "The Gaussian curvature for the indicatrix of a generalized Lagrange space", in M. Anastasiei, Peter L. Antonelli (Eds.) Finsler and Lagrange Geometries, Kluwer Academic Publishers, 2003, pp. 83-90.

[5] *P. Funk*, "Beiträge zur zweidimensional Finsler'schen Geometrie", Monatsh. Math. Phys. **52** (1948), pp. 194-216.

[6] *P.L. Antonelli, I. Bucataru, S.F. Rutz*, "Computer algebra and two and three dimensional Finsler geometry", Publ. Math. Debrecen, **62** (2003), No. 3-4, pp. 289-313.

[7] *T. Okada*, "On models of projectively flat Finsler spaces of constant negative curvature", Tensor, **40** (1983), pp. 117-123.

[8] *R. Miron, M. Anastasiei*, "The Geometry of Lagrange Spaces, Theory and Applications", Kluwer Academic Publishers, FTPH no. 59, 1994.

[9] *I. Bucataru, M.A. Slawinski*, "Generalized orthogonality between rays and wavefronts in anisotropic homogeneous media", Nonlinear Anal. Real World Appl. **6** (2005), No. 1, pp. 111-121.

[10] *Z. Shen*, "Landsberg curvature, $S$-curvature and Riemannian curvature", in D. Bao et al. (Eds.) A sampler of Riemann-Finsler geometry, Math. Sci. Res. Inst. Publ. 50, Cambridge Univ. Press, Cambridge, 2004.

[11] *Miron, H. Shimada, V.S. Sabău*, "Finsler spaces with $(\mu, \beta)$-metric", Publ. Math. Debrecen **62** (2003), No. 3-4, pp. 547-560.

[12] *M. Matsumoto*, "On Wagner's generalized Berwald spaces of dimension two", Tensor **36** (1982), No. 3, pp. 303-311.

[13] *V. Balan, S. Ikeda*, "Structural considerations on the theory of gravitational fields in Finsler spaces", Sci. Bull. Univ. Politehnica of Bucharest, Ser. A, **67** (2005), No. 3, pp. 3-10.

[14] *V. Balan, S. Ikeda*, "Structural considerations on the theory of gravitational fields in Finsler spaces II", Sci. Bull. Univ. Politehnica of Bucharest, Ser. A, **68** (2006), No. 4, pp. 3-8.

[15] *V. Balan, A. Pitea*, "Symbolic software for Y-energy extremal Finsler submanifolds", DGDS **11** (2009), 41-53.

[16] *V. Balan*, "Spectral properties and applications of the numerical multilinear algebra of $m$-root structures", Hypercomplex Numbers in Geometry and Physics, **2** (10), vol. 5 (2008), 101-108.

[17] *A. Bejancu*, "Geometry of Finsler Submanifolds", Kluwer Acad. Publishers, 2000.

[18] *D.G. Pavlov, Gh. Atanasiu, V. Balan (Eds)*, "Space-Time Structure. Algebra and Geometry", LILIA Print, Moscow 2007.

[19] *P.L. Antonelli, B.C. Lackey*, "The Theory of Finslerian Laplacians and Applications", Kluwer Academic Publishers, 1998.

[20] *L. Astola, L. Florack*, "Finsler Geometry on higher order tensor fields and applications to high angular resolution diffusion imaging", LNCS 5567 (2009), pp. 224-234.

[21] *M. Matsumoto, H. Shimada*, "On Finsler spaces with 1-form metric II. Berwald-Moor's metric $L = (y^1 y^2 \dots y^n)^{1/n}$", Tensor N.S. **32** (1978), pp. 275-278.

[22] *H. Shimada*, "On Finsler spaces with metric $L = \sqrt{n} a_{i_1 i_2 \dots i_n}(x) y^{i_1} \dots y^{i_n}$", Tensor, N.S. **33** (1979), pp. 365-372.

[23] *H. Shimada*, "On Finsler spaces with $1-$form metric II. Berwald-Moor's metric $L = \sqrt[n]{y^1 y^2 \dots y^n}$", Tensor N.S. **32** (1978), pp. 375-278.

[24] *G. Tzitzeica*, "Sur une nouvelle classe de surfaces", C.R. Acad. Sci., Paris, Ser. I, Math. **144** (1907), pp. 1257-1259.

[25] *C. Udrişte*, "Tzitzeica theory - opportunity for reflection in Mathematics", Balkan Journal of Geometry and Its Applications **10** (2005), No. 1, pp. 110-120.

[26] *C. Udrişte, N. Bîlă*, "Symmetry group of Tzitzeica surfaces PDE", Balkan J. Geom. Appl. **4** (1999), No. 2, pp. 123-140.

[27] *V. Balan and S.-A. Vâlcea*, "The Tzitzeica surface as solution of PDE systems", Balkan Journal of Geometry and Its Applications **10** (2005), No. 1, pp. 69-72.