Ricci solitons in manifolds with quasi-constant curvature

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This paper is dedicated to the memory of Professor Stere Ianus (1939–2010)

Abstract. The Eisenhart problem of finding parallel tensors treated already in
the framework of quasi-constant curvature manifolds in [Jia] is reconsidered for the
symmetric case and the result is interpreted in terms of Ricci solitons. If the generator
of the manifold provides a Ricci soliton then this is i) expanding on para-Sasakian spaces
with constant scalar curvature and vanishing $D$-concircular tensor field and ii) shrinking
on a class of orientable quasi-umbilical hypersurfaces of a real projective space=elliptic
space form.

1. Introduction

In 1923, Eisenhart [Eisenhart] proved that if a positive definite Riemannian
manifold $(M, g)$ admits a second order parallel symmetric covariant tensor other
than a constant multiple of the metric tensor, then it is reducible. In 1926, Levy
[Levy] proved that a parallel second order symmetric non-degenerated tensor $\alpha$
in a space form is proportional to the metric tensor. Note that this question
can be considered as the dual to the the problem of finding linear connections
making parallel a given tensor field; a problem which was considered by Wong
in [Wong]. Also, the former question implies topological restrictions namely if
the (pseudo) Riemannian manifold $M$ admits a parallel symmetric $(0, 2)$ tensor
field then $M$ is locally the direct product of a number of (pseudo) Riemannian

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manifolds, [Wu] (cited by [Zhao]). Another situation where the parallelism of $\alpha$ is involved appears in the theory of totally geodesic maps, namely, as is point out in [Oniciuc, p. 114]; $\nabla \alpha = 0$ is equivalent with the fact that $1 : (M, g) \to (M, \alpha)$ is a totally geodesic map.

While both Eisenhart and Levy work locally, Ramesh Sharma gives in [Sharma1] a global approach based on Ricci identities. In addition to space-forms, Sharma considered this Eisenhart problem in contact geometry [Sharma2]–[Sharma4], for example for $K$-contact manifolds in [Sharma3]. Since then, several other studies appeared in various contact manifolds, see for example, the bibliography of [CalinCrasm].

Another framework was that of quasi-constant curvature in [Jia]; recall that the notion of manifold with quasi-constant curvature was introduced by Bangyen Chen and Kentaro Yano in 1972, [ChenYano], and since then, was the subject of several and very interesting works, [Bernardini], [DeGhosh], [Wang], in both local and global approaches. Unfortunately, the paper of Jia contains some typos and we consider that a careful study deserves a new paper. There are two remarks regarding Jia result: i) it is in agreement with what happens in all previously recalled contact geometries for the symmetric case, ii) it is obtained in the same manner as in Sharma’s paper [Sharma1]. Our work improves the cited paper with a natural condition imposed to the generator of the given manifold, namely to be of torse-forming type with a regularity property.

Our main result is connected with the recent theory of Ricci solitons [Cao], a subject included in the Hamilton–Perelman approach (and proof) of Poincaré Conjecture. A connection between Ricci flow and quasi-constant curvature manifolds appears in [CaiZhao]; thus our treatment for Ricci solitons in quasi-constant curvature manifolds seems to be new.

Our work is structured as follows. The first section is a very brief review of manifolds with quasi-constant curvature and Ricci solitons. The next section is devoted to the (symmetric case of) Eisenhart problem in our framework and the relationship with the Ricci solitons is pointed out. A technical conditions appears, which we call regularity, and is concerning with the non-vanishing of the Ricci curvature with respect to the generator of the given manifold. Let us remark that in the Jia’s paper this condition is involved, but we present a characterization of these manifolds as well as some remarkable cases which are out of this condition namely: quasi-constant curvature locally symmetric and Ricci semi-symmetric metrics. A characterization of Ricci soliton is derived for dimension greater that 3.

Four concrete examples involved in possible Ricci solitons on quasi-constant manifolds are listed at the end. For the second example, we pointed out some
consequences which are yielded by the hypothesis of compacity, used in paper
[DragomirGrimaldi], in connection with (classic by now) papers of T. Ivey and Per-
elman.

2. Quasi-constant curvature manifolds. Ricci solitons

Fix a triple \((M, g, \xi)\) with \(M\) a smooth \(n(>2)\)-dimensional manifold, \(g\) a
Riemannian metric on \(M\) and \(\xi\) an unitary vector field on \(M\). Let \(\eta\) the 1-form
dual to \(\xi\) with respect to \(g\).

If there exist two smooth functions \(a, b \in C^\infty(M)\) such that:

\[
R(X,Y)Z = a[g(Y,Z)X - g(X,Z)Y] + bg(Y,Z)\eta(X) - g(X,Z)\eta(Y)]\xi
+ b\eta(Z)[\eta(Y)X - \eta(X)Y]
\] (2.1)

then \((M, g, \xi)\) is called manifold of quasi-constant curvature and \(\xi\) is the generator,
[ChenYano]. Using the notation of [DragomirTomassini, p. 325] let us denote \(M^a_{\xi,b}\) this manifold.

It follows:

\[
R(X,Y)\xi = (a + b)[\eta(Y)X - \eta(X)Y]
\] (2.2)
\[
R(X,\xi)Z = (a + b)[\eta(Z)X - g(X,Z)\xi]
\] (2.3)

while the Ricci curvature \(S(X,Y) = Tr(Z \to R(Z,X)Y)\) is:

\[
S(X,Y) = [a(n-1) + b]g(X,Y) + b(n-2)\eta(X)\eta(Y)
\] (2.4)

which means that \((M, g, \xi)\) is an eta-Einstein manifold; in particular, if \(a, b\) are
scalars, then \((M, g, \xi)\) is an quasi-Einstein manifold, [GhoshDeBinh]. The scalar
curvature is:

\[
r = (n - 1)(na + 2b),
\] (2.5)

and we derive:

\[
a = \frac{r - 2S(\xi,\xi)}{(n - 1)(n - 2)}, \quad b = \frac{nS(\xi,\xi) - r}{(n - 1)(n - 2)}.
\] (2.6)

Then \(a + b = \frac{S(\xi,\xi)}{n - 1}\). Let us consider also the Ricci (1,1) tensor field \(Q\) given by:

\[
S(X,Y) = g(QX,Y). \quad \text{From (2.4) we get:}
\]

\[
Q(X) = [a(n - 1) + b]X + b(n - 2)\eta(X)\xi
\] (2.7)

which yields:

\[
Q(\xi) = (a + b)(n - 1)\xi
\] (2.8)

and then \(\xi\) is an eigenvalue of \(Q\).
In the last part of this section we recall the notion of Ricci solitons according to [Sharma5, p. 139]. On the manifold $M$, a Ricci soliton is a triple $(g, V, \lambda)$ with $g$ a Riemannian metric, $V$ a vector field and $\lambda$ a real scalar such that:

$$\mathcal{L}_V g + 2S + 2\lambda g = 0.$$  \hfill (2.9)

The Ricci soliton is said to be shrinking, steady or expanding according as $\lambda$ is negative, zero or positive.

Also, we adopt the notion of $\eta$-Ricci soliton introduced in the paper [ChoKimura] as a data $(g, V, \lambda, \mu)$:

$$\mathcal{L}_V g + 2S + 2\lambda g + 2\mu \eta \otimes \eta = 0.$$  \hfill (2.10)

### 3. Parallel symmetric second order tensors and Ricci solitons

Fix $\alpha$ a symmetric tensor field of $(0, 2)$-type which we suppose to be parallel with respect to the Levi–Civita connection $\nabla$ i.e. $\nabla \alpha = 0$. Applying the Ricci identity $\nabla^2 \alpha(X, Y; Z, W) - \nabla^2 \alpha(X, Y; W, Z) = 0$ we obtain the relation (1.1) of [Sharma1, p. 787]:

$$\alpha(R(X, Y)Z, W) + \alpha(Z, R(X, Y)W) = 0$$  \hfill (3.1)

which is fundamental in all papers treating this subject. Replacing $Z = W = \xi$ and using (2.2) it results, by the symmetry of $\alpha$:

$$(a + b)[\eta(Y)\alpha(X, \xi) - \eta(X)\alpha(Y, \xi)] = 0.$$  \hfill (3.2)

**Definition 3.1.** $M_{a,b}^n(\xi)$ is called regular if $a + b \neq 0$.

In order to obtain a characterization of such manifolds we consider:

**Definition 3.2** ([RachunekMikes]). $\xi$ is called semi-torse forming vector field for $(M, g)$ if, for all vector fields $X$:

$$R(X, \xi)\xi = 0.$$  \hfill (3.3)

From (2.2) we get: $R(X, \xi)\xi = (a + b)(X - \eta(X)\xi)$ and therefore, if $X \in \ker \eta = \xi^\perp$, then $R(X, \xi)\xi = (a + b)X$ and we obtain:

**Proposition 3.3.** For $M_{a,b}^n(\xi)$ the following are equivalent:
i) is regular,
ii) $\xi$ is not semi-torse forming,
iii) $S(\xi, \xi) \neq 0$ i.e. $\xi$ is non-degenerate with respect to $S$,
iv) $Q(\xi) \neq 0$ i.e. $\xi$ does not belong to the kernel of $Q$.

In particular, if $\xi$ is parallel ($\nabla \xi = 0$) then $M$ is not regular.

Remarks 3.4.

i) From Theorems 2 and 3 of [Wang, p. 175] a regular $M^{a,b}_n(\xi)$ is neither recurrent nor locally symmetric.

ii) From Theorem 3 of [DragomirGrimaldi, p. 228] a regular $M^{a,b}_n(\xi)$ with $a$ and $b$ constants is not Ricci semi-symmetric.

In the following we restrict to the regular case. Returning to (3.2), with $X = \xi$ in:

$$\eta(Y)\alpha(X, \xi) = \eta(X)\alpha(Y, \xi)$$

we derive:

$$\alpha(Y, \xi) = \eta(Y)\alpha(\xi, \xi) = \alpha(\xi, \xi)g(Y, \xi).$$

(3.5)

The parallelism of $\alpha$ implies also that $\alpha(\xi, \xi)$ is a constant:

$$X(\alpha(\xi, \xi)) = 2\alpha(\nabla X \xi, \xi) = 2\alpha(\xi, \xi)g(\nabla X \xi, \xi) = 2\alpha(\xi, \xi) \cdot 0 = 0.$$  

(3.6)

Making $Y = \xi$ in (3.1) and using (2.3) we get:

$$\eta(Z)\alpha(X, W) - g(X, Z)\alpha(\xi, W) + \eta(W)\alpha(X, Z) - g(X, W)\alpha(\xi, Z) = 0$$

which yield, via (3.5) and $W = \xi$:

$$\alpha(X, Z) = \alpha(\xi, \xi)g(X, Z).$$

(3.7)

In conclusion:

**Theorem 3.5.** A parallel second order symmetric covariant tensor in a regular $M^{a,b}_n(\xi)$ is a constant multiple of the metric tensor.

At the end of this section we include some applications of the above Theorem to Ricci solitons:

Naturally, two remarkable situations appear regarding the vector field $V$: $V \in \operatorname{span} \xi$ or $V \perp \xi$ but the second class seems far too complex to analyse in practice. For this reason it is appropriate to investigate only the case $V = \xi$. So, we can apply the previous result for $\alpha := L_{\xi}g + 2S$ which yields $\lambda = -S(\xi, \xi)$.

**Theorem 3.6.** Fix a regular $M^{a,b}_n(\xi)$.

i) A Ricci soliton $(g, \xi, -S(\xi, \xi) \neq 0)$ can not be steady but is shrinking if the constant $S(\xi, \xi)$ is positive or expanding if $S(\xi, \xi) < 0$. 

ii) An $\eta$-Ricci soliton $(g, \xi, \lambda, \mu)$ provided by the parallelism of $\alpha + 2\mu \eta \otimes \eta$ is given by:

$$\lambda + \mu = -S(\xi, \xi) \neq 0.$$  \hfill (3.8)

iii) If $n \geq 4$ and $b \neq 0$ then $(g, \xi, -S(\xi, \xi))$ is a Ricci soliton if and only if $\xi$ is geodesic i.e. $\nabla_{\xi}\xi = 0$ and:

$$\frac{\xi(a + b)}{4b} + a(n - 1) + b = \frac{a + b}{n - 1}. \hfill (3.9)$$

**Proof.** iii) We have three cases:

I) $\alpha + 2\lambda g = 0$ on span $\xi$ yields the above expression of $\lambda$.

II) $\alpha + 2\lambda g = 0$ on ker $\eta = \xi^\perp$ gives:

$$\frac{\xi(a + b)}{4b} + \lambda + a(n - 1) + b = 0 \hfill (3.10)$$

where we use the formula (3.5) of [GanchevMihova, p. 123].

III) $\alpha + 2\lambda g = 0$ on $(U, \xi) \in$ ker $\eta \oplus$ span $\xi$ gives:

$$g(\nabla_U \xi, \xi) + g(U, \nabla_{\xi}\xi) = 0.$$  

But the first term is zero since $\xi$ is unitary while the second implies that $\nabla_{\xi}\xi \in$ span $\xi$. But again, $\xi$ being unitary we have that $\nabla_{\xi}\xi$ is orthogonal to $\xi$.  \hfill \Box

**Example 3.7.** A para-Sasakian manifold with constant scalar curvature and vanishing $D$-concircular tensor is an $M^{n,a,b}_{\varepsilon}(\xi)$ with [DragomirGrimaldi, p. 186]:

$$a = \frac{r + 2(n - 1)}{(n - 1)(n - 2)}, \quad b = \frac{-r - n(n - 1)}{(n - 1)(n - 2)}$$

and then, a Ricci soliton $(g, \xi)$ on it is expanding. This result can be considered as a version in para-contact geometry of the Corollary of [Sharma5, p. 140] which states that a Ricci soliton $g$ of a compact $K$-contact manifold is Einstein, Sasakian and shrinking.

From (3.9) we get $r = -n$ and returning to formulae above it results:

$$a = \frac{1}{n - 1}, \quad b = \frac{-n}{n - 1}.$$
Example 3.8. Let $N_{n+1}(c)$ be a space form with the metric $g$ and $M$ a quasi-umbilical hypersurface in $N$, [ChenYano], [Wang, p. 175], i.e. there exist two smooth functions $\alpha, \beta$ on $M$ and a 1-form $\eta$ of norm 1 such that the second fundamental form is:

$$h_{ij} = \alpha g_{ij} + \beta \eta_i \eta_j.$$

According to the cited papers $M$ is an $M_{a,b}^n(\xi)$ with:

$$a = c + \alpha^2, \quad b = \alpha \beta$$

and $\xi$ the $g$-dual of $\eta$. This $M_{a,b}^n(\xi)$ is regular if and only if $c + \alpha^2 + \alpha \beta \neq 0$. Therefore, a Ricci soliton $(g, \xi)$ on this $M_{a,b}^n(\xi)$ is shrinking if $c + \alpha^2 + \alpha \beta > 0$ and expanding if $c + \alpha^2 + \alpha \beta < 0$.

Inspired by Theorem 3 of [DragomirGrimaldi, p. 185] let $N = \mathbb{R}P^{n+1}(c)$, $c > 0$ and $M$ an orientable quasi-umbilical hypersurface with $b = \alpha \beta > 0$. Then:

i) a Ricci soliton $(g, \xi)$ on it is shrinking and $M$ is a real homology sphere (all Betti numbers vanish) if it is also compact,

ii) using the result of [Ivey], for $n = 3$ the manifold is of constant curvature being compact; so the case $n = 4$ is the first important in any conditions or the case $n = 3$ without compactness when we (possible) give up at the topology of real homology sphere,

iii) using again a classic result, now due to Perelman [Perelman], the compactness implies that the Ricci soliton is gradient i.e. $\eta$ is exact.

Example 3.9. Let $(M^n, \omega_0, B)$ be a generalized Hopf manifold [DragomirOrnea], and $M^n$ an $n$-dimensional anti-invariant and totally geodesic submanifold. We set $\|\omega_0\| = 2c$ and suppose that $B$ is unitary. Then, formula (12.40) of [DragomirOrnea, p. 162] gives that if $R^\perp = 0$ then $M^n$ is of quasi-constant curvature with $a = c^2$ and $b = -\frac{1}{2}$. Therefore, $M^n$ is regular for $\|\omega_0\| \neq 1$ and a Ricci soliton is shrinking if $\|\omega_0\| > 1$ and expanding if $\|\omega_0\| < 1$.

Example 3.10. Suppose that $\xi$ is a torse-forming vector field i.e. there exist a smooth function $f$ and a 1-form $\omega$ such that:

$$\nabla_X \xi = f X + \omega(X) \xi.$$ (3.11)

From the fact that $\xi$ has unitary length it results $f + \omega(\xi) = 0$ which means that $\xi$ is exactly a geodesic vector field.
Particular cases:

i) ([RachunekMikes]) If $\omega$ is exact then $\xi$ is called concircular; let $\omega = -du$ with $u$ a smooth function on $M$. Then $f = -\omega(\xi) = \xi(u)$.

ii) If $\omega = -f\eta$ then we call $\xi$ of Kenmotsu type since (3.11) becomes similar to an expression well-known in Kenmotsu manifolds, [CalinCrasm].

Let us restrict to ii). From (3.11) a straightforward computation gives:

$$R(X,Y)\xi = X(f)[Y-\eta(Y)\xi] - Y(f)[X-\eta(X)\xi] + f^2[\eta(X)Y - \eta(Y)X]$$

and a comparison with (2.2) yields $a + b = -f^2$ and $f$ must be a constant, different from zero from regularity of the manifold. So, a possible Ricci soliton in a Kenmotsu type case must be expanding and with $S(\xi,\xi)$ and the scalar curvature constants, a result similar to Propositions 3 and 4 of [CalinCrasm].

References


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