Turk J Math
35 (2011) , 717 – 728.

doi:10.3906/mat-0710-33

Geometrical objects associated to a substructure

Fatma Özdemir, Mircea Crâşmăreanu

Abstract

Several geometric objects, namely global tensor fields of $(1,1)$-type, linear connections and Riemannian metrics, associated to a given substructure on a splitting of tangent bundle, are studied. From the point of view of lifting to entire manifold, two types of polynomial substructures are distinguished according to the vanishing of not of the sum of the coefficients. Conditions of parallelism for the extended structure with respect to some remarkable linear connections are given in two forms, firstly in a global description and secondly using the decomposition in distributions. A generalization of both Hermitian and anti-Hermitian geometry is proposed.

Key Words: Polynomial substructure, Induced polynomial structure, Schouten and Vrănceanu connections, (anti)Hermitian metric, Shape operator.

1. Introduction

There are plenty of substructures in differential geometry. Usually we are interested in substructures generated by a given structure, e.g. the fruitful theory of submanifolds. In this paper we adopt a different point of view. Namely, we start with a fixed substructure, defined as a tensor field of $(1,1)$-type on a given distribution, and search what kind of geometrical objects can be associated in a natural way using a complementary distribution. Based on the fact that the bundle of tensor fields of $(1,1)$ provides a very interesting geometry (cf. [3], [4], [5]), we obtain a lot of geometric notions, namely: global tensor fields of $(1,1)$-type, linear connections, (anti)Hermitian metrics.

The paper is divided into three Sections. Firstly, polynomial structures generated by polynomial substructures are studied. The notion of *polynomial structure* was introduced by Samuel I. Golberg and Kentaro Yano in [10] as a generalization of f-structures (i.e. $f^3+f=0$) and studied by various authors in [11], [23], [24], [25], [26] and In our approach, we obtain two types of extensions according to the vanishing or non-vanishing for the sum of coefficients for the initial polynomial P. In the first case, the extension has the same polynomial, and in the second situation the polynomial $(X-1)\cdot P$.

The second section is devoted to some linear connections related to the given splitting of tangent bundle, namely Schouten and Vrănceanu. Conditions for the extension of a given substructure to be parallel with respect to these connections are given.

2000 *AMS Mathematics Subject Classification:* 53C15; 53C55.

Because Riemannian geometry is the most commonly used framework, in the last part we add a Riemannian metric, related with the given substructure in an (anti)Hermitian manner. In this way, a generalization of (anti)Hermitian geometry and the geometry of metric polynomial structures is obtained. The notion of *metric polynomial structures* was introduced by Jarolim Bures and Jiri Vanzura in [1] and intensively studied in [8], [17]-[20]. As main results of this section, conditions for the corresponding conjugate shape operator to be self-adjoint or skew-adjoint are obtained.

In first and last sections, the well known examples of almost complex and almost product structures are given in detail. Also, the results of first and last sections are generalizations of characterizations from [21] where the case of almost complex substructure and Hermitian metric is treated only and the second section extends the results of [7] where $f(2v+3,\varepsilon)$-structures are studied only. Moreover, other remarkable structures as almost tangent, $f(2\nu+3)$ and $\varphi(4,\pm 2)$ are considered.

As frameworks with possible applications for our results the Ehresmann connections, nonlinear connections and symplectic distributions are pointed out. Ehresmann connections appear in field theories, nonlinear connections are suited to geometrization of Lagrangian description of analytical mechanics and, as is well-known, symplectic distribution are natural tools in the dual point of view, namely Hamiltonian approach. Also, all our formulae are coordinates-free and suitable for any specific structure required by a concrete problem.

2. Extensions of polynomial substructures

Let M be a smooth, finite dimensional manifold and R, S two complementary distributions on M, i.e. for every $x \in M$ the following decomposition holds:

$$T_x M = R_x \oplus S_x. \tag{2.1}$$

Let r, s be the corresponding projections, both viewed as $(1,1)$-tensor fields; then

$$\begin{cases} r^2 = r, s^2 = s \\ rs = sr = 0, r + s = 1_{TM}. \end{cases} \tag{2.2}$$

Examples 2.1.

1) **Ehresmann connections.** Let $\pi : E \to N$ be a surjective submersion and $V = \ker T\pi$ *the vertical subbundle* of E. An *Ehresmann connection* on (E, π, M) is a smooth subbundle H of $M = TE$, also called *horizontal subbundle* ([13, p. 36]), such that: $TM = V \oplus H$.

2) **Nonlinear connections.** For the previous example with $E = TN$ an horizontal subbundle is usually called *nonlinear connection* ([15]). The decomposition $T(TN) = V \oplus H$ is useful for geometrization of Lagrangian mechanics on TN [14]. An extension of this notion to bundle-type tangent manifolds appears in [2].

3) **Symplectic distributions.** Recall, after [6, p. 272], that if W is a subspace of a symplectic vector space (V, σ), then $V = W + W^{\sigma}$, where $W^{\sigma} = \{v \in V; \sigma(v, w) = 0$ for every $w \in W\}$. Also ([6, p. 273]), the given subspace W is called *symplectic* if $\sigma\mid_{W \times W}$ is nondegenerate and then ([6, p. 274]) $W \cap W^{\sigma} = \{0\}$.

In conclusion, given any symplectic distribution R on a symplectic manifold (M,σ) (i.e. R_x is a symplectic subspace of T_xM, for all $x \in M$), we get another symplectic distribution $S = R^\sigma$, complementary to R.

We now return to general framework. Let us consider the following definiton.

Definition 2.1. (i) A tensor field J_R of $(1,1)$-type on R is called a *substructure* on M.
(ii) The above J_R is called *polynomial substructure* if there exists a polynomial P such that $P(J_R) = 0$. In the following, we shall suppose that P is the minimal polynomial with this property.

Let us remark that if J_R is a polynomial substructure and h is a nondegenerate tensor field of $(1,1)$-type on R, then the tensor field $h \circ J_R \circ h^{-1}$ is also a polynomial substructure of the same type as J_R.
(iii) For above J_R we define *a special extension*, denoted $E(J_R)$, to be the following tensor field of $(1,1)$-type on M:

$$E(J_R) = s + J_R \circ r. \tag{2.3}$$

The reason why we are interested in this case is to find out how the concept of extension can play a role in the following mathematical problem. The fact is that two types of polynomial substructures are distinguished according to the vanishing or non-vanishing of the sum of coefficients. (For the standard definition of to extension, see Remark 2.1 and Remark 2.2.)

A natural question appears: Is there a polynomial function $E(P)$ such that $E(P)(E(J_R)) = 0$? We prove that the answer is affirmative; more precisely the explicit form of $E(P)$ is obtained. Let us point out that the form of $E(J_R)$ depends on the vanishing of the sum of the coefficients of P. In the null case we denote $E(J_R)$ by $E_0(P)$ and for the nonzero case we denote it by $E^*(P)$. Therefore we call $E(J_R)$ the *polynomial structure generated by* J_R and $E_0(P)$ *the null extension* of P, and $E^*(P)$ *the non-null extension* of P.

Let us consider the expression of P

$$P(X) = X^k + a_1 X^{k-1} + \ldots + a_{k-1}X + a_k, \tag{2.4}$$

which yields

$$J_R^k + a_1 J_R^{k-1} + \ldots + a_{k-1} J_R + a_k = 0. \tag{2.5}$$

Let us denote $\Sigma(P) = 1 + a_1 + \ldots + a_k$ the sum of coefficients for P. Using the relations (2.3), we have

$$E(J_R)^2 = s + J_R^2 \circ r, \ldots, E(J_R)^k = s + J_R^k \circ r, \tag{2.6}$$

and then

$$P(E(J_R)) = (1 + a_1 + \ldots + a_k)\, s + P(J_R) \circ r = \Sigma(P)\, s. \tag{2.7}$$

The last relation implies the following classification:

A. $\Sigma(P) = 0$. Then

$$E_0(P) = P. \tag{2.8}$$

B. $\Sigma(P) \neq 0$. We get

$$E(J_R) \circ P(E(J_R)) = \Sigma(P)\, E(J_R) \circ s = \Sigma(P)\, s = P(E(J_R)). \tag{2.9}$$

In conclusion, by considering

$$E^{*}(P)(X)=X\cdot P(X)-P(X)=(X-1)P(X), \tag{2.10}$$

there results

$$E^{*}(P)(E(J_R))=0 \tag{2.11}$$

as final answer.

Remark 2.1. We study a special extension of polynomial substructure given in equation (2.3). Instead of this extension, one could start with the standard definition of extension

$$J_R\circ r=r\circ E(J_R), \tag{2.12}$$

and using the fact that

$$r(J_R\circ r)=J_R\circ r, \tag{2.13}$$

one could solve $E(J_R)$ as

$$E(J_R)=J_R\circ r+\sigma \tag{2.14}$$

for some linear transformation $\sigma: S\longrightarrow S$. However, in general

$$\sigma\neq s. \tag{2.15}$$

Since s is a projection, it should restrict to the identity map on S; however there is no such condition for σ. Hence, defining another linear transformation $J_S: S\longrightarrow S$ by

$$\sigma=J_S\circ s, \tag{2.16}$$

it is possible to define the extension

$$E(J_R)=J_R\circ r+J_S\circ s. \tag{2.17}$$

Note that it is not possible to get rid of J_S by redefining $\bar{s}=J_S\circ s$ for J_S may not be either surjective or injective. However, one could proceed by using the fact that

$$(E(J_R))^k=(J_R)^k\circ r+(J_S)^k\circ s\,. \tag{2.18}$$

Then it is obvious that

$$P(E(J_R))=P(J_R)\circ r+P(J_S)\circ s=P(J_S)\circ s\,. \tag{2.19}$$

Following the procedure, we obtain

$$E(J_R)\circ P(E(J_R))=(J_S\circ s)\circ(P(J_S)\circ s)=(J_S\circ s)\circ P(E(J_R)) \tag{2.20}$$

and

$$(E(J_R)-J_S\circ s)\circ P(E(J_R))=0, \tag{2.21}$$

which suggests that each choice of J_S defines an extension. In equations (2.3), we choose

$$J_S=Id_S \tag{2.22}$$

and derive our conditions on sum of the coefficients of the polynomial.

Remark 2.2. One can prefer to deal with a special case by choosing

$$J_S = \lambda Id_S \, , \tag{2.23}$$

and get, in turn,

$$P(E(J_R)) = P(\lambda)s. \tag{2.24}$$

For complex manifolds, one can choose λ to be the root of the polynomial and obtain

$$P(E(J_R)) = 0, \tag{2.25}$$

which implies that each distinct root of the polynomial defines a different extension of the same polynomial structure without any condition on the sum of the coefficients in the polynomial.

Examples 2.2.

1) $f\,(2v+3,\varepsilon)$**-substructures.** Let v be a natural number and $\varepsilon = \pm 1$. J_R is an $f\,(2v+3,\varepsilon)$-*substructure* if ([7])

$$J_R^{2v+3} + \varepsilon J_R = 0, \tag{2.26}$$

which means that

$$P\,(X) = X^{2v+3} + \varepsilon X\,, \tag{2.27}$$

$$E^*\,(P)\,(X) = (X-1)\left(X^{2v+3} + \varepsilon X\right) = X^{2v+4} - X^{2v+3} + \varepsilon X^2 - \varepsilon X. \tag{2.28}$$

Particular case: $v = 0$, i.e. $f\,(3,\varepsilon)$-*substructures* with

$$P\,(X) = X^3 + \varepsilon X, \quad E^*\,(P)\,(X) = X^4 - X^3 + \varepsilon X^2 - \varepsilon X. \tag{2.29}$$

(i) $\varepsilon = +1$. For example, if J_R is *almost complex substructure*, $J_R^2 = -1_R$, then

$$P\,(X) = X^2 + 1, \quad E^*\,(P)\,(X) = (X-1)\left(X^2+1\right) = X^3 - X^2 + X - 1. \tag{2.30}$$

It is interesting that in this case

$$E\,(J_R)^4 - 1_M = \left(E\,(J_R)^2 - 1\right)\left(E\,(J_R)^2 + 1\right) = (E\,(J_R) + 1)\,E^*\,(P)\,(E\,(J_R)) = 0, \tag{2.31}$$

which means that $E\,(J_R)$ is an *almost electromagnetic structure* on M; cf. [22]. This result was obtained in [21].

(ii) $\varepsilon = -1$. For example, if J_R is an *almost product substructure*, $J_R^2 = 1_R$, then we are in the case A and then

$$P\,(X) = E_0\,(P)\,(X) = X^2 - 1, \tag{2.32}$$

which means that $E\,(J_R)$ is also an almost product structure on M.

2) **Almost k-tangent substructures.** J_R is an *almost k-tangent substructure* if $J_R^{k+1}=0$ (cf. [16]), which yields

$$P(X)=X^{k+1},\quad E^*(P)(X)=(X-1)X^{k+1}=X^{k+2}-X^{k+1}. \tag{2.33}$$

3) $\varphi(4,\pm 2)$**-substructures**. J_R is an $\varphi(4,+2)$ (respectively $\varphi(4,-2)$) substructure if ([27],[9])

$$J_R^4+J_R^2=0\left(J_R^4-J_R^2=0\right) \tag{2.34}$$

and then

$$E^*(P)(X)=(X-1)\left(X^4+X^2\right)=X^5-X^4+X^3-X^2\,,$$

$$E_0(P)(X)=X^4-X^2\,,$$

respectively.

4) **Natural almost product structure** associated to the pair (R,S). Let us remark that $E(1_R)=1_{TM}$ and let us denote $T=E(-1_R)=s-r$. For $J_R=-1_R$ we have $P(X)=X+1$ and thus $E^*(P)(X)=(X-1)(X+1)=X^2-1$. Therefore $T^2=1_{TM}$ which means that T is an almost product structure on M naturally associated to the pair (R,S). An important remark is that T commutes with the extension $E(J_R)$ for every J_R

$$T\circ E(J_R)=E(J_R)\circ T=s-J_R\circ r. \tag{2.35}$$

3. Parallelism with respect to Schouten and Vrănceanu connections

In this section we obtain some commutation identities, more precisely conditions equivalent to the parallelism of the extension of a substructure with respect to Schouten and Vrănceanu connections.

For the rest of this section, let ∇ be a fixed linear connection on M. To the triple (R,S,∇) we associate two other linear connections ([7], [12]):

(i) *the Schouten connection*

$$\overset{Sc}{\nabla}_X Y=r(\nabla_X rY)+s(\nabla_X sY)\,; \tag{3.1}$$

(ii) *the Vrănceanu connection*

$$\overset{V}{\nabla}_X Y=r(\nabla_{rX} rY)+s(\nabla_{sX} sY)+r[sX,rY]+s[rX,sY]\,. \tag{3.2}$$

The following two propositions, generalizations of similar results from [7], show the importance of these connections from the point of view of splitting (2.1).

Proposition 3.1. *The projections r,s are parallel with respect to Schouten and Vrănceanu connection for every linear connection ∇ on M.*

Proof. Let $\mathcal{X}(M)$ denote the Lie algebra of vector fields on M. For every $X, Y \in \mathcal{X}(M)$

$$\left(\overset{Sc}{\nabla}_X r\right) Y = \overset{Sc}{\nabla}_X rY - r\left(\overset{Sc}{\nabla}_X Y\right) = r(\nabla_X rY) - r(\nabla_X rY) = 0 \tag{3.3}$$

$$\begin{aligned}\left(\overset{V}{\nabla}_X r\right) Y &= \overset{V}{\nabla}_X rY - r\left(\overset{V}{\nabla}_X Y\right) \\ &= r(\nabla_{rX} rY) + r[sX, rY] - r(\nabla_{rX} rY) - r[sX, rY] = 0.\end{aligned} \tag{3.4}$$

Similar relations hold for s. □

Recall that a distribution D on M is called *parallel with respect to the linear connection* ∇ if $X \in \mathcal{X}(M)$ and $Y \in D$ implies $\nabla_X Y \in D$.

Proposition 3.2. *The distributions R, S are parallel with respect to Schouten and Vrănceanu connection for every linear connection ∇ on M.*

Proof. Let $X \in \mathcal{X}(M)$ and $Y \in R$. Since, $sY = 0$ and $rY = Y$, we get

$$\overset{Sc}{\nabla}_X Y = r(\nabla_X Y) \in R$$

$$\overset{V}{\nabla}_X Y = r(\nabla_{rX} Y) + r[sX, Y] \in R,$$

Similar relations hold for S. □

For the following two characterization we use the relation

$$(\nabla_X J) Y = \nabla_X JY - J(\nabla_X Y) \tag{3.5}$$

for every tensor field of $(1,1)$-type J.

Proposition 3.3. *$E(J_R)$ is parallel with respect to Schouten connection if and only if ∇ satisfies*

$$r(\nabla_X J_R \circ rY) = J_R \circ r(\nabla_X rY) \tag{3.6}$$

for every $X, Y \in \mathcal{X}(M)$. Equivalently, for every $X \in X(M)$ and every $Y \in R$

$$r(\nabla_X J_R Y) = J_R \circ r(\nabla_X Y). \tag{3.7}$$

Proof. From

$$\overset{Sc}{\nabla}_X E(J_R) Y = r(\nabla_X r \circ E(J_R) Y) + s(\nabla_X s \circ E(J_R) Y)$$

$$= r(\nabla_X J_R \circ rY) + s(\nabla_X sY),$$

$$E(J_R)\left(\overset{Sc}{\nabla}_X Y\right) = E(J_R) \circ r(\nabla_X rY) + E(J_R) \circ s(\nabla_X sY)$$

$$= J_R \circ r\left(\nabla_X rY\right) + s\left(\nabla_X sY\right)$$

the conclusion follows. The second part is based on the fact that the equation (3.6) is obviously satisfied for $Y \in S$. □

Proposition 3.4. *$E\left(J_R\right)$ is parallel with respect to Vrănceanu connection if and only if ∇ satisfies*

$$r\left(\nabla_{rX} J_R \circ rY\right) + r\left[sX, J_R \circ rY\right] = J_R \circ r\left(\nabla_{rX} rY\right) + J_R \circ r\left[sX, rY\right] \tag{3.8}$$

for every $X, Y \in \mathcal{X}(M)$. Equivalently, for $X \in \mathcal{X}(M)$ and $Y \in R$

$$r\left(\nabla_{rX} J_R Y\right) + r\left[sX, J_R Y\right] = J_R \circ r\left(\nabla_{rX} Y\right) + J_R \circ r\left[sX, Y\right]. \tag{3.9}$$

Proof. The relations

$$\begin{aligned} \overset{V}{\nabla}_X E\left(J_R\right) Y &= r\left(\nabla_{rX} rE\left(J_R\right) Y\right) + s\left(\nabla_{sX} sE\left(J_R\right) Y\right) + r\left[sX, rE\left(J_R\right) Y\right] \\ &\quad + s\left[rX, sE\left(J_R\right) Y\right] \\ &= r\left(\nabla_{rX} J_R \circ rY\right) + s\left(\nabla_{sX} sY\right) + r\left[sX, J_R \circ rY\right] + s\left[rX, sY\right] \end{aligned}$$

$$\begin{aligned} E\left(J_R\right)\left(\overset{V}{\nabla}_X Y\right) &= E\left(J_R\right) r\left(\nabla_{rX} rY\right) + E\left(J_R\right) s\left(\nabla_{sX} sY\right) + E\left(J_R\right) r\left[sX, rY\right] \\ &\quad + E\left(J_R\right) s\left[rX, sY\right] \\ &= J_R \circ r\left(\nabla_{rX} rY\right) + s\left(\nabla_{sX} sY\right) + J_R \circ r\left[sX, rY\right] + s\left[rX, sY\right] \end{aligned}$$

yield the conclusion. □

Let us remark that:

(i) if $X \in R$ then (3.8) is exactly (3.6)

(ii) if $X \in S$ then (3.9) reads:

$$r\left[X, J_R Y\right] = J_R \circ r\left[X, Y\right]. \tag{3.10}$$

4. (anti)Hermitian metrics with respect to an ε-substructure

Let g be a fixed Riemannian metric on M such that the distributions R, S are g-orthogonal: $g(X,Y) = 0$, for every $X \in R, Y \in S$. Note that such metrics exist; if h is an arbitrary metric then

$$g(X,Y) = h(rX, rY) + h(sX, sY)$$

satisfies the above condition.

Definition 4.1. For λ a nonzero real number, the given Riemannian metric is called *λ-Hermitian with respect to J_R* if

$$g\mid_R\left(J_R X, J_R Y\right) = \lambda g\mid_R (X,Y) \tag{4.1}$$

for every $X, Y \in R$ where $g\,|_R$ is the restriction of g to R. For $\lambda = 1$, we call g a *Hermitian metric* and for $\lambda = -1$ we call g an *anti-Hermitian* metric w.r.t. J_R.

Hence

$$g\left(E\left(J_R\right)X, E\left(J_R\right)Y\right) = g\,|_S\,(sX, sY) + \lambda g\,|_R\,(rX, rY) \tag{4.2}$$

and then g is Hermitian w.r.t. $E\left(J_R\right)$ if and only if g is Hermitian w.r.t. J_R. The pair (f, g) with f a polynomial structure and g a Hermitian metric with respect to f was considered by Bures and Vanzura in [1] with the name *metric polynomial structure*. Hence, if g is Hermitian w.r.t. to J_R then $\left(E\left(J_R\right), g\right)$ is a metric polynomial structure.

In the following we restrict to the case when J_R is an ε-substructure, $\varepsilon \neq 0$, i.e. $J_R^2 = \varepsilon 1_R$. With the substitution $Y \to J_R Y$ in (4.1) we have

$$g\,|_R\,(J_R X, \varepsilon Y) = \lambda g\,|_R\,(X, J_R Y), \tag{4.3}$$

which means that the $g\,|_R$*-adjoint* of J_R is

$$J_R^* = \frac{\lambda}{\varepsilon} J_R. \tag{4.4}$$

In particular, for $\lambda = \varepsilon$, J_R is $g\,|_R$-self-adjoint and for $\lambda = -\varepsilon$, J_R is skew-adjoint.

For a fixed $Z \in S$, we define *the shape operator* $L_Z : R \to R$, $\quad L_Z(X) = r(\nabla_X Z)$, where ∇ is the Levi-Civita connection of g.

Proposition 4.1. *If R is integrable then L_Z is self-adjoint.*

Proof. Let $X, Y \in R$. From g-orthogonality of R and S there results

$$X\left(g(Y, Z)\right) = 0 = g\left(\nabla_X Y, Z\right) + g\left(Y, \nabla_X Z\right). \tag{4.5}$$

Also, because $Y \in R$

$$g\left(Y, \nabla_X Z\right) = g\left(Y, r\left(\nabla_X Z\right)\right) = g\left(Y, L_Z X\right). \tag{4.6}$$

The last two relations yield

$$g\left(\nabla_X Y, Z\right) = -g\left(Y, L_Z X\right) \tag{4.7}$$

and a similar one

$$g\left(\nabla_Y X, Z\right) = -g\left(X, L_Z Y\right).$$

By using the fact that the connection ∇ is torsionless and the integrability of R it follows that

$$g\left(X, L_Z Y\right) - g\left(Y, L_Z X\right) = g\left(\nabla_X Y, Z\right) - g\left(\nabla_Y X, Z\right) = g\left(\left[X, Y\right], Z\right) = 0.$$

□

In the definition of the shape operator the given substructure does not appear. Let $\overline{L}_Z = J_R \circ L_Z$ be *the conjugate shape operator*. For the rest of this section we are interested in conditions providing $\overline{L}_Z$ to be self-adjoint or skew-adjoint.

Proposition 4.2. *If R is integrable then the $g\,|_R$-adjoint of $\overline{L}_Z$ is*

$$\overline{L}_Z^* = \frac{\lambda}{\varepsilon} L_Z \circ J_R. \tag{4.8}$$

In consequence, if J_R commutes with L_Z then $\overline{L}_Z$ is self-adjoint(skew-adjoint) if and only if $\lambda = \varepsilon\,(\lambda = -\varepsilon)$.

Proof. For the first part,

$$\overline{L}_Z^* = (J_R \circ L_Z)^* = L_Z^* \circ J_R^* = L_Z \circ \left(\frac{\lambda}{\varepsilon} J_R\right).$$

For the second part,

$$\overline{L}_Z^* = \frac{\lambda}{\varepsilon} L_Z \circ J_R = \frac{\lambda}{\varepsilon} J_R \circ L_Z = \frac{\lambda}{\varepsilon} \overline{L}_Z.$$

□

Corollary 4.1. *Suppose that R is integrable.*

(i) *If J_R is an almost complex substructure($\varepsilon = -1$) and g is Hermitian w.r.t. $J_R\,(\lambda = 1)$, then $\overline{L}_Z$ is self-adjoint(skew-adjoint) if and only if J_R anticommutes(commutes) with L_Z.*

(ii) *If J_R is an almost complex substructure and g is anti-Hermitian w.r.t. $J_R\,(\lambda = -1)$, then $\overline{L}_Z$ is self-adjoint(skew-adjoint) if and only if J_R commutes(anticommutes) with L_Z.*

(iii) *If J_R is an almost product substructure($\varepsilon = 1$) and g is Hermitian w.r.t. J_R, then $\overline{L}_Z$ is self-adjoint(skew-adjoint) if and only if J_R commutes(anticommutes) with L_Z.*

(iv) *If J_R is an almost product substructure and g is anti-Hermitian w.r.t. J_R, then $\overline{L}_Z$ is self-adjoint(skew-adjoint) if and only if J_R anticommutes(commutes) with L_Z.*

A remarkable example is provided by the following definition.

Definition 4.2. The integral submanifolds of R are called *totally umbilical* if there exists a real number μ such that

$$L_Z = \mu 1_R \tag{4.9}$$

for every $Z \in S$. Then

$$\overline{L}_Z = \mu J_R, \tag{4.10}$$

which implies this next corollary.

Corollary 4.2. *Suppose that R is integrable with totally umbilical integral submanifolds.*

(i) *If J_R is an almost complex substructure and g is Hermitian(anti-Hermitian) w.r.t. J_R, then $\overline{L}_Z$ is skew-adjoint(self-adjoint).*

(ii) *If J_R is an almost product substructure and g is Hermitian(anti-Hermitian) w.r.t. J_R, then $\overline{L}_Z$ is self-adjoint(skew-adjoint).*

Remark 4.1. The cases (i) of Corollary 4.1 and 4.2 with the Hermitian metric g and the skew-adjoint shape operator $\overline{L}_Z$ appeared in [21].

5. Conclusions

In this paper we study a geometric object which is the global tensor fields of $(1,1)$-type associated to a given special extension of polynomial substructure on a splitting of tangent bundle. Two types of polynomial substructures are distinguished according to the vanishing of the sum of the coefficients or not. The parallelism problem and the case of (anti)Hermitian metrics with respect to an ε substructure is investigated by using this special extension developed in section 2. A generalization of both Hermitian and anti-Hermitian geometry is also proposed in this work. Starting from a more general extension of polynomial substructure and its relations with geometric objects will be one of the topics of later studies.

Acknowledgements

The authors would to thank the referee for his/her valuable suggestions and remarks (Remark 2.1–2.2) which improved the paper.

References

[1] Bures, J., Vanzura, J., Metric polynomial structures, Kōdai Mat. Sem. Rep., 27, 345-352, (1970).

[2] Crâşmăreanu, M., Nonlinear connections and semisprays on tangent manifolds, Novi Sad J. Math., 33, 11-22, (2003).

[3] Cruceanu, V., Le fibré des tenseurs de type $(1,1)$ sur une variété différentiable, An. Ştiinţ. Univ. "Al. I. Cuza", Iaşi, 28, 45-54, (1982).

[4] Cruceanu, V., Objects géométriques invariants sur le fibré des tenseurs de type $(1,1)$, An. Ştiinţ. Univ. "Al. I. Cuza", Iaşi, 28, 55-62, (1982).

[5] Cruceanu, V., Structures remarquables sur le fibré des tenseurs de type $(1,1)$, Mathematica(Cluj), 25, 21-28, (1983).

[6] Cushman, R. H., Bates, L. M., Global aspects of classical integrable systems, Birkhäuser, Basel, 1997.

[7] Demetropoulou-Psomopoulou, D., Linear connections on manifolds admitting an $f(2v+3,-1)$-structure or an $f(2v+3,1)$-structure, Tensor, 47, 235-239, (1988).

[8] Deszynski, K., Notes on polynomial structures equipped with a Lorentzian metric, Univ. Iagiel., Acta Math., 28, 11-18, (1991).

[9] Gadea, P. M., Cordero, L. A., On integrability conditions of a structure φ satisfying $\varphi^4 \pm \varphi^2 = 0$, Tensor, 28, 78-82, (1974).

[10] Goldberg, S. I., Yano, K., Polynomial structures on manifolds, Kōdai Math. Sem. Rep., 22, 199-218, (1970).

[11] Goldberg, S. I., Petridids, N. C., Differentiable solutions of algebraic equations on manifolds, Kōdai math. Sem. Rep., 25, 111-128, (1973).

[12] Ianuş, S., Sur les structures presque produit des varietes a connection lineaire, C. R. A. S., Paris, 272, 734-735, (1971).

[13] Marsden, J., Montgomery, R., Raţiu, T., Reduction, symmetry and phases in mechanics, Mem. Amer. Math. Soc., 88, 436, (1990).

[14] Martin, G., Almost complex structures that model nonlinear geometries, Journal of Geometry and Physics, 4, 21-38, (1987).

[15] Miron, R., Anastasiei, M., The geometry of Lagrange spaces: theory and applications, Fundamental Theories of Physics, 59. Dordrecht: Kluwer Academic Publishers, 1994.

[16] Miron, R., The geometry of higher-order Lagrange spaces. Applications to mechanics and physics, Fundamental Theories of Physics, 82. Dordrecht: Kluwer Academic Publishers, 1997.

[17] Opozda, B., A theorem on metric polynomial structures, Ann. Polon. Math., 41, 139-147, (1983).

[18] Opozda, B., . The f-sectional curvature of f-Kählerian manifolds, Ann. Polon. Math., 43, 141-150, (1983).

[19] Opozda, B., Almost product and almost complex structures generated by polynomial structures, Univ. Jagel. Acta Math., 24, 27-31, (1984).

[20] Opozda, B., Metric polynomial structures, Dissertationes Math., 249, (1986).

[21] Özdemir, F., Doğanaksoy, A., Crâşmăreanu, M., Almost complex integral submanifolds and totally umbilical submanifolds, Mem. Secţ. Ştiinţ. Acad. Rom., 45-49, (2002).

[22] Reyes, E., Cruceanu, V., Gadea, P. M., Structures of electromagnetic type on vector bundles, J. Phys. A, Math. Gen., 32, 3805-3814, (1999).

[23] Vanžura, J., Integrability conditions for polynomial structures, Kōdai Math. Sem. Rep., 27, 42-50, (1976).

[24] Vanžura, J., Vanzurová, A., Polynomial mappings of polynomial structures with simple root, Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math., 33, 157-164, (1994).

[25] Vanžurová, A., Polynomial structures with double roots, Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math., 36, 187-196, (1997).

[26] Walczak, P. G., Polynomial structures on principal fibre bundles, Colloq. Math., 35, 73-81, (1976).

[27] Yano, K., Houh, C., Chen, B., Structures defined by a tensor field φ of type $(1,1)$ satisfying $\varphi^4 \pm \varphi^2 = 0$, Tensor, 23, 81-87, (1972).

Fatma ÖZDEMİR
Department of Mathematics,
İstanbul Technical University,
34469, İstanbul-TURKEY
e-mail: fozdemir@itu.edu.tr

Mircea CRÂŞMĂREANU
Faculty of Mathematics,
Alexandru Ioan Cuza University,
Iaşi, 6600, ROMANIA
e-mail: mcrasm@uaic.ro

Received: 26.10.2007