Conservation laws generated by pseudosymmetries with applications to Lagrangian systems

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Abstract

In this paper we extend a result of Gerald L. Jones which give conservation laws for ordinary differential equations. Applications to Lagrangian systems are given.

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Introduction

The very well-known way to obtain conservation laws for a system of differential equations is Noether theorem which associates to every symmetry a conservation law. G. L. Jones gives in [3] another method based on a weaker generalization of notion of symmetry, namely *pseudosymmetries.* The advantages of this method are that it does not require any integration(if there are associated some natural invariants, see the Lagrangian case below) and does not require, as Noether theorem, that the differential equations follow from a variational principle.

In this paper we present a generalization of Jones result and applications to Lagrangian systems.

1 From pseudosymmetries to conservation laws

Let M be a smooth, n-dimensional manifold, $C^{\infty}(M)$ the ring of real-valued smooth functions, $\mathcal{X}(M)$ the Lie algebra of vector fields and $\Omega^p(M)$ the $C^{\infty}(M)$-module of p-differential forms, $1 \leq p \leq n$.

For $X \in \mathcal{X}(M)$ with local expression $X = X^i(x)\frac{\partial}{\partial x^i}$ one consider the system of differential equations which give the flow of X:

$$(1.1) \quad \dot{x}^i(t) = \frac{dx^i}{dt}(t) = X^i\left(x^1(t), \ldots, x^n(t)\right), \qquad i = 1, \ldots, n.$$

A solution of (1.1) is called *integral curve* of X.

Definition 1.1 A function $f \in C^{\infty}(M)$ is called *conservation law*(or *first integral,* or *constant of motion,* or *invariant function*) for X or (1.1) if f is constant along the solutions of (1.1) that is:

$$(1.2) \quad \frac{d(f \circ c)}{dt}(t) = 0$$

for every integral curve $c(t)$ of X.

Because for $f \in C^{\infty}(M)$ its rate of change along (1.1) is:

$$(1.3) \quad \frac{df}{dt} = \frac{\partial f}{\partial x^i}\dot{x}^i = \frac{\partial f}{\partial x^i}X^i = \mathcal{L}_X f$$

where the right-hand side means *the Lie derivative of f with respect to X* we get:

Proposition 1.2 *$f \in C^{\infty}(M)$ is conservation law for (1.1) if and only if:*

$$(1.4) \quad \mathcal{L}_X f = 0.$$

For our approach is necessary the following:

Definition 1.3 (i) $Y \in \mathcal{X}(M)$ is called *symmetry for X* if:

$$(1.5) \quad \mathcal{L}_X Y = 0.$$

(ii) If $Y \in \mathcal{X}(M)$ is fixed then $Z \in \mathcal{X}(M)$ is called *Y-pseudosymmetry for X* if there exists $f \in C^{\infty}(M)$ such that:

$$(1.6) \quad \mathcal{L}_X Z = fY.$$

(iii) $\omega \in \Omega^p(M)$ is called *invariant form for X* if:

$$(1.7) \quad \mathcal{L}_X \omega = 0.$$

Remark 1.4 (i) A 0-pseudosymmetry is obviously a symmetry.
(ii) A X-pseudosymmetry for X is called *pseudosymmetry for* X in ([3, p. 1055]).
(iii) If in (1.6) f is constant then $\mathcal{L}_X Z$ is symmetry for X.
(iv) If in (1.6) f is not constant then $\mathcal{L}_X Z$ is symmetry for X if and only if f is conservation law for X.

The result which give the association between pseudosymmetries and conservation laws is:

Theorem 1.5 *Let $X \in \mathcal{X}(M)$ be a fixed vector field and $\omega \in \Omega^p(M)$ be a p-form invariant for X. If $Y \in \mathcal{X}(M)$ is symmetry for X and $S_1, \ldots, S_{p-1} \in \mathcal{X}(M)$ are $(p-1)$ Y-pseudosymmetries for Y then:*

$$(1.8) \quad \phi = \omega(S_1, \ldots, S_{p-1}, Y)$$

or locally:

$$(1.9) \quad \phi = S_1^{i_1} \ldots S_{p-1}^{i_{p-1}} Y^{i_p} \omega_{i_1 i_2 \ldots i_p}$$

is a conservation law for X. Particularly, if $Y, S_1, \ldots, S_{p-1}$ are symmetries for X then ϕ given by (1.8) *is conservation law.*

Proof Applying the properties of Lie derivatives one have:

$$\mathcal{L}_X \phi = (\mathcal{L}_X S_1)^{i_1} \ldots + S_1^{i_1} (\mathcal{L}_X S_2)^{i_2} \ldots + \ldots + \ldots (\mathcal{L}_X Y)^{i_p} \omega_{\ldots} + \ldots Y^{i_p} (\mathcal{L}_X \omega)_{i_1 \ldots i_p} .$$

In this relation each of the first $p-1$ terms has a factor of the form $\mathcal{L}_X S_i = \lambda_i Y$ so that ω is contracted with two factors of Y and then each term vanishes by the antisymmetry of ω. The p-th term and $p+1$-th term vanishes since (1.5) and (1.7). □

Remark 1.6 (i) If the pseudosymmetries are linearly dependent then: $\phi = 0$ by the antisymmetry of ω.
(ii) For $Y = X$ one obtain the main result of G. L. Jones([3, p. 1056]).
(iii) If $p = 1$ one obtain theorem 2.5.10 of ten Eikelder([2, p. 48]).
(iv) The fact that the pseudosymmetries (1.6) with f =constant can be used to integrate planar($n = 2$) vector fields can be found in [7, p. 37-38, relation 8.13].

2 Applications to regular Lagrangian systems

The usual framework of Hamiltonian systems is the cotangent bundle T^*Q of a m-dimensional manifold Q. Using the Legendre transformation we obtain the Lagrangian point of view, which will be developed in this section.

Let Q be a m-dimensional manifold with TQ the tangent bundle and one consider a smooth function $L : TQ \to I\!R$ usually called *Lagrangian*.

On TQ an important structure is the $C^\infty(TQ)$-linear mapping([4, p. 108]) $J : \mathcal{X}(TQ) \to \mathcal{X}(TQ)$:

$$(2.1) \qquad J\left(\frac{\partial}{\partial q^i}\right) = \frac{\partial}{\partial \dot{q}^i}, \quad J\left(\frac{\partial}{\partial \dot{q}^i}\right) = 0$$

or, equivalently:

$$(2.2) \qquad J = \frac{\partial}{\partial \dot{q}^i} \otimes dq^i$$

where $(q^i) = q$ are the coordinates on Q and $(q^i, \dot{q}^i) = (q, \dot{q})$ the associated coordinates on TQ.

To the Lagrangian L we can associate two forms, usually called *Cartan forms*([1, p. 346]):

$$(2.3) \qquad \theta_L = J^*(dL)$$

where J^* is the adjoint of structure J given by (2.1) and:

$$(2.4) \qquad \omega_L = d\theta_L.$$

In local coordinates:

$$(2.5) \qquad \theta_L = \frac{\partial L}{\partial \dot{q}^i} dq^i$$

$$(2.6) \qquad \omega_L = \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j} dq^j \wedge dq^i + \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} d\dot{q}^i \wedge dq^j.$$

Recall that the L-extremals i.e. the extremals of the action:

$$(2.7) \qquad S = \int L(q(t), \dot{q}(t))\, dt$$

are the solutions of *Euler-Lagrange equations*([4, p. 160]):

$$(2.8) \qquad E_i(L) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0, \quad i = 1, \ldots, m.$$

If in Euler-Lagrange equations we compute the derivative with respect to time then:

$$(2.9) \qquad \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \ddot{q}^j + \frac{\partial^2 L}{\partial \dot{q}^i \partial q^k} \dot{q}^k - \frac{\partial L}{\partial q^i} = 0.$$

If we wish to write this equations in *the normal form,* that is the second derivatives appear explicitly, it must to consider:

$$(2.10) \qquad g_{ij} = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}$$

which is called *the metric of* L and (2.9) becomes:

$$(2.11) \quad g_{ij}\ddot{q}^j + \frac{\partial^2 L}{\partial \dot{q}^i \partial q^k}\dot{q}^k - \frac{\partial L}{\partial q^i} = 0.$$

Definition 2.1 The Lagrangian L is called *regular* or *nondegenerate* if the metric tensor (g_{ij}) is invertible that is:

$$(2.12) \quad rang\,(g_{ij}) = n$$

If L is regular denote by (g^{ij}) the inverse matrix of (g_{ij}). By multiplication of (2.11) with (g^{ij}) we get:

Proposition 2.2 *If the Lagrangian L is regular then the L-extremals are solutions of equations:*

$$(2.13) \quad \ddot{q}^i + G^i = 0, \quad i = 1, \ldots, m$$

where:

$$(2.14) \quad G^i = g^{ij}\left(\frac{\partial^2 L}{\partial \dot{q}^j \partial q^k}\dot{q}^k - \frac{\partial L}{\partial q^j}\right).$$

But the system (2.13) is exactly the flow system for the vector field $S \in \mathcal{X}\,(TQ)$ with:

$$(2.15) \quad S = \dot{q}^i \frac{\partial}{\partial q^i} - G^i \frac{\partial}{\partial \dot{q}^i}$$

which is called *the canonical semispray of L.*

This semispray generates a *nonlinear connection* on Q that is a distribution N on TQ with([4, p. 107]):

$$(2.16) \quad TTQ = N \oplus V\,(TQ)$$

where $V\,(TQ)$ is *the vertical distribution* of Q. Recall that V has as basis the vector fields $\left(\frac{\partial}{\partial \dot{q}^i}\right)$ and that a basis for N is given by the vector fields $\left(\frac{\delta}{\delta q^i}\right)$ where([4, p. 108]):

$$(2.17) \quad \frac{\delta}{\delta q^i} = \frac{\partial}{\partial q^i} - N_i^j \frac{\partial}{\partial \dot{q}^j}$$

with:

$$(2.18) \quad N_i^j = \frac{1}{2}\frac{\partial G^j}{\partial \dot{q}^i}.$$

Let $(dq^i, \delta\dot{q}^i)$ be the dual basis of $\left(\frac{\delta}{\delta q^i}, \frac{\partial}{\partial \dot{q}^i}\right)$ where:

$$(2.19) \quad \delta\dot{q}^i = d\dot{q}^i + N_j^i dq^j.$$

With respect to this basis one have:

$$(2.20) \quad \omega_L = g_{ij}\delta\dot{q}^i \wedge dq^j$$

or, in matrix notation:

$$(2.21) \quad \omega_L = \begin{pmatrix} 0 & -g_{ij} \\ g_{ij} & 0 \end{pmatrix}.$$

Returning to canonical semispray S one have another characterization, namely([1, p. 347]):

$$(2.22) \quad i_S\omega_L = -dH$$

where i_S denotes the interior product with respect to S and H is *the energy* of L :

$$(2.23) \quad H = \frac{\partial L}{\partial \dot{q}^i}\dot{q}^i - L.$$

This characterization yields a very important and well-known result in the theory of autonomous Lagrangian systems:

Theorem 2.3(Conservation of energy for autonomous Lagrangian systems) *If the Lagrangian L is time-independent then the energy H is a conservation law for L.*

Proof We have:

$$\mathcal{L}_S H = dH(S) = -(i_S\omega_L)(S) = \omega_L(S,S) = 0. \qquad \square$$

A straightforward computation give:

$$\mathcal{L}_S\theta_L = di_S\theta_L + i_S d\theta_L = d < \theta_L, S > + i_S\omega_L = d\left(\frac{\partial L}{\partial \dot{q}^i}\dot{q}^i\right) - dH = d\left(\frac{\partial L}{\partial \dot{q}^i}\dot{q}^i - H\right) = dL$$

and:

$$\mathcal{L}_S\omega_L = \mathcal{L}_S d\theta_L = d\mathcal{L}_S\theta_L = d(dL) = 0$$

that is:

Proposition 2.4([1, p. 348, ex. 56]) *If the Lagrangian L is regular then the Cartan 2-form is invariant for the canonical semispray S.*

Applying the results of section 1 we obtain:

Proposition 2.5 *Let L be a regular Lagrangian and $Y \in \mathcal{X}(TQ)$ be a symmetry of the canonical semispray S. If $Z \in \mathcal{X}(TQ)$ is a Y-pseudosymmetry for S then:*

$$(2.24) \quad \phi = \omega_L(Y,Z)$$

is a conservation law for the Lagrangian L that is a conservation law for Euler-Lagrange equations, equivalently for equations (2.13). *Particularly, if* Y *and* Z *are symmetries for the canonical spray* S *then* ϕ *given by* (2.24) *is conservation law for* L.

If:

$$Y = Y^i \frac{\delta}{\delta q^i} + \tilde{Y}^i \frac{\partial}{\partial \dot{q}^i}$$

$$Z = Z^i \frac{\delta}{\delta q^i} + \tilde{Z}^i \frac{\partial}{\partial \dot{q}^i}$$

then (2.24) becomes:

$$(2.25) \quad \phi = \left(Y^i, \tilde{Y}^i\right) \begin{pmatrix} 0 & -g_{ij} \\ g_{ij} & 0 \end{pmatrix} \begin{pmatrix} Z^j \\ \tilde{Z}^j \end{pmatrix} = g_{ij} \tilde{Y}^i Z^j - g_{ij} Y^i \tilde{Z}^j.$$

Corollary 2.6 *If the Lagrangian* L *is regular and* $Z \in \mathcal{X}(TQ)$ *is pseudosymmetry for the canonical semispray* S *then:*

$$(2.26) \quad \phi = \omega_L(S, Z) = -\mathcal{L}_Z H$$

is a conservation law for the Lagrangian.

Remark that:
(i) If in (2.26) we take $Z = S$ we obtain $\phi = 0$.
(ii)

$$(2.27) \quad S = \dot{q}^i \frac{\partial}{\partial q^i} - G^i \frac{\partial}{\partial \dot{q}^i} = \dot{q}^i \left(\frac{\delta}{\delta q^i} + N_i^k \frac{\partial}{\partial \dot{q}^k}\right) - G^k \frac{\partial}{\partial \dot{q}^k} = \dot{q}^i \frac{\delta}{\delta q^i} + \left(\dot{q}^i N_i^k - G^k\right) \frac{\partial}{\partial \dot{q}^k}$$

and then (2.25) and (2.26) give:

$$(2.28) \quad \phi = g_{ij} \left(\dot{q}^k N_k^i - G^i\right) Z^j - g_{ij} \dot{q}^i \tilde{Z}^j.$$

Definition 2.7 The semispray S is called *spray* if the functions (G^i) are 2-positive homogeneous with respect to velocity that is:

$$(2.29) \quad G^i(q, \lambda \dot{q}) = \lambda^2 G^i(q, \dot{q}), \quad i = 1, \ldots, m, \forall \lambda > 0.$$

Applying (2.18) and Euler Theorem on homogeneous functions we get:
Proposition 2.8 *The semispray is spray if and only if*

$$(2.30) \quad G^k = \dot{q}^i N_i^k.$$

Applying (2.27) we obtain:

Corollary 2.9 *If S is spray then:*

$$S = \dot{q}^i \frac{\delta}{\delta q^i} \tag{2.31}$$

and ϕ given by (2.28) *is:*

$$\phi = g_{ij}\dot{q}^i \tilde{Z}^j. \tag{2.32}$$

Another characterization of sprays is:

Proposition 2.10([4, p. 112]) *A semispray S is spray if and only if:*

$$\left[S, \dot{q}^i \frac{\partial}{\partial \dot{q}^i}\right] = S. \tag{2.33}$$

But relation (2.33) means that $\Upsilon = \dot{q}^i \frac{\partial}{\partial \dot{q}^i}$, usualy called *the Liouville vector field,* is a pseudosymmetry for S and then relation (2.32) yields:

Proposition 2.11 *If the Lagrangian L is regular and S is spray then:*

$$\phi = g_{ij}\dot{q}^i \dot{q}^j \tag{2.34}$$

is a conservation law for L.

The expression $\mathcal{E}(g) = \frac{1}{2}g_{ij}\dot{q}^i\dot{q}^j$ is usually called *the kinetic energy of metric* g_{ij}. Then the proposition 2.11 give conservation of kinetic energy.

A class of Lagrangians which generates sprays is given by:

Proposition 2.12 *If the Lagrangian L is regular and r-positively homogeneous with respect to velocity then G^i are 2-positively homogeneous i.e. S is spray for any r.*

The most important cases are([4]) Finslerian and Riemannian case for which $r = 2$.

But for a r-positively homogeneous Lagrangian L a straightforward computation give:

$$H = (r-1)\,L \tag{2.35}$$

$$r\,(r-1)\,L = 2\mathcal{E}(g) \tag{2.36}$$

that is:

$$rH = 2\mathcal{E}(g) \tag{2.37}$$

and then these relations via the theorem 2.1.3 give another proof of conservation of kinetic energy for r-positively homogeneous Lagrangians with $r \neq 0$.

3 From conservation laws to exact Cartan symmetries. Classical Noether theorem

For the Lagrangian system (M, L) let the following type of symmetry:

Definition 3.1([1, p. 349]) $X \in \mathcal{X}(TM)$ is called *Cartan symmetry for* (M, L) if:

$$(3.1) \quad \mathcal{L}_X \omega_L = 0$$

and:

$$(3.2) \quad \mathcal{L}_X H = 0.$$

Therefore:

Proposition 3.2 *The canonical spray is Cartan symmetry.*

Proposition 3.3([1, p. 349]) *If (M, L) is a Lagrange space and $X \in \mathcal{X}(TM)$ is Cartan symmetry for L then X is symmetry for the canonical spray of L that is:*

$$(3.3) \quad \mathcal{L}_X S = 0.$$

Let $X \in \mathcal{X}(TM)$ be a Cartan symmetry. By (3.1) we have:

$$(3.4) \quad d\mathcal{L}_X \theta_L = 0$$

that is $L_X \theta_L$ is closed.

Definition 3.4([1, p. 349]) If $\mathcal{L}_X \theta_L$ is exact then X is called *exact Cartan symmetry.*

A key result in the theory of conservation laws is:

Proposition 3.5([1, p. 349]; **Generalized Noether theorem**) *If $X \in \mathcal{X}(TM)$ is an exact Cartan symmetry with:*

$$(3.5) \quad \mathcal{L}_X \theta_L = df$$

then:

$$(3.6) \quad P_X = J(X) L - f$$

is a conservation law for L. Conversely if $F \in C^\infty(TM)$ is conservation law for L then $X \in \mathcal{X}(TM)$ uniquely defined by:

$$(3.7) \quad i_X \omega_L = -dF$$

is exact Cartan symmetry.

This result say that there is a bijective correspondence between exact Cartan symmetries and conservation laws for L.

In this section we give the local expression of exact Cartan symmetry X given by (3.7) which we not found in literature. Suppose that $X = X^i \frac{\delta}{\delta q^i} + \tilde{X}^i \frac{\partial}{\partial \dot{q}^i}$. Because:

$$(3.8) \quad dF = \frac{\delta F}{\delta q^i} dq^i + \frac{\partial F}{\partial \dot{q}^i} \delta \dot{q}^i$$

with:

$$(3.9) \quad \frac{\delta F}{\delta q^i} = \frac{\partial F}{\partial q^i} - N_i^j \frac{\partial F}{\partial \dot{q}^j}$$

relation (3.7) becomes:

$$\left(X^i, \tilde{X}^i\right) \begin{pmatrix} 0 & -g_{ij} \\ g_{ij} & 0 \end{pmatrix} = -\left(\frac{\delta F}{\delta q^j}, \frac{\partial F}{\partial \dot{q}^j}\right)$$

which give:

Proposition 3.6 *If $X \in \mathcal{X}(TM)$ is exact Cartan symmetry with conservation law $F \in C^\infty(TM)$ then:*

$$\begin{aligned} (3.10a) \quad X^i &= g^{ij} \frac{\partial F}{\partial \dot{q}^j} \\ (3.10b) \quad \tilde{X}^i &= -g^{ij} \frac{\delta F}{\delta q^j}. \end{aligned}$$

The original Noether theorem ([5]) covered the case in which X is the complete lift of a vector field on M. Let $X \in \mathcal{X}(M)$ and denote by $(\phi_t)_t$ the flow it generates. This flow lifts to a flow $(\psi_t)_t$ on TM given by:

$$(3.11) \qquad \psi_t(q, \dot{q}) = \left(\phi_t(q), (\phi_t)_{*,q}(\dot{q})\right)$$

Definition 3.7 The generator of the flow $(\psi_t)_t$, denoted by X^C, is called *the complete lift of X*.

Definition 3.8 $X \in \mathcal{X}(M)$ is *an invariant of L*(or *L-invariant*) if:

$$(3.12) \quad L \circ \psi_t = L \qquad \forall t$$

Because $(\psi_t)_t$ is generated by the complete lift of X we have:

Proposition 3.9 *$X \in \mathcal{X}(M)$ is invariant of L if and only if:*

$$(3.13) \quad \mathcal{L}_{X^C} L = 0.$$

Then we have:

Proposition 3.10(Characterization of invariant vector fields) *If* $X = X^i \frac{\partial}{\partial q^i}$ *then* X *is invariant of* L *if and only if:*

$$(3.14) \qquad X^k \frac{\partial L}{\partial q^k} + \dot{q}^s \frac{\partial X^k}{\partial q^s} \frac{\partial L}{\partial \dot{q}^k} = 0.$$

Proposition 3.11([1, p. 349]; **E. Noether)** *If* X *is invariant of* L *then* X^C *is an exact Cartan symmetry with:*

$$(3.15) \quad f = 0$$

that is the quantity:

$$(3.16) \qquad P_X = X^i \frac{\partial L}{\partial \dot{q}^i} = X^i p_i$$

is conservation law for L.

The conservation laws obtained with this last result will be called "classical".

4 A nonclassical conservation law generated by symmetries

Let the 2-dimensional istropic harmonic oscillator:

$$\begin{aligned} (4.1a) \quad \ddot{q}^1 + \omega^2 q^1 &= 0 \\ (4.1b) \quad \ddot{q}^2 + \omega^2 q^2 &= 0 \end{aligned}$$

a toy model for many methods to finding conservation laws.

The Lagrangian is:

$$(4.2) \quad L = \frac{1}{2}\left[\left(\dot{q}^1\right)^2 + \left(\dot{q}^2\right)^2\right] - \frac{\omega^2}{2}\left[\left(q^1\right)^2 + \left(q^2\right)^2\right]$$

and then applying the conservation of energy(theorem 2.3) we have two conservation laws:

$$\begin{aligned} (4.3a) \quad \phi_1 &= \left(\dot{q}^1\right)^2 + \omega^2 \left(q^1\right)^2 \\ (4.3b) \quad \phi_2 &= \left(\dot{q}^2\right)^2 + \omega^2 \left(q^2\right)^2. \end{aligned}$$

A straightforward computation give that the unique L-invariant vector field is:

$$(4.4) \quad X = q^2 \frac{\partial}{\partial q^1} - q^1 \frac{\partial}{\partial q^2}$$

and then the associate classical Noetherian conservation law is:

$$(4.5) \quad \phi_3 = P_X = q^2\dot{q}^1 - q^1\dot{q}^2.$$

But we can obtain a nonclassical conservation law with symmetries. The spray of (4.1) is:

$$(4.6) \quad S = \dot{q}^1\frac{\partial}{\partial q^1} + \dot{q}^2\frac{\partial}{\partial q^2} - \omega^2 q^1\frac{\partial}{\partial \dot{q}^1} - \omega^2 q^2\frac{\partial}{\partial \dot{q}^2}$$

and another calculus give that:

$$(4.7) \quad Y = \dot{q}^2\frac{\partial}{\partial q^1} + \dot{q}^1\frac{\partial}{\partial q^2} - \omega^2 q^2\frac{\partial}{\partial \dot{q}^1} - \omega^2 q^1\frac{\partial}{\partial \dot{q}^2}$$

is a symmetry for S. Also, because S is total 1-homogeneous, i.e. with respect to all variables $(q, \dot{q})$ it result that:

$$(4.8) \quad Z = q^1\frac{\partial}{\partial q^1} + q^2\frac{\partial}{\partial q^2} + \dot{q}^1\frac{\partial}{\partial \dot{q}^1} + \dot{q}^2\frac{\partial}{\partial \dot{q}^2}$$

is symmetry for S. We have: $\mathcal{L}_Y H = 0, \quad \mathcal{L}_Z H = 2H$ and then $\phi = \omega_L(S, Y) = 0, \quad \phi = \omega_L(S, Z) = 2H$ i.e. we not have new conservation law applying corollary 2.6. But:

$$(4.9) \quad \phi_4 = \omega_L(Y, Z) = \dot{q}^1\dot{q}^2 + \omega^2 q^1 q^2$$

is a new conservation law given by proposition 2.5. We remark that
(i) ϕ_4 is nonclassical conservation law
(ii) ϕ_4 represent the energy of a new Lagrangian of (4.1), that is:

$$(4.10) \quad L^* = \dot{q}^1\dot{q}^2 - \omega^2 q^1 q^2$$

a result very important from the point of view of Inverse Problem of Analytical Mechanics([6]). Our Lagrangian L^* appear in [6, p. 122].

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