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# RECURRENT METRICS IN THE GEOMETRY OF SECOND ORDER DIFFERENTIAL EQUATIONS

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Abstract. Given a pair (semispray $S$, metric $g$) on a tangent bundle, the family of nonlinear connections $N$ such that $g$ is recurrent with respect to $(S, N)$ with a fixed recurrent factor is determined by using the Obata tensors. In particular, we obtain a characterization for a pair $(N, g)$ to be recurrent as well as for the triple $(S, \overset{c}{N}, g)$, where $\overset{c}{N}$ is the canonical nonlinear connection of the semispray $S$. Also, the Weyl connection of conformal gauge theories is obtained as a particular case.

## 1. **Introduction**

In two cited papers, [5] and [6], Yung-Chow Wong derived several properties of a recurrent tensor field $T$ on a manifold $M$ endowed with a linear connection $\nabla$. Recall that this means the existence of a 1-form $\alpha_T$ on $M$ such that

$$\nabla T = \alpha_T \otimes T, \tag{1.1}$$

where $\nabla$ is the covariant differential of $\nabla$. For $\alpha_T = 0$, we recover the notion of *parallel* or *covariant constant* tensor field.

The aim of our work here is to extend the notion of recurrence to the geometry of systems of second order differential equations on $M$. More precisely, given such a system $S$, on short *semispray*, we can derive a type of differential $\nabla$ if $S$ is considered as a vector field on the tangent bundle $TM$. A main tool in the definition of $\nabla$ is given by a splitting

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of the iterated tangent bundle $T(TM)$ provided by a distribution $N$ on $TM$. Such an object $N$ is called *nonlinear connection*. A remarkable result is that every $S$ yields such a nonlinear connection, $\overset{c}{N}$, indexed by us with $c$ for canonical.

We treat in detail the recurrence of tensor fields $T$ of type $(0,2)$ as generalization of the results from [1], where the metrizability problem is considered; therefore, we can say that we search recurrent metrics for a given system of second order differential equations, the recurrence factor being fixed. In fact, given a pair (semispray $S$, metric $g$) on $TM$, the family of nonlinear connections $N$ such that $g$ is recurrent with respect to $(S,N)$ with a fixed recurrent factor is determined by using the Obata tensors. In particular, we obtain a characterization for a pair $(N,g)$ to be recurrent as well as for the triple $(S,\overset{c}{N},g)$, where $\overset{c}{N}$ is the canonical nonlinear connection of the semispray $S$. In the former case, we arrive at the unique Weyl connection ([4]) associated to a Riemannian metric via recurrence, which is the main object in the theory of Weyl structures initiated by Gerald B. Folland in 1970. A last remark is necessary here: although all our main results are globally expressed, we will work also locally in order to treat the examples and in this way we recover the usual Christoffel process of the Riemannian geometry as well as the Weyl connection.

## 2. Nonlinear connections and semisprays on tangent bundles

Let $M$ be a smooth, $n$-dimensional manifold for which we denote: $C^{\infty}(M)$-the algebra of smooth real functions on $M$, $\mathcal{X}(M)$-the Lie algebra of vector fields on $M$, and $T^r_s(M)$-the $C^{\infty}(M)$-module of tensor fields of $(r,s)$-type on $M$.

A local chart $x=(x^i)=(x^1,...,x^n)$ on $M$ lifts to a local chart on the tangent bundle $TM$, given by: $(x,y)=(x^i,y^i)$. If $\pi: TM \to M$, is the canonical projection, then the kernel of the differential of $\pi$ is an integrable distribution $V(TM)$ with local basis $(\frac{\partial}{\partial y^i})$. An important element of $V(TM)$ is the *Liouville vector field* $\mathbb{C}=y^i\frac{\partial}{\partial y^i}$. $V(TM)$ is called *the vertical distribution* and its elements are *vertical vector fields*.

The tensor field $J\in T^1_1(TM)$ given by $J=\frac{\partial}{\partial y^i}\otimes dx^i$ is called *the tangent structure*. Two of this properties are: the nilpotence $J^2=0$ and $imJ\,(=\ker J)=V(TM)$.

A well-known notion in the tangent bundles geometry is given by the following definition.

**Definition 2.1.** *A supplementary distribution $N$ to the vertical distribution $V(TM)$,*

$$T(TM) = N \oplus V(TM), \tag{2.1}$$

*is called horizontal distribution or nonlinear connection. A vector field belonging to $N$ is called horizontal.*

*A nonlinear connection has a local basis,*

$$\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j}, \tag{2.2}$$

*and the functions $\left(N_j^i(x,y)\right)$ are called the coefficients of $N$. So, a basis of $\mathcal{X}(TM)$ adapted to the decomposition* (1.1) *is $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})$ called the Berwald basis. The dual of the Berwald basis is: $(dx^i, \delta y^i = dy^i + N_j^i dx^j)$.*

A second remarkable structure on $TM$ is provided by following definition.

**Definition 2.2.** *$S \in \mathcal{X}(TM)$ is called semispray if*

$$J(S) = \mathbb{C}. \tag{2.3}$$

*In canonical coordinates,*

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}, \tag{2.4}$$

*and the functions $\left(G^i(x,y)\right)$ are the coefficients of $S$. The flow of $S$ is a system of second order differential equations: $\frac{d^2x^i}{dt^2} = 2G^i(x, \frac{dx}{dt})$.*

*An important result is that a nonlinear connection $N = (N_j^i)$ yields an unique horizontal semispray denoted $S(N)$ with*

$$G^i = \frac{1}{2} N_j^i y^j. \tag{2.5}$$

*In other words,*

$$S(N) = y^i \frac{\delta}{\delta x^i}. \tag{2.6}$$

*The converse of this result is that a semispray $S$ yields a nonlinear connection $\overset{c}{N}$ given by*

$$\overset{c}{N}{}_j^i = \frac{\partial G^i}{\partial y^j}. \tag{2.7}$$

**Definition 2.3.** *A semispray $S$ for which the coefficients $\left(G^i\right)$ are homogeneous of degree 2 with respect to the variables $\left(y^i\right)$ is called a spray.*

Locally, this means, via Euler theorem:

$$2G^i = y^j \frac{\partial G^i}{\partial y^j}, \tag{2.8}$$

and then $\overset{c}{N}$ is 1-homogeneous:

$$\overset{c}{N}{}^i_j = y^a \frac{\partial \overset{c}{N}{}^i_j}{\partial y^a}, \tag{2.9}$$

which yields that $S$ is horizontal with respect to $\overset{c}{N}$, i.e., $S$ has the expression (1.6).

## 3. Recurrence metrics

3.1. **The general problem of recurrent triples.** Let us fix a semispray $S = \left(G^i\right)$ and a nonlinear connection $N = \left(N^i_j\right)$. Following [1, p. 337] consider the following definition.

**Definition 3.1.** *The dynamical derivative associated to the pair* $(S, N)$ *is the map*
$\overset{SN}{\nabla} : V(TM) \to V(TM)$ *given by:*

$$\overset{SN}{\nabla} X = \overset{SN}{\nabla}\left(X^i \frac{\partial}{\partial y^i}\right) := \left(S\left(X^i\right) + N^i_j X^j\right) \frac{\partial}{\partial y^i}, \tag{3.1}$$

*with properties:*
*(I)* $\overset{SN}{\nabla}\left(\frac{\partial}{\partial y^i}\right) = N^j_i \frac{\partial}{\partial y^j}$,
*(II)* $\overset{SN}{\nabla}(X+Y) = \overset{SN}{\nabla} X + \overset{SN}{\nabla} Y$,
*(III)* $\overset{SN}{\nabla}(fX) = S(f)X + f \overset{SN}{\nabla} X$.

It is straightforward to extend the action of $\overset{SN}{\nabla}$ to general vertical tensor fields by requirement of preserving the tensor product. Moreover, we extend $\overset{SN}{\nabla}$ to a special class of tensor fields.

**Definition 3.2.** *A d-tensor field (d for distinguished) on* $TM$ *is a tensor field whose change of components, under a change of canonical coordinates* $(x, y) \to (\widetilde{x}, \widetilde{y})$ *on* $TM$*, involves only factors of type* $\frac{\partial \widetilde{x}}{\partial x}$ *and (or)* $\frac{\partial x}{\partial \widetilde{x}}$.

**Example 3.3.** *(i)* $\left(\frac{\delta}{\delta x^i}\right)$ *and* $\left(\frac{\partial}{\partial y^i}\right)$ *are components of d-tensor fields of* $(1,0)$*-type.*
*(Iii)* $\left(dx^i\right)$ *and* $\left(\delta y^i\right)$ *are components of d-tensor fields of* $(0,1)$*-type.*
*(ii)* $\left(G^i\right)$ *are not components of a d-tensor field, since a change of coordinates implies:*

$$2\widetilde{G}^i = 2\frac{\partial \widetilde{x}^i}{\partial x^j}G^j - \frac{\partial \widetilde{y}^i}{\partial x^j}y^j,$$

*but it results that given two semisprays* $\overset{1}{S}$ *and* $\overset{2}{S}$ *their difference* $X = \overset{2}{S} - \overset{1}{S}$ *is a vertical (and then d-) vector field. Then, the set of semisprays is an* $C^\infty(TM)$*-affine module, associated to the* $C^\infty(TM)$*-linear module* $\mathcal{X}^v(TM)$ *of vertical vector fields.*
*(iv)* $\left(N_j^i\right)$ *are not components of a d-tensor field, since a change of coordinates implies:*

$$\frac{\partial \widetilde{x}^j}{\partial x^k}N_i^k = \widetilde{N}_k^j\frac{\partial \widetilde{x}^k}{\partial x^i} + \frac{\partial \widetilde{y}^j}{\partial x^i}.$$

*It follows that given two nonlinear connections* $\overset{1}{N}$ *and* $\overset{2}{N}$*, their difference* $F = \overset{2}{N} - \overset{1}{N} = \left(F_j^i = \overset{2}{N}{}_j^i - \overset{1}{N}{}_j^i\right)$ *is a d-tensor field of* $(1,1)$*-type. In other words, the set* $\mathcal{N}(TM)$ *of nonlinear connections is an* $C^\infty(TM)$*-affine module, associated to the* $C^\infty(TM)$*-linear module* $T_1^1(TM)$*.*

**Definition 3.4.** *A metric* $g$ *on* $TM$ *is a d-tensor field of* $(0,2)$*-type which is symmetric and non-degenerated.*

*For the components* $g_{ij} = g(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j})$ *the following properties hold:*
(*1*) (*symmetry*) $g_{ij} = g_{ji}$,
(*2*) (*non-degeneracy*) $\det(g_{ij}) > 0$*; then, there exists the d-tensor field of* $(2,0)$*-type* $g^{-1} = (g^{ij})$.

The name is justified from the fact that $g_{ij}dx^i \otimes dx^j + g_{ij}\delta y^i \otimes \delta y^j$ is a Riemannian metric on $TM$ for which $N$ and $V(TM)$ are orthogonal distributions.

Therefore, the *dynamical derivative of metric* $g$ with respect to the pair $(S, N)$ is:
$\overset{SN}{\nabla} g : V(TM) \times V(TM) \to V(TM)$, given by:

$$\overset{SN}{\nabla} g(X,Y) = S(g(X,Y)) - g(\overset{SN}{\nabla} X, Y) - g(X, \overset{SN}{\nabla} Y). \tag{3.2}$$

The main notion of this subsection is given next

**Definition 3.5.** *Let $\alpha \in C^{\infty}(TM)$. The metric $g$ is called $\alpha$-recurrent with respect to the pair $(S, N)$ if*

$$\overset{SN}{\nabla} g = \alpha g. \tag{3.3}$$

*Also, the triple $(S, N, g)$ is called an $\alpha$-recurrent structure.*

The aim of this section is to find all nonlinear connections which together with fixed $(S, g)$ form an $\alpha$-recurrent structure. In order to address this a look at example 2.3 (iv) necessitates a study of two operators denoted by $O$ and $\overset{*}{O}$, and called *Obata*, in the following, acting on the space of d-tensor fields of $(1,1)$-type:

$$O^{ij}_{kl} = \frac{1}{2}\left(\delta^i_k\delta^j_l - g^{ij}g_{kl}\right), \quad \overset{*}{O}{}^{ij}_{kl} = \frac{1}{2}\left(\delta^i_k\delta^j_l + g^{ij}g_{kl}\right). \tag{3.4}$$

The Obata operators are supplementary projectors:

$$O^{ia}_{bj}\overset{*}{O}{}^{bk}_{la} = \overset{*}{O}{}^{ia}_{bj}O^{bk}_{la} = 0, \qquad O^{ia}_{bj}O^{bk}_{la} = O^{ik}_{lj}, \qquad \overset{*}{O}{}^{ia}_{bj}\overset{*}{O}{}^{bk}_{la} = \overset{*}{O}{}^{ik}_{lj}, \tag{3.5}$$

and then tensorial equations involving these operators has solutions as follows. The system of equations

$$\overset{*}{O}{}^{ia}_{bj}\left(X^b_a\right) = A^i_j, \qquad \left(O^{ia}_{bj}(X^b_a) = A^i_j\right) \tag{3.6}$$

with $X$ as unknown has solutions if and only if

$$O^{ia}_{bj}\left(A^b_a\right) = 0, \qquad \left(\overset{*}{O}{}^{ia}_{bj}\left(A^b_a\right) = 0\right), \tag{3.7}$$

and then, the general solution is:

$$X^i_j = A^i_j + O^{ia}_{bj}\left(Y^b_a\right), \qquad \left(X^i_j = A^i_j + \overset{*}{O}{}^{ia}_{bj}\left(Y^b_a\right)\right) \tag{3.8}$$

with $Y$ an arbitrary d-tensor field of $(1,1)$-type.

We are ready for one of the main results of paper:

**Theorem 3.6.** *Set $S$ and $g$. The family $\mathcal{N}(S, g, \alpha)$ of all nonlinear connections $N$ such that $(S, N, g)$ is $\alpha$-recurrent is given by*

$$N = \overset{c}{N} + \frac{1}{2}g^{-1}(\overset{S}{\nabla} g) - \frac{\alpha}{2}I + O(X), \tag{3.9}$$

*where $X$ is an arbitrary d-tensor field of $(1,1)$-type, $\overset{S}{\nabla}$ is the dynamical derivative with respect to the pair $(S, \overset{c}{N})$ and $I$ is the Kronecker tensor.*

*Therefore, $\mathcal{N}(S,g,\alpha)$ is an affine submodule of $\mathcal{N}(TM)$ passing to the nonlinear connection $\overset{c}{N} + \frac{1}{2}g^{-1}(\overset{S}{\nabla} g) - \frac{\alpha}{2}I$ and having the direction given by the linear submodule $ImO$ of $T_1^1(TM)$.*

*Proof.* We search $N$ of the form

$$N = \overset{c}{N} + F \tag{3.10}$$

with $F = \left(F_j^i\right)$ a d-tensor field of $(1,1)$-type to be determined. The local expression of equation (2.3) is:

$$S\left(g_{uv}\right) - g_{um}N_v^m - g_{mv}N_u^m = \alpha g_{uv}, \tag{3.11}$$

and inserting (2.10) in (2.11) gives:

$$(\overset{S}{\nabla} g)_{uv} = g_{um}F_v^m + g_{mv}F_u^m + \alpha g_{uv}.$$

Multiplying the last relation with $g^{ku}$, we get

$$g^{ku}(\overset{S}{\nabla} g)_{uv} - \alpha\delta_v^k = F_v^k + g^{ku}g_{mv}F_u^m = 2\overset{*}{O}{}_{av}^{kb}\left(F_b^a\right), \tag{3.12}$$

which means:

$$\overset{*}{O}(F) = \frac{1}{2}g^{-1}(\overset{S}{\nabla} g) - \frac{\alpha}{2}I.$$

Let us search for the condition (2.7):

$$O_{av}^{kb}\left(g^{am}S\left(g_{mb}\right) - \overset{c}{N}{}_b^a - g^{am}g_{bl}\overset{c}{N}{}_m^l - \alpha\delta_b^a\right) =$$

$$= g^{km}S\left(g_{mv}\right) - \overset{c}{N}{}_v^k - g^{km}g_{vl}\overset{c}{N}{}_m^l - g^{km}S\left(g_{mv}\right) + g^{km}g_{vl}\overset{c}{N}{}_m^l + \overset{c}{N}{}_v^k = 0.$$

It follows that

$$F = \frac{1}{2}g^{-1}(\overset{S}{\nabla} g) - \frac{\alpha}{2}I + O(X),$$

and returning to (2.10), we have the result. □

A more detailed formula for (2.9) is:

$$N_j^i = \tfrac{1}{2}\overset{c}{N}{}_j^i - \tfrac{1}{2}g^{ia}g_{jb}\overset{c}{N}{}_a^b + \tfrac{1}{2}g^{ia}S\left(g_{aj}\right) - \tfrac{\alpha}{2}\delta_j^i + O_{bj}^{ia}\left(X_a^b\right). \tag{3.13}$$

In the spray case, the above equation admits a simplification as follows.

**Corollary 3.7.** *Fix a spray $S$ and a metric $g$. The family $\mathcal{N}(S,g,\alpha)$ is:*

$$N_j^i = \frac{1}{2}\overset{c}{N}{}_j^i - \frac{1}{2}g^{ia}g_{jb}\overset{c}{N}{}_a^b + \frac{1}{2}g^{ia}y^m\frac{\delta g_{aj}}{\delta x^m} - \frac{\alpha}{2}\delta_j^i + O_{bj}^{ia}\left(X_a^b\right). \tag{3.14}$$

3.2. **Recurrence of a pair (nonlinear connection, metric).** Fix a nonlinear connection $N = \left(N^i_j\right)$ and associate to $N$, the semispray $S(N)$.

**Definition 3.8.** *The pair $(N, g)$ is $\alpha$-recurrent if the triple $(S(N), N, g)$ is so.*

We have

$$\overset{c}{N} = \frac{1}{2}N + \frac{1}{2}F_N, \tag{3.15}$$

with

$$(F_N)^i_j = \frac{\partial N^i_a}{\partial y^j} y^a. \tag{3.16}$$

Then, we have the following result.

**Corollary 3.9.** *The pair $(N, g)$ is $\alpha$-recurrent if and only if*

$$\overset{*}{O}\left(N - F_N - g^{-1}(\overset{S(N)}{\nabla} g)\right) = -\alpha I. \tag{3.17}$$

*Proof.* From (3.9) and (3.13) it results that $(N, g)$ is $\alpha$-recurrent if and only if:

$$N = \frac{1}{2}N + \frac{1}{2}F_N + \frac{1}{2}g^{-1}(\overset{S(N)}{\nabla} g) - \frac{\alpha}{2}I + O(X),$$

which means

$$N - F_N - g^{-1}(\overset{S(N)}{\nabla} g) + \alpha I = O(2X),$$

and from $\overset{*}{O}(I) = I$, we get the result. □

**Example 3.10. Riemannian metrics.** *Let $g = g(x)$ be a Riemannian metric on $M$. Recall that a symmetric linear connection on $M$ with coefficients $\left(\Gamma^i_{jk}(x)\right)$ yields the nonlinear connection with the coefficients*

$$N^i_j = \Gamma^i_{ja} y^a. \tag{3.18}$$

*Then the associated semispray $S(N)$ is a spray:*

$$G^i = \frac{1}{2}\Gamma^i_{jk} y^j y^k, \tag{3.19}$$

*and $F_N = N$. The condition (3.17) becomes:*

$$\overset{*}{O}\left(g^{-1}(\overset{S(N)}{\nabla} g)\right) = \alpha I. \tag{3.20}$$

*This means:*

$$\left(g^{im}\frac{\partial g_{mj}}{\partial x^a}-\Gamma^i_{ja}-g^{iu}g_{jv}\Gamma^v_{ua}\right)y^a=\alpha\delta^i_j. \tag{3.21}$$

*But multiplying the last equation with $g_{ik}$ we arrive at:*

$$\left(\frac{\partial g_{jk}}{\partial x^a}-g_{ki}\Gamma^i_{ja}-g_{ji}\Gamma^i_{ka}\right)y^a=\alpha g_{jk} \tag{3.22}$$

*which is the usual Christoffel process replaced in the recurrent framework on $TM$. So, we verified the condition* (3.17) *in the Riemannian setting. Let us point out the rôle of homogeneity of the spray* (3.19)*. We remark from* (3.22) *that $\alpha$ must be a* 1*-homogeneous on $y$, i.e. $\alpha(x,y)=\alpha_a(x)y^a$, and then we have:*

$$\frac{\partial g_{jk}}{\partial x^a}-g_{ki}\Gamma^i_{ja}-g_{ji}\Gamma^i_{ka}=\alpha_a g_{jk} \tag{3.23}$$

*for all $a,j,k\in\{1,...,n\}$. By considering the* 1*-form $\alpha_g=\alpha_a(x)dx^i$ we recover the starting formula (1.1) from Introduction for $T=g$.*
*It is well known that the solution of* (2.21) *is the unique Weyl connection* [3, p. 147]*:*

$$\Gamma^i_{jk}=\overset{c}{\Gamma}{}^i_{ij}+\tfrac{1}{2}\left(\alpha^i g_{jk}-\delta^i_j\alpha_k-\delta^i_k\alpha_j\right) \tag{3.23W}$$

*where $\overset{c}{\Gamma}$ is the Levi-Civita connection of $g$ and $\alpha^i=g^{ia}\alpha_a$ is the $g$-contravariant version of $\alpha$.*

3.3. **Recurrence of a pair (semispray, metric).** Let us fix a semispray $S=\left(G^i\right)$ and a metric $g$.

**Definition 3.11.** *The pair $(S,g)$ is called $\alpha$-recurrent if the triple $\left(S,\overset{c}{N},g\right)$ is so.*

Inserting $\overset{c}{N}$ in the left-hand-side of (2.13), we get the following result.

**Corollary 3.12.** *i) The pair $(S,g)$ is $\alpha$-recurrent if and only if:*

$$\overset{*}{O}\left(g^{-1}\cdot\overset{S}{\nabla}g\right)=\alpha I, \tag{3.24}$$

*which means locally*

$$\frac{\partial G^i}{\partial y^j}+g^{ia}g_{jb}\frac{\partial G^b}{\partial y^a}-g^{ia}S(g_{aj})=-\alpha\delta^i_j, \tag{3.25}$$

*for all* $i, j \in \{1, \ldots, n\}$.
*(ii) The spray* $S$ *makes* $\alpha$*-recurrent the metric* $g$ *if and only if*

$$\frac{\partial G^i}{\partial y^j} + g^{ia} g_{jb} \frac{\partial G^b}{\partial y^a} - g^{ia} y^m \frac{\delta g_{aj}}{\delta x^m} = -\alpha \delta^i_j, \tag{3.26}$$

*for all* $i, j \in \{1, \ldots, n\}$.

*Proof.* From (3.9), we must have

$$\overset{*}{O} (g^{-1} \overset{S}{\nabla} g - \alpha I) = 0,$$

which gives (3.24). □

**Example 3.13. Euclidean metrics**. *Let us consider the tangent bundle* $TM$ *with* $g$ *a constant metric, i.e.,* $g_{ij}$ *does not depend on* $(x, y)$. *The condition* (3.25) *is:*

$$\overset{c}{N}{}^i_j + g^{ia} g_{jb} \overset{c}{N}{}^b_a = -\alpha \delta^i_j. \tag{3.27}$$

*If* $M = \mathbb{R}^n$ *and* $g$ *is the usual Euclidean metric, then* (3.27) *reads:*

$$\overset{c}{N}{}^i_j + \overset{c}{N}{}^j_i = -\alpha \delta^i_j. \tag{3.28}$$

*Let* $G$ *be the structural Lie group of the recurrent pair* $(S, g)$ *seen as a* $G$*-structure. Then, the last formula says that the Lie algebra of* $G$ *is* $L(G) = -\alpha \cdot o(n)$, *with* $o(n)$ *being the Lie algebra of skew-symmetric matrices which is the Lie algebra of the Euclidean geometry; in other words,* $G$ *is the conformal group* $CO(n)$. *This way, a verification of* (3.24) *is at hand.*

**Remark 3.14.** $(i)$ *For* $\alpha = 0$, *the general formula* (3.9) *first appeared in* [1, p. 339] *and also in* [2, p. 172], *while the results of subsections 3.2 and 3.3 are generalizations of those in* [2].
$(ii)$ *Since we recovered the Weyl connection of conformal gauge theory, it is investigate to what conformal transformation* $g \to \lambda g$ *the class* $\mathcal{N}(S, g, \alpha)$ *is invariant, with* $\lambda$ *a strictly positive function on* $TM$. *From* (3.13), *it results that* $\mathcal{N}(S, \lambda g, \alpha) = \mathcal{N}(S, g, \alpha)$, *for every strictly positive first integral of the semispray* $S$, *i.e.,* $S(\lambda) = 0$.

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