International Journal of Geometric Methods in Modern Physics
Vol. 9, No. 4 (2012) 1220005 (7 pages)

DOI: 10.1142/S0219887812200058

DIRAC STRUCTURES FROM LIE INTEGRABILITY

MIRCEA CRASMAREANU

Faculty of Mathematics
University "Al. I. Cuza"
Iaşi 700506, România
mcrasm@uaic.ro

Received 21 June 2011
Accepted 9 November 2011
Published 4 April 2012

We prove that a pair (F = vector sub-bundle of TM, its annihilator) yields an almost Dirac structure which is Dirac if and only if F is Lie integrable. Then a flat Ehresmann connection on a fiber bundle ξ yields two complementary, but not orthogonally, Dirac structures on the total space M of ξ. These Dirac structures are also Lagrangian sub-bundles with respect to the natural almost symplectic structure of the big tangent bundle of M. The tangent bundle in Riemannian geometry is discussed as particular case and the 3-dimensional Heisenberg space is illustrated as example. More generally, we study the Bianchi–Cartan–Vranceanu metrics and their Hopf bundles.

Keywords: Big tangent bundle; Courant bracket; (almost) Dirac structure; Lie integrability.

Mathematics Subject Classification 2010: 53C60, 53C05, 53C15, 53C20, 53C55, 53D18

Fix M a smooth and connected $m \geq 2$-dimensional manifold with the tangent bundle TM and cotangent bundle T^*M. The $C^\infty(M)$-module of sections of these bundles are: $\mathcal{X}(M) = \Gamma(TM)$ the Lie algebra of vector fields on M and $\Omega^1(M) = \Gamma(T^*M)$ the real linear space of 1-forms on M.

The manifold $T^{\text{big}}M := TM \oplus T^*M$ is called *big tangent bundle* in [8] and *the Pontryagin bundle* of M in [5]; for simplicity we will use the first name. This manifold is the total space of a vector bundle $\pi : T^{\text{big}}M \to TM$ with π the first projection and $\Gamma(T^{\text{big}}M) = \mathcal{X}(M) \otimes \Omega^1(M)$ is endowed with *the Courant structure* $(\langle\, , \rangle, [\, ,])$, [2]:

(1) the (neutral) inner product:

$$\langle (X, \alpha), (Y, \beta) \rangle = \frac{1}{2}(\beta(X) + \alpha(Y)), \tag{1}$$

(2) the (skew-symmetric) Courant bracket:

$$[(X,\alpha),(Y,\beta)] = \left([X,Y], \mathcal{L}_X\beta - \mathcal{L}_Y\alpha - \frac{1}{2}d(\beta(X)-\alpha(Y))\right), \tag{2}$$

where $\mathcal{L}_X$ is the Lie derivative with respect to the vector field $X \in \mathcal{X}(M)$.

Definition 1 ([2]). A vector sub-bundle L of $T^{\mathrm{big}}M$ which is $\langle\, ,\, \rangle$-isotropic and of maximal rank m is called *almost Dirac structure* on M. If, in addition, the $C^\infty(M)$-module $\Gamma(L)$ of sections of L is $[\, ,\,]$-integrable then L is *a Dirac structure.*

Two associated notions are:

(1) a pair of complementary Dirac subspaces is called *a reflector* ([4, 8]) or *Manin triple* in [6],
(2) if L is a Dirac structure on M then the triple $(L, [,]|_L, pr : L \to TM)$ is *a Lie algebroid* on M ([2, Theorem 2.3.4]).

Fix F a vector sub-bundle of TM and denote by F^0 its annihilator, that means the vector sub-bundle of T^*M which vanish on F. Then $L_F = F \oplus F^0$ is a vector sub-bundle of rank m of $T^{\mathrm{big}}M$.

The main result of this short note is a version of [8, Example 2.3].

Theorem 2. *L_F is an almost Dirac structure which is Dirac on M if and only if F is Lie integrable.*

Proof. L_F is maximally isotropic with respect to the inner product (1). Suppose now that F is Lie integrable; then it defines a foliation $\mathcal{F}$ of M. The elements of $\Gamma(F^0)$ are 1-forms which vanish by restriction to each leaf of $\mathcal{F}$. Also, their exterior derivative induce zero 2-forms on the leaves of $\mathcal{F}$. Using these facts, it results that the Courant bracket of any two sections of L_F is:

$$[(X,\alpha),(Y,\beta)] = ([X,Y],0),$$

which yields the conclusion about the Dirac structure of L_F.

Conversely, if L_F is a Dirac structure from the Lie algebroid structure of L_F we get that $pr(L_F)$ is an integrable generalized distribution in the Stefan–Sussman sense. But here $pr(L_F)$ being exactly F is a true vector sub-bundle of TM i.e. of constant rank and then the Stefan–Sussman integrability is exactly the Lie-Frobenius one. □

Example 3 (Ehresmann connection). Let $\xi = (M, \pi, B, F)$ be a fiber bundle; we use the approach of [1, p. 105]. Let n be the dimension of B and $x = (x^i; 1 \leq i \leq n)$, respectively $y = (y^\alpha; 1 \leq \alpha \leq m-n)$ the coordinates on B respectively F. Then, $(x,y) = (x^i, y^\alpha)$ are the coordinates on M. Applying the tangent functor to the bundle ξ we get $T\pi : TM \to TB$ and then the kernel of $T\pi$ is a vector bundle over M denoted $T_{V,\pi}M$ and called *the vertical bundle of* ξ. Its space of sections $\Gamma(T_{V,\pi}M)$ is spanned by $\{\frac{\partial}{\partial y^\alpha}\}$.

Let an *Ehresmann connection* E on ξ with the horizontal vector sub-bundle $T_{H,\pi}M$ over M, complementary to $T_{V,\pi}M$ in TM, [1, p. 107]. It results the existence of smooth functions $\{N^{\alpha}_{i}(x,y)\}$ on M, called *the coefficients of connection*, such that $\Gamma(T_{H,\pi}M)$ is spanned by $\{\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^{\alpha}_{i}\frac{\partial}{\partial y^{\alpha}}\}$.

Let $T^{*}_{V,\pi}M$ be the annihilator of $F_1 = T_{V,\pi}M$; then $\Gamma(T^{*}_{V,\pi}M)$ is spanned by $\{dx^i\}$. Similarly, let $T^{*}_{H,\pi}M$ be the annihilator of $F_2 = T_{H,\pi}M$; it results $\Gamma(T^{*}_{H,\pi}M)$ is spanned by $\{\delta y^{\alpha} = dy^{\alpha} + N^{\alpha}_{i}dx^i\}$. Our main objects in the following are the vector bundles over $M: L_{F_1} = V_D(\pi, E) = (T_{V,\pi}M, T^{*}_{V,\pi}M)$, respectively $L_{F_2} = H_D(\pi, E) = (T_{H,\pi}M, T^{*}_{H,\pi}M)$. The rank of both is m.

We compute:

$$\mathcal{L}_{\frac{\delta}{\delta x^i}}\delta y^{\alpha} = R^{\alpha}_{ik}dx^k - \frac{\partial N^{\alpha}_{i}}{\partial y^{\rho}}\delta y^{\rho}, \quad \mathcal{L}_{\frac{\partial}{\partial y^{\alpha}}}dx^j = 0, \tag{3}$$

where R is *the curvature* of the Ehresmann connection E:

$$R^{\alpha}_{ij} = \frac{\delta N^{\alpha}_{i}}{\delta x^j} - \frac{\delta N^{\alpha}_{j}}{\delta x^i} \tag{4}$$

provided by the condition of Lie integrability of the distribution $\Gamma(T_{H,\pi}M)$:

$$\left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right] = R^{\alpha}_{ij}\frac{\partial}{\partial y^{\alpha}}. \tag{5}$$

Since $T_{V,\pi}M$ is integrable it results:

Corollary 4. *$V_D(\pi, E)$ is a Dirac structure on M. $H_D(\pi, E)$ is an almost Dirac structure which is Dirac if and only if the connection E is flat.*

It results that a flat connection yields a Manin triple:

$$T^{\mathrm{big}}M = V_D(\pi, E) \oplus H_D(\pi, E) \tag{6}$$

in analogy with the decomposition of the tangent bundle provided by the Ehresmann connection:

$$TM = T_{V,\pi}M \oplus T_{H,\pi}M. \tag{7}$$

From:

$$\left\langle \left(\frac{\partial}{\partial y^{\alpha}}, dx^i\right), \left(\frac{\delta}{\delta x^j}, \delta y^{\beta}\right)\right\rangle = \frac{1}{2}(\delta^i_j + \delta^{\beta}_{\alpha}) \tag{8}$$

we get that the decomposition (6) is not $\langle\, ,\rangle$-orthogonally. Therefore, we obtain two Lie algebroids over M provided by a flat connection.

On this example we will treat other two main constructions of [2]. Firstly, let $\pi^* : T^{\mathrm{big}}TM \to T^*M$ be the canonical second projection. Relation $(2.2.3-4)$ of [2, p. 645] states that for a general almost Dirac structure we have: $\pi(L) = L \cap T^*M$ and $\pi^*(L) = (L \cap TM)^0$ and a skew-symmetric map $\Omega : \pi(L) \to \pi^*(L)$ is derived with the formula: $\Omega(\pi(u)) = \pi^*(u)|_{\pi(L)}$. In our framework we get: $\pi(L_F) = F$ and

$\pi^*(L_F) = F^0$. Also, we have:

(1) $\Omega_V : T_{V,\pi}M \to T^*_{V,\pi}M$ is $\Omega(\frac{\partial}{\partial y^\alpha}|_{(x,y)}) = dx^i|_{(x,y)}$,
(2) $\Omega_H : T_{H,\pi}M \to T^*_{H,\pi}M$ is $\Omega(\frac{\delta}{\delta x^i}|_{(x,y)}) = \delta y^\alpha|_{(x,y)}$,

for every $(x,y) \in M$ since as it is pointed out several times in [2] in the manifold case we need to work pointwise.

Secondly, a function $f \in C^\infty(M)$ is called *admissible* ([2, p. 650]) or *Hamiltonian* ([8]) if there exists a vector field $X_f \in \mathcal{X}(M)$ such that $(X_f, df) \in \Gamma(L)$ and then X_f is called *the Hamiltonian vector field* of f and a bracket on admissible functions is given by $\{f,g\} = X_f(g)$. So, for $T_{V,\pi}M$ we get that the coordinate functions x^i are admissible functions and the bracket is trivial:

$$\{x^i, x^j\} = \frac{\partial x^j}{\partial y^\alpha} = 0, \tag{9}$$

where $\frac{\partial}{\partial y^\alpha}$ is the Hamiltonian vector field of $f = x^i$ since $(\frac{\partial}{\partial y^\alpha}, dx^i) \in \Gamma(T_{V,\pi}M)$. $T_{H,\pi}M$ has $\{y^\alpha\}$ as admissible functions only if the Ehresmann connection has vanishing coefficients, i.e. $N^\alpha_i = 0$, in which case $\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i}$ are the corresponding Hamiltonian vector fields and again the bracket is trivial.

Particular Case 5 (Riemannian geometry). Suppose that ξ is exactly the tangent bundle of B i.e. $M = TB$. Then $m = 2n$, $F = \mathbb{R}^n$ and $\alpha = k$. Every Riemannian metric g on B gives rise to the Levi-Civita connection E where $N^k_i = \Gamma^k_{ia}(x)y^a$, with (Γ^k_{ij}) being the Christoffel symbols of g. Then, a flat Riemannian metric on B yields two complementary Dirac structures on TB.

Let us remark that some classes of Dirac structures naturally associated to Lagrangian systems appear in [11].

Remark 6 (The almost symplectic point of view). Let us recall that the big tangent bundle has also a non-degenerate skew-symmetric 2-form, [8]:

$$\Omega((X,\alpha),(Y,\beta)) = \frac{1}{2}(\beta(X) - \alpha(Y)). \tag{10}$$

From:

$$\Omega\left(\left(\frac{\partial}{\partial y^\alpha}, dx^i\right), \left(\frac{\partial}{\partial y^\beta}, dx^j\right)\right) = 0, \quad \Omega\left(\left(\frac{\delta}{\delta x^i}, \delta y^\alpha\right), \left(\frac{\delta}{\delta x^j}, \delta y^\beta\right)\right) = 0 \tag{11}$$

we get that the complementary Dirac structures $V_D(\pi, E)$ and $H_D(\pi, E)$ are Ω-Lagrangian sub-bundles of $T^{\rm big}M$.

Example 7. Let H_3 be the 3-dimensional *Heisenberg space*, [3]; recall that $H_3 = (\mathbb{R}^3, g)$ with the Riemannian metric:

$$g = dx^2 + dy^2 + \left(dz + \frac{y}{2}dx - \frac{x}{2}dy\right)^2. \tag{12}$$

An orthonormal basis of left-invariant vector fields on this Lie group is:

$$X_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}, \quad X_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}, \quad X_3 = \frac{\partial}{\partial z} \tag{13}$$

with the dual frame:

$$\theta_1 = dx, \quad \theta_2 = dy, \quad \theta_3 = dz + \frac{y}{2}dx - \frac{x}{2}dy. \tag{14}$$

Let (x, y, z, X, Y, Z) be the coordinates on TH_3 and (x, y, z, p_x, p_y, p_z) the coordinates on T^*H_3; then $T^{\text{big}}H_3$ has the coordinates $(x, y, z, X, Y, Z, p_x, p_y, p_z)$. Let the locally trivial fiber bundle $\pi : M = H_3 \to B = \mathbb{R}^2$, $\pi(x, y, z) = (x, y)$, then $m = 3$ and $n = 2$. Fix the connection $E = E_g$ such that $T_{H,\pi}H_3$ is the g-orthogonal complement of $T_{V,\pi}H_3$.

Then $\Gamma(T_{V,\pi}H_3)$ is spanned by X_3 while $\Gamma(T^*_{V,\pi}H_3)$ is spanned by $\{\theta_1, \theta_2\}$; $\Gamma(T_{H,\pi}H_3)$ is spanned by $\{\frac{\delta}{\delta x} = X_1, \frac{\delta}{\delta y} = X_2\}$ and $\Gamma(T^*_{H,\pi}H_3)$ is spanned by $\{\theta_3\}$. We get the coefficients of the connection:

$$N^1_1 = \frac{y}{2}, \quad N^1_2 = -\frac{x}{2} \tag{15}$$

and therefore the curvature is:

$$R^1_{12} = -R^1_{21} = 1. \tag{16}$$

(Of course, we know that g is not a flat metric but we include these computations in order to illustrate completely this example.) In conclusion, the only Dirac structure provided by Corollary 4 on H_3 is:

$$V_D(\pi, E_g) = \{(x, y, z, 0, 0, Z, p_x, p_y, 0) \in T^{\text{big}}H_3\} \tag{17}$$

while:

$$H_D(\pi, E_g) = \left\{\left(x, y, z, X - \frac{y}{2}Z, Y + \frac{x}{2}Z, 0, 0, 0, p_z + \frac{y}{2}p_x - \frac{x}{2}p_y\right) \in T^{\text{big}}H_3\right\} \tag{18}$$

is an almost Dirac structure on H_3.

Example 8. Fix k and τ two real numbers and denotes by M^3_k the manifold $\{(x, y, z) \in \mathbb{R}^3; F(x, y, z) = 1 + k(x^2 + y^2) > 0\}$. We shall consider on M^3_k *the Bianchi–Cartan–Vranceanu metric*, [9, p. 343]:

$$g_{k,\tau} = \frac{1}{F^2}dx^2 + \frac{1}{F^2}dy^2 + \left(dz + \frac{\tau y}{2F}dx - \frac{\tau x}{2F}dy\right)^2. \tag{19}$$

For $k = 0$ and $\tau = 1$ we get the previous example; for other remarkable examples as well as the history of these metrics see [9].

An orthonormal basis in $\mathcal{X}(M^3_k)$ is:

$$X_1 = F\frac{\partial}{\partial x} - \frac{\tau y}{2}\frac{\partial}{\partial z}, \quad X_2 = F\frac{\partial}{\partial y} + \frac{\tau x}{2}\frac{\partial}{\partial z}, \quad X_3 = \frac{\partial}{\partial z} \tag{20}$$

with the dual frame:

$$\theta_1 = \frac{dx}{F}, \quad \theta_2 = \frac{dy}{F}, \quad \theta_3 = dz + \frac{\tau y}{2F}dx - \frac{\tau x}{2F}dy. \tag{21}$$

Let $M^2_k = \{(x, y) \in \mathbb{R}^2; F(x, y) > 0\}$ with the metric $g_k = \frac{dx^2+dy^2}{F^2}$. Then the map $\pi : (M^3_k, g_{k,\tau}) \to (M^2_k, g_k)$, $\pi(x, y, z) = (x, y)$ being a Riemannian submersion yields a fiber bundle called *Hopf bundle* in [9, p. 344] since for $k = \frac{\tau^2}{4}$ we get the Hopf fibration $\pi : S^3(k) \to S^2(4k)$. Again, let the connection $E = E_g$ such that $T_{H,\pi}M^3_k$ is the $g_{k,\tau}$-orthogonal complement of $T_{V,\pi}M^3_k$.

Then $\Gamma(T_{V,\pi}M^3_k)$ is spanned by X_3 while $\Gamma(T^*_{V,\pi}M^3_k)$ is spanned by $\{\delta x = F\theta_1, \delta y = F\theta_2\}$; $\Gamma(T_{H,\pi}M^3_k)$ is spanned by $\{\frac{\delta}{\delta x} = \frac{1}{F}X_1, \frac{\delta}{\delta y} = \frac{1}{F}X_2\}$ and $\Gamma(T^*_{H,\pi}M^3_k)$ is spanned by $\{\theta_3\}$. We get the coefficients of the connection:

$$N^1_1 = \frac{\tau y}{2F}, \quad N^1_2 = -\frac{\tau x}{2F} \tag{22}$$

and therefore the curvature is:

$$R^1_{12} = -R^1_{21} = \frac{\tau}{F^2}. \tag{23}$$

In conclusion, the only Dirac structure provided by Corollary 4 on M^3_k is:

$$V_D(\pi, E_g) = \{(x, y, z, 0, 0, Z, p_x, p_y, 0) \in T^{\text{big}}M^3_k\} \tag{24}$$

while:

$$H_D(\pi, E_g) = \left\{\left(x, y, z, X - \frac{\tau y}{2F}Z, Y + \frac{\tau x}{2F}Z, 0, 0, 0, p_z + \frac{\tau y}{2F}p_x - \frac{\tau x}{2F}p_y\right) \in T^{\text{big}}M^3_k\right\} \tag{25}$$

is an almost Dirac structure on M^3_k.

Remark 9 (A relationship with previous works). Returning to the general framework, let us recall that a connection on ξ is equivalent with an almost product structure P on M such that $T_{V,\pi}M$ is the eigenbundle corresponding to the eigenvalue -1 of P while $T_{H,\pi}M$ is the eigenbundle corresponding to the eigenvalue $+1$ of P:

$$P\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial x^i} - 2N^\alpha_i \frac{\partial}{\partial y^\alpha}, \quad P\left(\frac{\partial}{\partial y^\alpha}\right) = -\frac{\partial}{\partial y^\alpha}. \tag{26}$$

So, our Corollary 4 can be thought in correspondence with Example 4 from [10, p. 894]; the present result is more efficient than Proposition 2.3 of [10, p. 891] since one of the generated almost Dirac structure is in fact already Dirac. The flatness of the connection is equivalent with the integrability of P expressed as the vanishing of the Nijenhuis tensor of P and then we recover the Theorem 3.3 of [10, p. 892].

References

[1] A. M. Bloch, *Nonholonomic Mechanics and Control*, With the collaboration of J. Baillieul, P. Crouch and J. Marsden, With scientific input from P. S. Krishnaprasad, R. M. Murray and D. Zenkov. Interdisciplinary Applied Mathematics; Vol. 24, Systems and Control (Springer-Verlag, New York, 2003), MR1978379 (2004e:37099).

[2] T. J. Courant, Dirac manifolds, *Trans. Amer. Math. Soc.* **319**(2) (1990) 631–661, MR0998124 (90m:58065).

[3] A. Ibort, T. Rodríguez de La Peña and R. Salmoni, Dirac structures and reduction of optimal control problems with symmetries, preprint (2010), arXiv: 1004.1438v1.

[4] G. R. Jensen and M. Rigoli, Neutral surfaces in neutral four-spaces, *Matematiche* (Catania) **45**(2) (1990) 407–443, MR1152942 (93d:53084).

[5] M. Jotz, T. S. Ratiu and J. Śniatycki, Singular reduction of Dirac structures, *Trans. Amer. Math. Soc.* **363** (2011) 2967–3013, MR2775795, http://arXiv.org/abs/0901.3062v5.

[6] Z.-J. Liu, A. Weinstein and P. Xu, Manin triples for Lie bialgebroids, *J. Differential Geom.* **45**(3) (1997) 547–574, MR1472888 (98f:58203).

[7] H. J. Sussmann, Orbits of families of vector fields and integrability of distributions, *Trans. Amer. Math. Soc.* **180** (1973) 171–188, MR0321133 (47 #9666).

[8] I. Vaisman, Isotropic subbundles of $TM \oplus T^*M$, *Int. J. Geom. Meth. Mod. Phys.* **4**(3) (2007) 487–516, MR2343378 (2008g:53102).

[9] J. Van der Veken, Higher order parallel surfaces in Bianchi–Cartan–Vranceanu spaces, *Results Math.* **51**(3–4) (2008) 339–359, MR2400172 (2009h:53128).

[10] A. Wade, Dirac structures and paracomplex manifolds, *C. R. Math. Acad. Sci. Paris* **338**(11) (2004) 889–894, MR2059668 (2005a:53140).

[11] H. Yoshimura and J. E. Marsden, Dirac structures in Lagrangian mechanics. I, II, *J. Geom. Phys.* **57**(1) (2006) 133–156, 209–250, MR2265464 (2007j:37087), MR2265469 (2007j:37088).