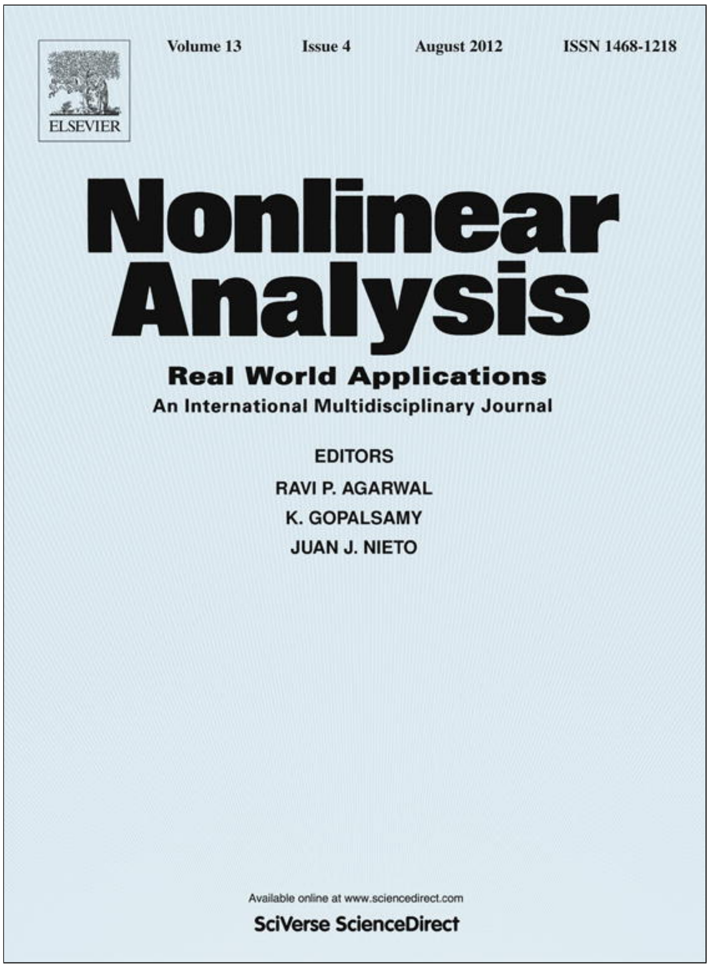
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Rayleigh dissipation from the general recurrence of metrics in path spaces

Mircea Crasmareanu

Faculty of Mathematics, University "Al. I. Cuza", Iaşi, 700506, Romania

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ABSTRACT

Given a pair (metric g, symmetric 2-covariant tensor field H though as a Rayleigh dissipation) on a path space (manifold M, semispray S), the family of nonlinear connections N such that H equals the dynamical derivative of g with respect to (S, N) is determined by using the Obata tensors. In this way, we generalize the case of metric nonlinear connections as well as that of recurrent metrics. As applications, we treat firstly the case of Finslerian (α, β)-metrics finding all nonlinear connections for which the associated Finsler–Sasaki metric is exactly the dynamical derivative of the Riemannian–Sasaki metric. Secondly, we apply our results for the case of Beil metrics used in Relativity and field theories.

0. Introduction

In two many cited papers, [1,2], Yung-chow Wong derived several properties of a recurrent tensor field T on a manifold M endowed with a linear connection ∇. Recall that this means the existence of a 1-form α_T on M such that:

$$\nabla T = \alpha_T \otimes T \tag{0.1}$$

where ∇ is the covariant differential of ∇. For $\alpha_T = 0$ we recover the notion of *parallel* or *covariant constant* tensor field. The most general case to be consider is as follows: let us given T a tensor field of (p, q)-type and $\tilde{T}$ of $(p, q + 1)$-type. Our problem is to find the class of linear connections ∇ such that:

$$\nabla T = \tilde{T}. \tag{0.2}$$

We call this last problem *general recurrence*.

The aim of the present paper is to extend the notion of general recurrence to the geometry of systems of second order differential equations on M. More precisely, given such a system S, on short *semispray*, we can derive a type of covariant derivative ∇ if S is considered as a vector field on the tangent bundle TM. A main tool in the definition of ∇ is given by a splitting of the iterated tangent bundle $T(TM)$ provided by a distribution N on TM. Such an object N is called *nonlinear connection*. A remarkable result is that every S yields such a nonlinear connection, $\overset{c}{N}$, indexed by us with c from canonical.

E-mail address: mcrasm@uaic.ro.
URL: http://www.math.uaic.ro/~mcrasm.

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Given a triple (semispray S, generalized metric g, symmetric two-covariant tensor field H) on TM, we use the Obata tensor fields to determine the family of all nonlinear connections N such that $\nabla g = H$. In particular, we obtain a characterization for a triple (N, g, H) to be general recurrent as well as for a triple (S, g, H). In the former case, we arrive at the unique Weyl connection associated to a Riemannian metric via usual recurrence, which is the main object in the theory of Weyl structures initiated by Folland in 1970. At this moment let us explain the title: after [3, p. 198] the tensor field H can be called Rayleigh dissipation if in addition it is positive-semidefinite. We adopt this general name without any constraint about the signature of H and the applications of our approach to find a Rayleigh dissipation function for the given semispray are discussed at the end of the second section. In our framework the tensor field H must be negative-semidefinite on the Liouville vector field for the case of homogeneous semisprays also called *sprays*.

A setting where we have such pairs (g, H) is the Finslerian geometry of (α, β)-metrics. More precisely, starting with a Riemannian metric a and a 1-form b we get on the tangent bundle TM two Riemannian metrics: g_a the Sasaki lift of a as well as the Finsler metric H generated by a and b in the form of (α, β)-metrics. As a consequence, we express all nonlinear connections N such that H is the dynamical derivative of g_a with respect to the Finslerian spray of the (α, β)-metric and N. As examples, we treat the case of Ransders, Kropina and "Riemann"-type (α, β)-metrics.

In fact, together with a view toward the geometrization of Control Theory (as in [3], who suggest the name of *Rayleigh structure*), the case of Finsler geometry of (α, β)-metrics was the main motivation of this study. More precisely, required again by some physical theories, there is a strong need to express the Finslerian metric H by means of the Riemannian metric a (in fact g_a) and one such possibility is with *nonholonomic frames* considered in [4]. The present paper gives a second approach to this problem, i.e. we express the (α, β)-metric in terms of its spray and the support Riemannian metric.

A second class of applications consists of Beil metrics, a class of generalized metrics used in some physical theories, [5–7]. An example of a natural pair, in the sense of the Atanasiu–Hashiguchi–Miron paper [8], is obtained for this type of deformation of a given metric. An important feature of a natural pair is that the corresponding Obata tensors commutes.

1. Nonlinear connections and semisprays on tangent bundles

Let M be a smooth, n-dimensional manifold for which we denote: $C^\infty(M)$—the algebra of smooth real functions on M, $\mathcal{X}(M)$—the Lie algebra of vector fields on M, $T^r_s(M)$—the $C^\infty(M)$-module of tensor fields of (r, s)-type on M.

A local chart $x = (x^i) = (x^1, \ldots, x^n)$ on M lifts to a local chart on the tangent bundle TM given by: $(x, y) = (x^i, y^i)$. If $\pi : TM \to M$ is the canonical projection then the kernel of the differential of π is an integrable distribution $V(TM)$ with local basis $(\frac{\partial}{\partial y^i})$. An important element of $V(TM)$ is the *Liouville vector field* $\mathbb{C} = y^i \frac{\partial}{\partial y^i}$. $V(TM)$ is called *the vertical distribution* and its elements are *vertical vector fields*.

The tensor field $J \in T^1_1(TM)$ given by $J = \frac{\partial}{\partial y^i} \otimes dx^i$ is called *the tangent structure*. Two of its properties are: the nilpotence $J^2 = 0$ and $\mathrm{Im}\, J\ (=\mathrm{Ker}\, J) = V(TM)$.

A well-known notion in the tangent bundles geometry is:

Definition 1.1 *([9, p. 336])*. A supplementary distribution N to the vertical distribution $V(TM)$:

$$T(TM) = N \oplus V(TM) \tag{1.1}$$

is called *horizontal distribution* or *nonlinear connection*. A vector field belonging to N is called *horizontal*.

A nonlinear connection has a local basis:

$$\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^j} \tag{1.2}$$

and the functions $\left(N^i_j(x, y)\right)$ are called *the coefficients* of N. So, a basis of $\mathcal{X}(TM)$ adapted to the decomposition (1.1) is $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})$ called the *Berwald basis*. The dual of the Berwald basis is: $(dx^i, \delta y^i = dy^i + N^i_j dx^j)$.

A second remarkable structure on TM is provided by:

Definition 1.2 *([9, p. 336])*. $S \in \mathcal{X}(TM)$ is called *semispray* if:

$$J(S) = \mathbb{C}. \tag{1.3}$$

In canonical coordinates:

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i} \tag{1.4}$$

and the functions $\left(G^i(x, y)\right)$ are *the coefficients* of S. The flow of S is a system of second order differential equations: $\frac{d^2x^i}{dt^2} = 2G^i(x, \frac{dx}{dt})$ and then the pair (M, S) is called *path space*.

An important result is that a nonlinear connection $N = (N^i_j)$ yields an unique horizontal semispray denoted $S(N)$ with:

$$G^i = \frac{1}{2} N^i_j y^j. \tag{1.5}$$

In other words:

$$S(N) = y^i \frac{\delta}{\delta x^i}. \tag{1.6}$$

The converse of this result is that a semispray S yields a nonlinear connection $\overset{c}{N}$ given by:

$$\overset{c}{N}{}^i_j = \frac{\partial G^i}{\partial y^j}. \tag{1.7}$$

Definition 1.3. A semispray S for which the coefficients $\left(G^i\right)$ are homogeneous of degree 2 with respect to the variables $\left(y^i\right)$ will be called *spray*.

Locally this means, via the Euler theorem:

$$2G^i = y^j \frac{\partial G^i}{\partial y^j} \tag{1.8}$$

and then $\overset{c}{N}$ is 1-homogeneous:

$$\overset{c}{N}{}^i_j = y^a \frac{\partial \overset{c}{N}{}^i_j}{\partial y^a} \tag{1.9}$$

which yields that S is horizontal with respect to $\overset{c}{N}$, i.e. S has the expression (1.6).

Another approach of spray theory can be given by using the KCC-invariants of semisprays, [10]. Namely, for a semispray S, its first KCC-invariant is the vertical vector field:

$$\mathcal{E}(S) = \mathcal{E}^i \frac{\partial}{\partial y^i} \tag{1.10}$$

with the components:

$$\mathcal{E}^i = 2G^i - \frac{\partial G^i}{\partial y^j} y^j. \tag{1.11}$$

Therefore, S is a spray if and only if $\mathcal{E}(S) = 0$.

For a later use, we introduce as the following generalization of the above object: $\mathcal{E}(S, N)$ will be given by (1.10) but with the components:

$$\mathcal{E}^i = 2G^i - N^i_j y^j. \tag{1.12}$$

Thus $\mathcal{E}(S) = \mathcal{E}(S, \overset{c}{N})$.

2. Rayleigh structures for semisprays and metrics

2.1. The general problem

Let us fix a semispray $S = \left(G^i\right)$ and a nonlinear connection $N = \left(N^i_j\right)$. Following [9, p. 337] let us consider:

Definition 2.1. The *dynamical derivative* associated to the pair (S, N) is the map $\overset{SN}{\nabla}: V(TM) \to V(TM)$ given by:

$$\overset{SN}{\nabla} X = \overset{SN}{\nabla}\left(X^i \frac{\partial}{\partial y^i}\right) := \left(S\left(X^i\right) + N^i_j X^j\right) \frac{\partial}{\partial y^i}. \tag{2.1}$$

The dynamical derivative with respect to $(S, \overset{c}{N})$ will be denoted $\overset{S}{\nabla}$.

Properties:

(I) $\overset{SN}{\nabla}\left(\frac{\partial}{\partial y^i}\right) = N^j_i \frac{\partial}{\partial y^j}$,

(II) $\overset{SN}{\nabla}(X + Y) = \overset{SN}{\nabla} X + \overset{SN}{\nabla} Y$,

(III) $\overset{SN}{\nabla}(fX) = S(f) X + f \overset{SN}{\nabla} X$.

It is straightforward to extend the action of $\overset{SN}{\nabla}$ to general vertical tensor fields by requiring to preserve the tensor product and commute with contraction. Moreover, we will extend $\overset{SN}{\nabla}$ to a special class of tensor fields:

Definition 2.2. A d-*tensor field* (d from distinguished) on *TM* is a tensor field whose change of components, under a change of canonical coordinates $(x, y) \to (\widetilde{x}, \widetilde{y})$ on *TM*, involves only factors of type $\frac{\partial \widetilde{x}}{\partial x}$ and (or) $\frac{\partial x}{\partial \widetilde{x}}$.

Example 2.3. (i) $\left(\frac{\delta}{\delta x^i}\right)$ and $\left(\frac{\partial}{\partial y^i}\right)$ are components of d-tensor fields of (1, 0)-type.
(ii) $\left(dx^i\right)$ and $\left(\delta y^i\right)$ are components of d-tensor fields of (0, 1)-type.
(iii) $\left(G^i\right)$ are not components of a d-tensor field since a change of coordinates implies:

$$2\widetilde{G}^i = 2\frac{\partial \widetilde{x}^i}{\partial x^j} G^j - \frac{\partial \widetilde{y}^i}{\partial x^j} y^j$$

but it results that given two semisprays $\overset{1}{S}$ and $\overset{2}{S}$ their difference $X = \overset{2}{S} - \overset{1}{S}$ is a vertical (and then *d*-) vector field.
(iv) $\left(N^i_j\right)$ are not components of a d-tensor field since a change of coordinates implies:

$$\frac{\partial \widetilde{x}^j}{\partial x^k} N^k_i = \widetilde{N}^j_k \frac{\partial \widetilde{x}^k}{\partial x^i} + \frac{\partial \widetilde{y}^j}{\partial x^i}.$$

It follows that given two nonlinear connections $\overset{1}{N}$ and $\overset{2}{N}$ their difference $F = \overset{2}{N} - \overset{1}{N} = \left(F^i_j = \overset{2}{N}{}^i_j - \overset{1}{N}{}^i_j\right)$ is a d-tensor field of (1, 1)-type. Therefore, the set $\mathcal{N}(S, g)$ of all nonlinear connections is an $C^\infty(TM)$-affine module associated to the $C^\infty(TM)$-linear module of d-tensor fields of (1, 1)-type.

Definition 2.4. A (*generalized*) or (*vertical*) *metric* g on *TM* is a d-tensor field of (0, 2)-type which is symmetric and non-degenerated.

It results for the components $g_{ij} = g(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j})$ the following properties:

(1) (symmetry) $g_{ij} = g_{ji}$,
(2) (non-degeneration) $\det\left(g_{ij}\right) \neq 0$ i.e. g has the rank n; then there exists the d-tensor field of (2, 0)-type $g^{-1} = (g^{ij})$.

The name is justified from the fact that $g_{ij}dx^i \otimes dx^j + g_{ij}\delta y^i \otimes \delta y^j$ is a Riemannian metric on *TM* for which N and $V(TM)$ are orthogonal distributions.

Definition 2.5. The *dynamical derivative of metric* g with respect to the pair (S, N) is $\overset{SN}{\nabla} g : V(TM) \times V(TM) \to V(TM)$ given by:

$$\overset{SN}{\nabla} g(X, Y) = S(g(X, Y)) - g(\overset{SN}{\nabla} X, Y) - g(X, \overset{SN}{\nabla} Y). \tag{2.2}$$

The main notion of this subsection is:

Definition 2.6. Let H be a symmetric d-tensor field of (0, 2)-type on *TM*, in particular a metric. Then H is called (S, N, g)-*Rayleigh dissipation* if the general recurrence relation holds:

$$\overset{SN}{\nabla} g = H. \tag{2.3}$$

Also, the data (S, N, g, H) will be called a *general Rayleigh structure*.

The aim of this section is to find all nonlinear connections which together with fixed (S, g, H) form a general Rayleigh structure. In order to answer this question, a look at Example 2.3 (iv) gives a necessary study of two operators, called *Obata*, acting on the space of d-tensor fields of (1, 1)-type:

$$O^{ij}_{kl} = \frac{1}{2}\left(\delta^i_k\delta^j_l - g^{ij}g_{kl}\right), \qquad \overset{*}{O}{}^{ij}_{kl} = \frac{1}{2}\left(\delta^i_k\delta^j_l + g^{ij}g_{kl}\right). \tag{2.4}$$

The Obata operators are supplementary projectors:

$$O^{ia}_{bj}\overset{*}{O}{}^{bk}_{la} = \overset{*}{O}{}^{ia}_{bj}O^{bk}_{la} = 0, \qquad O^{ia}_{bj}O^{bk}_{la} = O^{ik}_{lj}, \qquad \overset{*}{O}{}^{ia}_{bj}\overset{*}{O}{}^{bk}_{la} = \overset{*}{O}{}^{ik}_{lj} \tag{2.5}$$

and the tensorial equations involving these operators has solutions as follows:

Proposition 2.7. *The system of equations:*

$$\overset{*}{O}{}^{ia}_{bj}\left(X^b_a\right) = A^i_j, \qquad \left(O^{ia}_{bj}(X^b_a) = A^i_j\right) \tag{2.6}$$

with X as unknown has solutions if and only if:

$$O^{ia}_{bj}\left(A^b_a\right)=0,\qquad \left(\overset{*}{O}{}^{ia}_{bj}\left(A^b_a\right)=0\right) \tag{2.7}$$

and then, the general solution is:

$$X^i_j=A^i_j+O^{ia}_{bj}\left(Y^b_a\right),\qquad \left(X^i_j=A^i_j+\overset{*}{O}{}^{ia}_{bj}\left(Y^b_a\right)\right) \tag{2.8}$$

with Y an arbitrary d-tensor field of $(1, 1)$*- type.*

We are ready for one of the main results of paper:

Theorem 2.8. *Set S and* (g, H)*. The family* $\mathcal{N}(S, g, H)$ *of all nonlinear connections* $N=\left(N^i_j\right)$ *such that* (S, N, g, H) *is a general Rayleigh structure is given by:*

$$N^i_j=\frac{1}{2}\overset{c}{N}{}^i_j-\frac{1}{2}g^{ia}g_{jb}\overset{c}{N}{}^b_a+\frac{1}{2}g^{ia}S\left(g_{aj}\right)-\frac{1}{2}g^{ia}H_{aj}+O^{ia}_{bj}\left(X^b_a\right) \tag{2.9}$$

with $X=\left(X^b_a\right)$ *an arbitrary d-tensor field of* $(1, 1)$*-type. Therefore, writing:*

$$N=\overset{c}{N}+\frac{1}{2}g^{-1}(\overset{S}{\nabla}g-H)+O(X) \tag{2.9G}$$

it results that $\mathcal{N}(S, g, H)$ *is an affine submodule of* $\mathcal{N}(TM)$ *passing through the nonlinear connection* $\overset{c}{N}+\frac{1}{2}g^{-1}(\overset{S}{\nabla}g-H)$ *and having the direction given by the linear submodule* ImO *of* $T^1_1(TM)$*.*

Proof. We search $\left(N^i_j\right)$ of the form:

$$N^i_j=\overset{c}{N}{}^i_j+F^i_j \tag{2.10}$$

with $\left(F^i_j\right)$ a d-tensor field of $(1, 1)$-type to be determined. The local expression of Eq. (2.3) is:

$$S\left(g_{uv}\right)-g_{um}N^m_v-g_{mv}N^m_u=H_{uv} \tag{2.11}$$

and inserting (2.10) in (2.11) gives:

$$S\left(g_{uv}\right)-g_{um}\overset{c}{N}{}^m_v-g_{mv}\overset{c}{N}{}^m_u=g_{um}F^m_v+g_{mv}F^m_u+H_{uv}.$$

Multiplying the last relation with g^{ku} we get:

$$g^{ku}S\left(g_{uv}\right)-\overset{c}{N}{}^k_v-g^{ku}g_{mv}\overset{c}{N}{}^m_u-g^{ku}H_{uv}=F^k_v+g^{ku}g_{mv}F^m_u=2\overset{*}{O}{}^{kb}_{av}\left(F^a_b\right). \tag{2.12}$$

Let us search for the condition (2.7):

$$O^{kb}_{av}\left(g^{am}S\left(g_{mb}\right)-\overset{c}{N}{}^a_b-g^{am}g_{bl}\overset{c}{N}{}^l_m-g^{am}H_{mb}\right)=g^{km}S\left(g_{mv}\right)-\overset{c}{N}{}^k_v-g^{km}g_{vl}\overset{c}{N}{}^l_m-g^{km}S\left(g_{mv}\right)+g^{km}g_{vl}\overset{c}{N}{}^l_m+\overset{c}{N}{}^k_v=0.$$

It follows:

$$F^i_j=\frac{1}{2}g^{im}S\left(g_{mj}\right)-\frac{1}{2}\overset{c}{N}{}^i_j-\frac{1}{2}g^{ia}g_{jb}\overset{c}{N}{}^b_a-\frac{1}{2}g^{ia}H_{aj}+O^{ib}_{aj}\left(X^a_b\right)$$

and returning to (2.10) we have the conclusion. □

In the spray case the Eq. (2.9) admits a simplification:

Proposition 2.9. *Fix a spray S and a pair* (g, H)*. The family* $\mathcal{N}(S, g, H)$ *is:*

$$N^i_j=\frac{1}{2}\overset{c}{N}{}^i_j-\frac{1}{2}g^{ia}g_{jb}\overset{c}{N}{}^b_a+\frac{1}{2}g^{ia}y^m\frac{\delta g_{aj}}{\delta x^m}-\frac{1}{2}g^{ia}H_{aj}+O^{ia}_{bj}\left(X^b_a\right). \tag{2.13}$$

Remark. The Obata operators split the space of d-tensor fields of $(1, 1)$-type into a g-symmetric part ImO^* = Ker$O$, of dimension $\frac{n(n-1)}{2}$, and a g-skew-symmetric part ImO = KerO^*, of dimension $\frac{n(n+1)}{2}$. The general formula (2.9G) says that the general recurrence relation (2.3) fixes the symmetric part of the tensor field $N-\overset{c}{N}$ as $\frac{1}{2}(\overset{S}{\nabla}g-H)$. An interesting open problem is to consider remarkable geometrical conditions which fix the skew-symmetrical part. For example, the Frobenius integrability of the horizontal distributions N may be a way to solve this question. Since:

$$\left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right] = R^k_{ij}\frac{\partial}{\partial y^k}$$

the integrability of N means to find X such that *the curvature* of N:

$$R^k_{ij} = \frac{N^k_j}{\delta x^i} - \frac{N^k_i}{\delta x^j}$$

to vanish.

2.2. Raylegh triples (nonlinear connection, metric, Rayleigh dissipation)

Fix a nonlinear connection $N = \left(N^i_j\right)$ and associate to N the semispray $S\left(N\right)$.

Definition 2.10. The triple (N, g, H) is *Rayleigh* if the data $(S\left(N\right), N, g, H)$ is so.

Theorem 2.11. *The triple* (N, g, H) *is Rayleigh if and only if:*

$$N^i_j + g^{ia}g_{jb}N^b_a + g^{im}\frac{\partial g_{mj}}{\partial y^b}N^b_a y^a + g^{ia}H_{aj} = g^{im}\frac{\partial g_{mj}}{\partial x^a}y^a \tag{2.14}$$

for all $i, j \in \{1, \dots, n\}$.

Proof. From (2.9) it results that (N, g, H) is a Rayleigh triple if and only if:

$$\overset{*}{O}{}^{iv}_{uj}\left(N^u_v + g^{ua}g_{vb}N^b_a + g^{um}\frac{\partial g_{mv}}{\partial y^b}N^b_a y^a + g^{um}H_{mv}\right) = \overset{*}{O}{}^{iv}_{uj}\left(g^{um}\frac{\partial g_{mv}}{\partial x^a}y^a\right)$$

and a straightforward computation yields the conclusion. □

Example 2.12 (*Riemannian Metrics*)**.** Let us consider $g = g(x)$ a Riemannian metric on M. The relation (2.14) becomes:

$$N^i_j + g^{ia}g_{jb}N^b_a = g^{im}\frac{\partial g_{mj}}{\partial x^a}y^a - g^{im}H_{mj}. \tag{2.15}$$

Recall that a symmetric linear connection on M with coefficients $\left(\Gamma^i_{jk}(x)\right)$ yields the nonlinear connection with the coefficients:

$$N^i_j = \Gamma^i_{ja}y^a. \tag{2.16}$$

Then the associated semispray $S(N)$ is a spray:

$$G^i = \frac{1}{2}\Gamma^i_{jk}y^j y^k. \tag{2.17}$$

Inserting (2.16) in (2.15) we get:

$$\left(g^{im}\frac{\partial g_{mj}}{\partial x^a} - \Gamma^i_{ja} - g^{iu}g_{jv}\Gamma^v_{ua}\right)y^a = g^{im}H_{mj}. \tag{2.18}$$

But multiplying the last equation with g_{ik} we arrive at:

$$\left(\frac{\partial g_{jk}}{\partial x^a} - g_{ki}\Gamma^i_{ja} - g_{ji}\Gamma^i_{ka}\right)y^a = H_{jk} \tag{2.19}$$

which is the usual Christoffel process replaced in the general framework on TM. So, we verified the condition (2.14) in the Riemannian setting.

Let us point out the role of homogeneity of the spray (2.17). We remark from (2.19) that H must be a 1-homogeneous on y, i.e. $H_{jk}(x, y) = \tilde{g}_{ajk}(x)y^a$, and then we have:

$$\frac{\partial g_{jk}}{\partial x^a} - g_{ki}\Gamma^i_{ja} - g_{ji}\Gamma^i_{ka} = \tilde{g}_{ajk} \tag{2.20}$$

for all $a, j, k \in \{1, \dots, n\}$. So we recover the starting formula (0.2) from the Introduction in terms of g and $\tilde{g}$.

If H is a conformal version of g i.e. there exists a smooth function α such that $H = \alpha g$ then it results that α has the form $\alpha(x, y) = \alpha_a(x)y^a$. It is well known that the solution of (2.20) is the unique *Weyl connection* [11, p. 147]:

$$\Gamma^i_{jk} = \overset{c}{\Gamma}{}^i_{ij} + \frac{1}{2}\left(\alpha^i g_{jk} - \delta^i_j\alpha_k - \delta^i_k\alpha_j\right) \tag{2.20W}$$

where $\overset{c}{\Gamma}$ is the Levi-Civita connection of g and $\alpha^i = g^{ia}\alpha_a$ is the g-contravariant version of α.

2.3. Rayleigh triples (semispray, metric, Rayleigh dissipation)

Let us fix the semispray $S = \left(G^i\right)$.

Definition 2.13. The triple (S, g, H) is called *Rayleigh* if the data $\left(S, \overset{c}{N}, g, H\right)$ is so.

Inserting $\overset{c}{N}$ in the left-hand-side of (2.9) we get:

Theorem 2.14. (i) *The triple (S, g, H) is Rayleigh if and only if:*

$$\frac{\partial G^i}{\partial y^j} + g^{ia} g_{jb} \frac{\partial G^b}{\partial y^a} + g^{ia} H_{aj} = g^{ia} S(g_{aj}) \tag{2.21}$$

for all $i, j \in \{1, \dots, n\}$.

(ii) *The spray S makes Rayleigh the pair (g, H) if and only if:*

$$\frac{\partial G^i}{\partial y^j} + g^{ia} g_{jb} \frac{\partial G^b}{\partial y^a} + g^{ia} H_{aj} = g^{ia} y^m \frac{\delta g_{aj}}{\delta x^m} \tag{2.22}$$

for all $i, j \in \{1, \dots, n\}$.

Proof. The left-hand-side of (2.9) becomes:

$$\overset{*}{O}{}^{iv}_{uj} \left(\overset{c}{N}{}^{u}_{v} + g^{ua} g_{vb} \overset{c}{N}{}^{b}_{a} - g^{um} S\left(g_{mv}\right) \right) = -g^{ia} H_{aj} \tag{2.23}$$

and the computations give (2.21). □

Example 2.15 (*Euclidean Metrics*)**.** Let us consider the tangent bundle TM with g a constant metric i.e. g_{ij} does not depend of (x, y). The condition (2.21) is:

$$\overset{c}{N}{}^{i}_{j} + g^{ia} g_{jb} \overset{c}{N}{}^{b}_{a} = -g^{ia} H_{aj}. \tag{2.24}$$

If $M = \mathbb{R}^n$ and g is the usual Euclidean metric then (2.24) reads:

$$\overset{c}{N}{}^{i}_{j} + \overset{c}{N}{}^{j}_{i} = -H^i_j. \tag{2.25}$$

Let $\mathcal{G}$ be the structural Lie group of the Rayleigh triple (S, g, H) seen as a G-structure; then the last formula gives the Lie algebra of $\mathcal{G}$ as $L(\mathcal{G})$ = the Lie algebra of H-skew-symmetric matrices. If G is a conformal variant of g then $L(\mathcal{G}) = -\alpha \cdot o(n)$ which is the Lie algebra of the conformal Euclidean geometry; in other words $\mathcal{G}$ is the conformal group $CO(n)$. Thus a verification of (2.21) results.

Remarks 2.16. (i) For $\alpha = 0$ the general formula (2.9) appears firstly in [9, p. 339] and also in [12, p. 172] while the results of Sections 2.2 and 2.3 are generalizations of those from [13].

(ii) Since we recover the Weyl connection of conformal gauge theory it is interesting to what conformal transformation $g \to \lambda g$ the class $\mathcal{N}(S, g, H)$ is invariant, with λ a strictly positive function on TM. From (2.9) it results that $\mathcal{N}(S, \lambda g, H) = \mathcal{N}(S, g, \lambda H)$ for every strictly positive first integral of the semispray S, i.e. $S(\lambda) = 0$.

2.4. Rayleigh dissipation

Let us end this section with a dynamical interpretation of the recurrence relation (2.3). Namely, for a symmetric d-tensor field of $(0, 2)$-type g we define its *energy* as the smooth function on TM:

$$\mathcal{E}(g) = g_{ij} y^i y^j = g(\mathbb{C}, \mathbb{C}). \tag{2.26}$$

Let us point out also that for $L \in C^\infty(TM)$ we have:

$$\overset{SN}{\nabla}(L) = S(L) \tag{2.27}$$

and we consider:

Definition 2.17. L is a *Rayleigh dissipation function* on the triple (path space (M, S), nonlinear connection N) if:

$$\overset{SN}{\nabla}(L) \leq 0. \tag{2.28}$$

More particularly, L is called the *Rayleigh dissipation function* for the path space (M, S) if:

$$\overset{SN}{\overset{c}{\nabla}}(L) \leq 0. \tag{2.29}$$

Then, another main result is the following:

Theorem 2.18. *Let (S, N, g, H) be a general Rayleigh structure with:*

$$\mathcal{E}(H) \leq 2g(\mathbb{C}, \mathcal{E}(S, N)). \tag{2.30}$$

Then $\mathcal{E}(g)$ is a Rayleigh dissipation function for the triple (M, S, N).

Proof. A direct computation gives:

$$\overset{SN}{\nabla}(\mathcal{E}(g)) = \mathcal{E}(H) + 2g_{ij}y^j(N^i_a y^a - 2G^i) \tag{2.31}$$

which can be expressed as:

$$\overset{SN}{\nabla}(\mathcal{E}(g)) = \mathcal{E}(H) - 2g(\mathcal{C}, \mathcal{E}(S, N)) \tag{2.32}$$

and the conclusion is obtained. □

A more simple case is the spray case:

Corollary 2.19. *Let S be a spray such that $\overset{c}{N}$ belongs to the family $\mathcal{N}(S, g, H)$ from (2.13). If H is vertical negative-semidefinite, i.e. $\mathcal{E}(H) = H(\mathbb{C}, \mathbb{C}) \leq 0$ then $\mathcal{E}(g)$ is a Rayleigh dissipation function for the path space (M, S).*

3. Applications to Finsler (α, β)-metrics

The most general framework providing a couple (semispray S, metric g) is given by the variational calculus associated to a *regular* Lagrangian L when S is exactly the Euler–Lagrange system of equations of L and the regularity yields the metric g, [9]. In order to handle completely several examples we restrict to the case of 2-homogeneous Lagrangians of Finsler geometry.

So, let us recall:

Definition 3.1. A *Finsler fundamental function* is a map $F : TM \to \mathbb{R}_+$ such that:

(F1) F is smooth on the slit tangent bundle $T_0M = TM \setminus \{0\}$ and only continuous on the null section $\{0\}$ of the projection $\pi : TM \to M$,
(F2) F is positive homogeneous of order one with respect to the fibre coordinates, i.e. $F(x, \lambda y) = \lambda F(x, y)$, for $\lambda > 0$,
(F3) For any $(x, y) \in T_0M$ the symmetric bilinear form $H(x, y)$ is non-degenerated and has constant signature:

$$H_{(x,y)}(u, v) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t}[F^2(x, y + su + tv)]|_{s=t=0}, \quad y, u, v \in T_xM. \tag{3.1}$$

From homogeneity it results that $F^2 = H_{ij}y^iy^j$ with:

$$H_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}. \tag{3.2}$$

Then, $H = (H_{ij})$ is called *the Finsler metric* generated by F.

To H we associate the Christoffel symbols:

$$\gamma^i_{jk}(x, y) = \frac{1}{2}H^{ia}\left(\frac{\partial H_{ak}}{\partial x^j} + \frac{\partial H_{ja}}{\partial x^k} - \frac{\partial H_{jk}}{\partial x^a}\right) \tag{3.3}$$

and then we obtain the Finslerian spray S_F:

$$G^i = \frac{1}{2}\gamma^i_{jk}y^jy^k. \tag{3.4}$$

The canonical nonlinear connection is called the *Cartan nonlinear connection* and has the expression:

$$\overset{c}{N}{}^i_j = \gamma^i_{jk}y^k - C^i_{mj}\gamma^m_{ks}y^ky^s \tag{3.5}$$

with the vertical Christoffel symbols $C^i_{jk} = H^{ia}C_{ajk}$ where:

$$C_{ajk} = \frac{1}{4}\frac{\partial^3 F^2}{\partial y^a \partial y^j \partial y^k} = \frac{1}{2}\frac{\partial H_{jk}}{\partial y^a}. \tag{3.6}$$

Consider now $a = (a_{ij}(x))$ a Riemannian metric on the base manifold M and $b = (b_i(x))$ a 1-form, also on M to which we associate:

- the function $\alpha(x, y) = \sqrt{a_{ij}y^iy^j}$,
- the function $\beta(x, y) = b_iy^i$.

Definition 3.2. The Finsler space (M, F) is (α, β)-*type* if there exists a 2-homogeneous function $L(\cdot, \cdot)$ of two variables such that:

$$F^2 = L(\alpha, \beta). \tag{3.7}$$

Lifting the Riemannian metric a to TM we obtain *the Riemann–Sasaki metric*:

$$g_a = a_{ij}dx^i \otimes dx^j + a_{ij}dy^i \otimes \delta y^i \otimes \delta y^j \tag{3.8}$$

which will be considered in pair with *the Finsler–Sasaki metric* (3.2):

$$H = H_{ij}dx^i \otimes dx^j + H_{ij}\delta y^i \otimes \delta y^j. \tag{3.9}$$

For a Finsler space of (α, β)-type we consider the following four invariants, [14, p. 890]:

$$\begin{cases} p = \dfrac{1}{2\alpha}\dfrac{\partial F^2}{\partial \alpha}, \qquad p_0 = \dfrac{1}{2}\dfrac{\partial^2 F^2}{\partial \beta^2}, \\ p_1 = \dfrac{1}{2\alpha}\dfrac{\partial^2 F^2}{\partial\alpha\partial\beta}, \qquad p_2 = \dfrac{1}{2\alpha^2}\left(\dfrac{\partial^2 F^2}{\partial\alpha^2} - \dfrac{1}{\alpha}\dfrac{\partial F^2}{\partial\alpha}\right) \end{cases} \tag{3.10}$$

where the subscripts denote the minus of degree of homogeneity of these invariants and which connect the Riemannian metric a with the Finsler metric H through the relation:

$$H_{ij} = pa_{ij} + p_0b_ib_j + p_1\left(b_iy_j + b_jy_i\right) + p_2y_iy_j. \tag{3.11}$$

Here (y_a) is the a-covariant version of the Liouville vector field: $y_i = a_{ij}(x)y^j$.

Applying the Theorem 2.8 we get:

Proposition 3.3. *The family $\mathcal{N}(a, b)$ of all nonlinear connections N such that (S_F, N, g_a, H) is a general Rayleigh structure is given by:*

$$2N^i_j = 2\gamma^i_{jm}y^m + (a^{iu}a_{jv}C^v_{mu} - C^i_{mj})\gamma^m_{ks}y^ky^s - p\delta^i_j - p_0b^ib_j - p_1(b^iy_j + b_jy^i) - p_2y^iy_j + O^{iu}_{vj}(X^v_u). \tag{3.12}$$

If H is a locally Minkowski metric i.e. $H = H(y)$ then the Christoffel symbols vanish and then:

$$2N^i_j = -p\delta^i_j - p_0b^ib_j - p_1(b^iy_j + b_jy^i) - p_2y^iy_j + O^{iu}_{vj}(X^v_u). \tag{3.12M}$$

Example 3.4.

- Randers metrics: $F^2 = (\alpha + \beta)^2$

$$p = 1 + \frac{\beta}{\alpha}, \qquad p_0 = 1, \qquad p_1 = \frac{1}{\alpha}, \qquad p_2 = -\frac{\beta}{\alpha^3}. \tag{3.13}$$

- Kropina: $F^2 = \frac{\alpha^4}{\beta^2}$

$$p = \frac{2\alpha^2}{\beta^2}, \qquad p_0 = \frac{3\alpha^4}{\beta^4}, \qquad p_1 = -\frac{4\alpha^2}{\beta^3}, \qquad p_2 = \frac{4}{\beta^2}. \tag{3.14}$$

- "Riemann" type (α, β)-metric: $F^2 = 1 + \alpha^2$

$$p = p_0 = 1, \qquad p_1 = p_2 = 0. \tag{3.15}$$

4. Applications to Beil metrics

Consider the metric $g = (g_{ij})$ in the sense of Definition 2.4 and two functions $a, b \in C^\infty(TM)$ with $a \neq 0$ and $b \geq 0$. Let also $B = B_i(x, y)dx^i$ be a vertical 1-form. It results that:

$$H_{ij} = ag_{ij} + bB_iB_j \tag{4.1}$$

is a new metric called the *Beil metric* or sometimes the *Beil deformation* of the metric g. The case g semi-Riemannian (more precisely Minkowski or Lorentz) on the base M, $a = 1$ with various choices of b and B was introduced and studied by Beil for constructing a new unified field theory in [5,6].

Applying directly the Theorem 2.8 we get:

Proposition 4.1. *Fix the semispray S. The family $\mathcal{N}(S, g, a, b, B)$ of all nonlinear connections N such that (S, N, g, H) is a general Rayleigh structure is given by:*

$$N^i_j = \frac{1}{2}\overset{c}{N}{}^i_j - \frac{1}{2}g^{ia}g_{jb}\overset{c}{N}{}^b_a + \frac{1}{2}g^{ia}S\left(g_{aj}\right) - \frac{1}{2}(a\delta^i_j + bB^iB_j) + O^{ia}_{bj}\left(X^b_a\right). \tag{4.2}$$

Example 4.2.

(i) the classical Beil metrics: $g = g(x)$

$$N^i_j = \frac{1}{2}\overset{c}{N}{}^i_j - \frac{1}{2}g^{ia}g_{jb}\overset{c}{N}{}^b_a + \frac{1}{2}g^{ia}y^u\frac{\partial g_{aj}}{\partial x^u} - \frac{1}{2}(a\delta^i_j + bB^iB_j) + O^{ia}_{bj}\left(X^b_a\right). \tag{4.3}$$

(ii) Miron–Tavakol metrics useful in General Relativity: $g = g(x)$, $a = \exp(2\sigma(x, y))$ and $b = 0$

$$N^i_j = \frac{1}{2}\overset{c}{N}{}^i_j - \frac{1}{2}g^{ia}g_{jb}\overset{c}{N}{}^b_a + \frac{1}{2}g^{ia}y^u\frac{\partial g_{aj}}{\partial x^u} - \frac{1}{2}\exp(2\sigma)\delta^i_j + O^{ia}_{bj}\left(X^b_a\right). \tag{4.4}$$

(iii) Finslerian case: suppose that g is a Finsler metric (3.2) and S is exactly the corresponding Finslerian semispray S_F. Then $\overset{c}{N}$ is a metric nonlinear connection for the pair (g, S_F), [9], and then (4.2) becomes:

$$N^i_j = -\frac{1}{2}(a\delta^i_j + bB^iB_j) + O^{ia}_{bj}\left(X^b_a\right). \tag{4.5}$$

Let us end this paper with a short discussion of a topic connected with our tools namely pairs of metrics on TM. In [8, p. 3] the pair (g, H) of metrics is called *natural* if there exists a smooth and non-vanishing function $\mu = \mu(x, y)$ such that:

$$H_{ir}g^{rs}H_{sj} = \mu g_{ij} \tag{4.6}$$

for all indices i, j. It results that if the pair (g, H) is μ-natural then:

(i) the pair (H, g^{-1}) is $\frac{1}{\mu}$-natural,

(ii) μ is strictly positive being the determinant of $(g^{-1}\cdot H)^2$.

An important feature of these metrics connected with our present study is that their Obata operators commutes. Unfortunately, no example of such a pair has been provided in twenty years [15].

So, we search for a μ-natural Beil pair (g, H); the last equation becomes:

$$a^2g_{ij} + (2ab + b^2\|B\|^2_g)B_iB_j = \mu g_{ij} \tag{4.7}$$

with $\|B\|_g$ the norm of 1-form B with respect to g i.e. $\|B\|^2_g = g^{ij}B_iB_j$. Since the rank of g_{ij} is n while the rank of B_iB_j is $n - 1$ it results:

$$\begin{cases} a^2 = \mu, \\ 2ab + b^2\|B\|^2_g = 0 \end{cases} \tag{4.8}$$

and then we get the solution:

$$\begin{cases} a = -\dfrac{b}{2}\|B\|^2_g, \\ \mu = \dfrac{b^2}{4}\|B\|^4_g \end{cases} \tag{4.9}$$

which pointed out that in addition to the trivial case (g, ag) corresponding to the conformal class of g (when $a = \pm\sqrt{\mu}$) we have another 1-parametric case corresponding to $b > 0$. An important remark is that in the last case $a < 0$ while the case of positive definite Beil metrics (4.1) implies $a > 0$.

In dimension two we present the general solution of (4.6) with prescribed diagonal (positive definite) metric $a = \mathrm{diag}(a_{11}, a_{22})$. Namely, let u and v be smooth functions with $v \neq 0$. Then:

$$H_{u,v} = \sqrt{\mu}\begin{pmatrix} ua_{11} & \dfrac{1-u^2}{v}a_{11} \\ va_{22} & -ua_{22} \end{pmatrix} \tag{4.10}$$

is the solution of (4.6). Let us remark that $\det H_{u,v} = -\mu a_{11}a_{22} = -\mu \det a$ and then $\det H_{u,v}$ is independent of u and v and strictly negative.

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