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NONHOLONOMIC DYNAMICS OF SECOND-ORDER AND THE HEISENBERG SPINNING PARTICLE

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The equations of motion for the associated constrained Lagrangian to a nonholonomic Lagrangian of second-order are computed. The spinning particle subject to the Heisenberg constraint is treated as example and its dynamics is completely described.

Keywords: Lagrangian of second-order; Euler–Lagrange equations; nonholonomic constraints; constrained Lagrangian; spinning particle; Heisenberg constraint.

Mathematics Subject Classification: 70H25, 70H35, 58F05

0. Introduction

In the last years it has been shown an increasing interest in nonholonomic mechanics especially from a geometrical point of view. Following the methodology of [1], where the nonholonomic Lagrangians of first-order are treated, we obtain the equations of motion of a nonholonomic dynamical system involving accelerations by using the associated constrained Lagrangian. A very interesting fact is that these dynamical equations involve the curvature of the horizontal distribution characterized by the nonholonomic constraints; this curvature is non-vanishing due to the nonholonomic character of the constraints.

This type of Lagrangian is illustrated in Sec. 2 by the spinning particle subject to the Heisenberg constraint, [1, p. 29]. The corresponding dynamical system will be called then Heisenberg spinning particle. Let us note that the free spinning particle is a 12-dimensional system while this Heisenberg spinning particle is a nine-dimensional one and the complete solution is presented in Sec. 3.

This paper is dedicated to the Academician Radu Miron on his 85th anniversary since he devoted more than fifty years to the subject of nonholonomic geometry (see [4, 6]).

1. Equations of Motion in Second-Order Nonholonomic Dynamics

The starting point of our approach is a configuration space given by an n-dimensional manifold Q, for which we consider the tangent bundle of order two T^2Q (see [3, p. 4139; 5]). The coordinates $(q^i)_{1\leq i\leq n}$ on Q yield the induced coordinates $(q^i, q^{i(1)} = \frac{dq^i}{dt}, q^{i(2)} = \frac{d^2q^i}{dt^2})$ on T^2Q.

Let us suppose that the evolution of the considered dynamical system is described by the following objects:

(1) a second-order Lagrangian, that is a smooth map $L : T^2Q \longrightarrow \mathbb{R}$ (see [3, p. 4139; 5]),
(2) a set of m independent one-forms $\{\omega^a(q)\}_{1\leq a\leq m}$ whose vanishing gives the constraints of the system.

These one-forms defines an $(n-m)$-dimensional distribution D on Q i.e. $\{\omega^a\}$ is a local basis for the annihilator D^0 of D. Also, these constraints means that the only allowable velocities are the tangent vectors belonging to D or, in other words, the motion is constrained to the distribution D.

The Lagrangian L gives the Euler–Lagrange equations of order two (see [3, p. 4140]):

$$\delta L = (\mathrm{EL})_i^{\mathrm{free}} \delta q^i = 0 \tag{1.1a}$$

with classical Euler–Lagrange equations:

$$(\mathrm{EL})_i^{\mathrm{free}} = \frac{\partial L}{\partial q^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial q^{i(1)}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial q^{i(2)}}\right) \tag{1.1b}$$

and supposing that the constraints are of nonholonomic type, we can choose a local coordinate chart and a local basis for the constraints such that (see [1, p. 217]):

$$\omega^a(q) = ds^a + \overset{1}{A}{}^a_\alpha (r,s)dr^\alpha, \quad 1 \leq a \leq m, \tag{1.2}$$

where $q = (r,s) \in \mathbb{R}^{n-m} \times \mathbb{R}^m$ and $\alpha \in \{1,\ldots,n-m\}$.

From (1.2) it results that:

$$\delta s^a + \overset{1}{A}{}^a_\alpha \,\delta r^\alpha = 0 \tag{1.3}$$

which, by substitution into (1.1) yields:

$$\frac{\partial L}{\partial r^\alpha} - \frac{d}{dt}\left(\frac{\partial L}{\partial r^{\alpha(1)}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial r^{\alpha(2)}}\right) = \overset{1}{A}{}^a_\alpha \left[\frac{\partial L}{\partial s^a} - \frac{d}{dt}\left(\frac{\partial L}{\partial s^{a(1)}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial s^{a(2)}}\right)\right]. \tag{1.4}$$

Equations (1.4) combined with the constraint equations:

$$s^{a(1)} = -\overset{1}{A}{}^{a}_{\alpha}\, r^{\alpha(1)}, \tag{1.5a}$$

$$s^{a(2)} = -\frac{d}{dt}(\overset{1}{A}{}^{a}_{\alpha})r^{\alpha(1)} - \overset{1}{A}{}^{a}_{\alpha}\, r^{\alpha(2)} \tag{1.5b}$$

gives a complete description of the equations of motion; notice that they consist of $(n-m)$ fourth-order equations and m first-order equations. Remark that another form for (1.5b) is:

$$s^{a(2)} = \overset{2}{A}{}^{a}_{\alpha\beta}\, r^{\alpha(1)}r^{\beta(1)} - \overset{1}{A}{}^{a}_{\alpha}\, r^{\alpha(2)}, \tag{1.5b$'$}$$

where:

$$\overset{2}{A}{}^{a}_{\alpha\beta}\,(r,s) = \frac{\partial \overset{1}{A}{}^{a}_{\alpha}}{\partial s^b}\,\overset{1}{A}{}^{b}_{\beta} - \frac{\partial \overset{1}{A}{}^{a}_{\alpha}}{\partial r^\beta}. \tag{1.6}$$

Following [1, p. 31] we define an associated *constrained* Lagrangian L_c by substituting the constraints (1.5) into the Lagrangian L:

$$\begin{aligned} &L_c(r^\alpha, s^a, r^{\alpha(1)}, r^{\alpha(2)}) \\ &\quad := L(r^\alpha, s^a, r^{\alpha(1)}, -\overset{1}{A}{}^{a}_{\alpha}\, r^{\alpha(1)}, r^{\alpha(2)}, \overset{2}{A}{}^{a}_{\alpha\beta}\, r^{\alpha(1)}r^{\beta(1)} - \overset{1}{A}{}^{a}_{\alpha}\, r^{\alpha(2)}). \end{aligned} \tag{1.7}$$

A direct coordinates calculation get:

$$\begin{aligned} \frac{\partial L_c}{\partial r^\alpha} &= \frac{\partial L}{\partial r^\alpha} - \frac{\partial L}{\partial s^{b(1)}}\frac{\partial \overset{1}{A}{}^{b}_{\beta}}{\partial r^\alpha} r^{\beta(1)} \\ &\quad + \frac{\partial L}{\partial s^{b(2)}}\left(\frac{\partial \overset{2}{A}{}^{b}_{\beta\gamma}}{\partial r^\alpha} r^{\beta(1)}r^{\gamma(1)} - \frac{\partial \overset{1}{A}{}^{b}_{\beta}}{\partial r^\alpha} r^{\beta(2)}\right), \end{aligned} \tag{1.8a}$$

$$\begin{aligned} \frac{\partial L_c}{\partial s^a} &= \frac{\partial L}{\partial s^a} - \frac{\partial L}{\partial s^{b(1)}}\frac{\partial \overset{1}{A}{}^{b}_{\beta}}{\partial s^a} r^{\beta(1)} \\ &\quad + \frac{\partial L}{\partial s^{b(2)}}\left(\frac{\partial \overset{2}{A}{}^{b}_{\beta\gamma}}{\partial s^a} r^{\beta(1)}r^{\gamma(1)} - \frac{\partial \overset{1}{A}{}^{b}_{\beta}}{\partial s^a} r^{\beta(2)}\right), \end{aligned} \tag{1.8b}$$

$$\frac{\partial L_c}{\partial r^{\alpha(1)}} = \frac{\partial L}{\partial r^{\alpha(1)}} - \frac{\partial L}{\partial s^{b(1)}}\,\overset{1}{A}{}^{b}_{\alpha} + \frac{\partial L}{\partial s^{b(2)}}(\overset{2}{A}{}^{b}_{\alpha\beta} + \overset{2}{A}{}^{b}_{\beta\alpha})r^{\beta(1)}, \tag{1.8c}$$

$$\frac{\partial L_c}{\partial r^{\alpha(2)}} = \frac{\partial L}{\partial r^{\alpha(2)}} - \frac{\partial L}{\partial s^{b(2)}}\,\overset{1}{A}{}^{b}_{\alpha}\,. \tag{1.8d}$$

A long, but straightforward computation gives the equations of motion for L_c:

$$(\mathrm{EL})^c_\alpha = \left(\frac{\partial L}{\partial s^{b(1)}} - \frac{d}{dt}\left(\frac{\partial L}{\partial s^{b(2)}}\right)\right)\overset{1}{B}{}^{b}_{\alpha\beta}\, r^{\beta(1)} + \frac{\partial L}{\partial s^{b(2)}}\,\overset{2}{B}{}^{b}_{\alpha\beta\gamma}\, r^{\beta(1)}r^{\gamma(1)}, \tag{1.9a}$$

where:

$$(\mathrm{EL})^c_\alpha = \frac{\partial L_c}{\partial r^\alpha} - \frac{d}{dt}\left(\frac{\partial L_c}{\partial r^{\alpha(1)}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L_c}{\partial r^{\alpha(2)}}\right) - \overset{1}{A}{}^a_\alpha \frac{\partial L_c}{\partial s^a}, \tag{1.9b}$$

$$\overset{1}{B}{}^b_{\alpha\beta} = \overset{2}{A}{}^b_{\beta\alpha} - \overset{2}{A}{}^b_{\alpha\beta}, \tag{1.10a}$$

$$\overset{2}{B}{}^b_{\alpha\beta\gamma} = \frac{\partial \overset{2}{A}{}^b_{\beta\gamma}}{\partial r^\alpha} - \frac{\partial \overset{2}{A}{}^b_{\beta\alpha}}{\partial r^\gamma} + \overset{1}{A}{}^a_\gamma \frac{\partial \overset{2}{A}{}^b_{\beta\alpha}}{\partial s^a} - \overset{1}{A}{}^a_\alpha \frac{\partial \overset{2}{A}{}^b_{\beta\gamma}}{\partial s^a}. \tag{1.10b}$$

Several remarks are necessary:

(1) The coefficients B does not depend on Lagrangian but only of constraints.

(2) Let $\{\frac{\delta}{\delta r^\alpha} := \frac{\partial}{\partial r^\alpha} - \overset{1}{A}{}^a_\alpha \frac{\partial}{\partial s^a}\}$ be the dual basis of $\{\omega^a\}$. Then:

$$\left[\frac{\delta}{\delta r^\alpha}, \frac{\delta}{\delta r^\beta}\right] = \left(\frac{\delta \overset{1}{A}{}^a_\alpha}{\delta r^\beta} - \frac{\delta \overset{1}{A}{}^a_\beta}{\delta r^\alpha}\right)\frac{\partial}{\partial s^a} = \overset{1}{B}{}^a_{\alpha\beta}\frac{\partial}{\partial s^a}. \tag{1.11}$$

So, by vanishing of $\overset{1}{B}{}^a_{\alpha\beta}$ we get the integrability of the distribution D and then $\overset{1}{B}$ is *the curvature* of *the nonlinear connection* spanned by $\{\frac{\delta}{\delta r^\alpha}\}$. Also, (1.9b) becomes:

$$(\mathrm{EL})^c_\alpha = \frac{\delta L_c}{\delta r^\alpha} - \frac{d}{dt}\left(\frac{\partial L_c}{\partial r^{\alpha(1)}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L_c}{\partial r^{\alpha(2)}}\right). \tag{1.9b$'$}$$

(3) Similarly, we have:

$$\overset{2}{B}{}^a_{\alpha\beta\gamma} = \frac{\delta \overset{2}{A}{}^a_{\beta\gamma}}{\delta r^\alpha} - \frac{\delta \overset{2}{A}{}^a_{\beta\alpha}}{\delta r^\gamma} \tag{1.12}$$

for:

$$\overset{2}{A}{}^a_{\alpha\beta} = -\frac{\delta \overset{1}{A}{}^b_\alpha}{\delta r^\beta}. \tag{1.13}$$

In conclusion:

$$\overset{2}{B}{}^a_{\alpha\beta\gamma} = \left[\frac{\delta}{\delta r^\alpha}, \frac{\delta}{\delta r^\gamma}\right](\overset{1}{A}{}^a_\beta) = \overset{1}{B}{}^b_{\alpha\gamma}\frac{\partial \overset{1}{A}{}^a_\beta}{\partial s^b} \tag{1.14}$$

and then $\overset{1}{B}{}^b_{\alpha\alpha} = \overset{2}{B}{}^b_{\alpha\beta\alpha} = 0$ for every α.

(4) If $L = L(q, \dot{q})$ is a first-order Lagrangian then (1.9a) reduces to the equation (5.2.7) of [1, p. 217].

2. Example: The Heisenberg Spinning Particle

According to [3, p. 4147] the Lagrangian of classical spinning particle is:

$$L(q, q^{(1)}, q^{(2)}) = \frac{1}{2}\sum_{i=1}^{3}(q^{i(1)})^2 - \frac{1}{2}\sum_{i=1}^{3}(q^{i(2)})^2 \tag{2.1}$$

on $Q = \mathbb{R}^3$ where we will use the classical notation $(q^i) = (x, y, z)$.

The Euler–Lagrange equations for the free Lagrangian (2.1) are:

$$(\mathrm{EL})_i^{\mathrm{free}} := \frac{d^2q^i}{dt^2} + \frac{d^4q^i}{dt^4} = 0, \quad 1 \leq i \leq 3. \tag{2.2}$$

Consider the nonholonomic constraint of Heisenberg-type (see [1, p. 29]):

$$z^{(1)} = yx^{(1)} - xy^{(1)} \tag{2.3}$$

which gives:

$$z^{(2)} = yx^{(2)} - xy^{(2)}, \tag{2.4a}$$

$$\overset{1\ 1}{A_1} = -y, \quad \overset{1\ 1}{A_2} = x, \quad \overset{1\ 1}{B_{12}} = -2. \tag{2.4b}$$

The constrained Lagrangian is:

$$L_c(y, x^{(1)}, y^{(1)}, x^{(2)}, y^{(2)}) = \frac{1}{2}[{x^{(1)}}^2 + {y^{(1)}}^2 + (yx^{(1)} - xy^{(1)})^2] - \frac{1}{2}[{x^{(2)}}^2 + {y^{(2)}}^2 + (yx^{(2)} - xy^{(2)})^2]. \tag{2.5}$$

We have: $m = 1, s^1 = z, r^1 = x, r^2 = y, \overset{2\ 1}{A_{12}} = 1 = -\overset{2\ 1}{A_{21}}$ and:

$$\frac{\partial L_c}{\partial x} = y^{(2)}z^{(2)} - y^{(1)}z^{(1)}, \quad \frac{\partial L_c}{\partial y} = x^{(1)}z^{(1)} - x^{(2)}z^{(2)}, \quad \frac{\partial L_c}{\partial z} = 0, \tag{2.6a}$$

$$\frac{\partial L_c}{\partial x^{(1)}} = x^{(1)} + yz^{(1)}, \quad \frac{\partial L_c}{\partial y^{(1)}} = y^{(1)} - xz^{(1)}, \tag{2.6b}$$

$$\frac{\partial L_c}{\partial x^{(2)}} = -x^{(2)} - yz^{(2)}, \quad \frac{\partial L_c}{\partial y^{(2)}} = -y^{(2)} + xz^{(2)}, \tag{2.6c}$$

where $z^{(2)}$ is given by (2.4a). Also:

$$z^{(3)} = yx^{(3)} + y^{(1)}x^{(2)} - xy^{(3)} - x^{(1)}y^{(2)}, \tag{2.7a}$$

$$z^{(4)} = 2y^{(1)}x^{(3)} + yx^{(4)} - 2x^{(1)}y^{(3)} - xy^{(4)}. \tag{2.7b}$$

Therefore:

$$\frac{\partial L_c}{\partial x} - \frac{d}{dt}\left(\frac{\partial L_c}{\partial x^{(1)}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L_c}{\partial x^{(2)}}\right) - \overset{1\ 1}{A_1}\frac{\partial L_c}{\partial z} = -x^{(4)} - yz^{(2)} - x^{(2)} - 2y^{(1)}z^{(1)} - 2y^{(1)}z^{(3)} - yz^{(4)} \tag{2.8a}$$

$$\frac{\partial L_c}{\partial y} - \frac{d}{dt}\left(\frac{\partial L_c}{\partial y^{(1)}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L_c}{\partial y^{(2)}}\right) - \overset{1}{A}{}^1_2 \frac{\partial L_c}{\partial z}$$
$$= -y^{(4)} - y^{(2)} + 2x^{(1)}z^{(1)} + xz^{(2)} + 2x^{(1)}z^{(3)} + xz^{(4)}. \tag{2.8b}$$

The right-hand side of (1.9a) is:

$$(\mathrm{EL})^c_1 = -2y^{(1)}(z^{(1)} + z^{(3)}), \tag{2.9a}$$
$$(\mathrm{EL})^c_2 = 2x^{(1)}(z^{(1)} + z^{(3)}), \tag{2.9b}$$

since $\overset{2}{B} = 0$ and then the Eqs. (1.9a) and (1.9b) gives:

$$(\mathrm{EL})^c_1 : (1+y^2)\left(\frac{d^4x}{dt^4} + \frac{d^2x}{dt^2}\right) - xy\left(\frac{d^2y}{dt^2} + \frac{d^4y}{dt^4}\right)$$
$$+ 2y\left(\frac{dy}{dt}\frac{d^3x}{dt^3} - \frac{dx}{dt}\frac{d^3y}{dt^3}\right) = 0, \tag{2.10a}$$
$$(\mathrm{EL})^c_2 : (1+x^2)\left(\frac{d^4y}{dt^4} + \frac{d^2y}{dt^2}\right) - xy\left(\frac{d^2x}{dt^2} + \frac{d^4x}{dt^4}\right)$$
$$- 2x\left(\frac{dy}{dt}\frac{d^3x}{dt^3} - \frac{dx}{dt}\frac{d^3y}{dt^3}\right) = 0. \tag{2.10b}$$

Another form of these equation is:

$$\begin{cases} x^{(2)} + x^{(4)} = \dfrac{-2y}{1+x^2+y^2}(y^{(1)}x^{(3)} - x^{(1)}y^{(3)}), \\ y^{(2)} + y^{(4)} = \dfrac{2x}{1+x^2+y^2}(y^{(1)}x^{(3)} - x^{(1)}y^{(3)}). \end{cases} \tag{2.11}$$

With notations of [3, p. 4147] the free Euler–Lagrange equations have the first integrals:

$$p_1 = x^{(1)} + x^{(3)}, \quad p_2 = y^{(1)} + y^{(3)}, \quad p_3 = z^{(1)} + z^{(3)}, \tag{2.12}$$

provided by the fact that the commutative group $\mathbb{R}^3$ is a Lie group of symmetries of L. The constrained equations (2.10) can be written as:

$$\begin{cases} \dfrac{dp_1}{dt} = -y\dfrac{dp_3}{dt}, \\ \dfrac{dp_2}{dt} = x\dfrac{dp_3}{dt}. \end{cases} \tag{2.13}$$

Both the Lagrangian L and the constraint (2.3) are invariant by z-translations in $\mathbb{R}^3$ i.e to the action $(\mathbb{R}, +) \times \mathbb{R}^3 \to \mathbb{R}^3$, $(\lambda, x, y, z) \to (x, y, z + \lambda)$; this means that the Heisenberg spinning particle is a $\mathbb{R}$-*Chaplygin system* conform [2, p. 103].

3. The Complete Dynamics of the Heisenberg Spinning Particle

The complete solution of the free spinning particle (2.2) is:

$$\begin{cases} x(t) = x_{01}\cos t + x_{02}\sin t + x_{03}t + x_{04}, \\ y(t) = y_{01}\cos t + y_{02}\sin t + y_{03}t + y_{04}, \\ z(t) = z_{01}\cos t + z_{02}\sin t + z_{03}t + z_{04} \end{cases} \tag{3.1}$$

with all $x_{0.}, y_{0.}, z_{0.}$ real constants. The presence of the trigonometric functions as well as the t-part of this solution show indeed that the particle describe a rotation movement along a center in translation.

The Heisenberg constraint (2.3) reads:

$$\begin{aligned} -z_{01}\sin t + z_{02}\cos t + z_{03} &= (y_{01}\cos t + y_{02}\sin t + y_{03}t + y_{04})(-x_{01}\sin t + x_{02}\cos t + x_{03}) \\ &\quad - (x_{01}\cos t + x_{02}\sin t + x_{03}t + x_{04})(-y_{01}\sin t + y_{02}\cos t + y_{03}). \end{aligned} \tag{3.2}$$

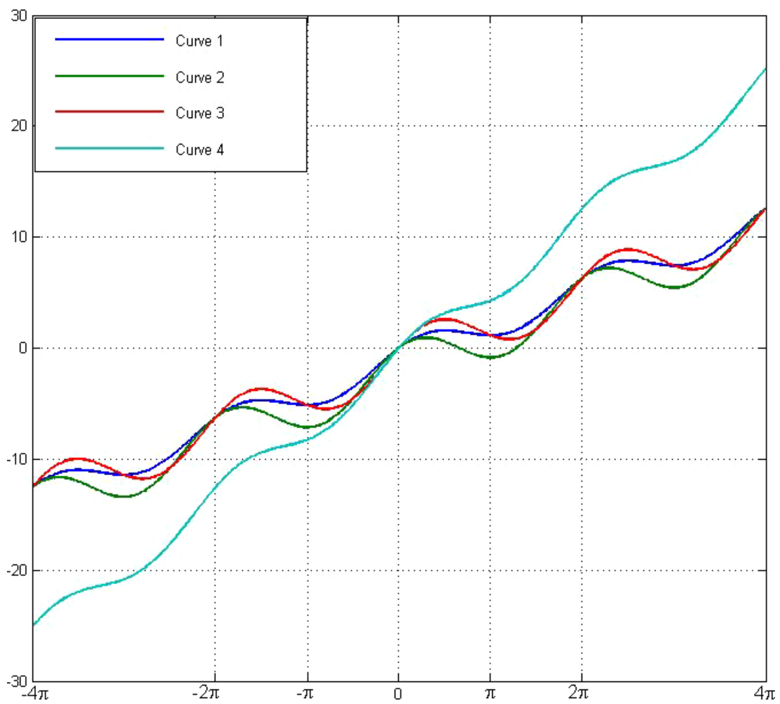

Fig. 1. The graph for $x(t)$.

In the right-hand side of this equation we have the term $(x_{03}y_{01} - x_{01}y_{03})t\sin t + (x_{02}y_{03} - x_{03}y_{02})t\cos t$ and then by vanishing of this expression we derive the existence of a real scalar λ such that:

$$\frac{y_{01}}{x_{01}} = \frac{y_{03}}{x_{03}} = \frac{y_{02}}{x_{02}} = \lambda. \tag{3.3}$$

In conclusion, the general dynamics is:

$$\begin{cases} x(t) = x_{01}\cos t + x_{02}\sin t + x_{03}t + x_{04}, \\ y(t) = \lambda x(t) + y_{04} - \lambda x_{04}, \\ z(t) = (y_{04} - \lambda x_{04})(x(t) - x_{04}) + z_{04} \end{cases} \tag{3.4}$$

which means that the Heisenberg spinning particle is a seven-dimensional dynamical system.

Suppose that the initial position is the origin $\bar{0} = (0, 0, 0)$ of $\mathbb{R}^3$. Then (3.4) becomes via $x_{01} + x_{04} = 0$ and $y_{01} + y_{04} = 0$:

$$\begin{cases} x(t) = x_{01}(\cos t - 1) + x_{02}\sin t + x_{03}t, \\ y(t) = \lambda x(t), \\ z(t) = 0 \end{cases} \tag{3.5}$$

which means that the dynamical system modeling this Heisenberg spinning particle has four degrees of freedom. An important remark here is that the trajectory is a line in the xOy plane.

Of course, the relation (3.5) is written according to the non-vanishing of the coefficients x_{01}, x_{02} and x_{03}. The curve $t \to x(t)$ is plotted in Fig. 1 for various values of these coefficients. For the first curve we have chosen $x_{01} = x_{02} = x_{03} = 1$, for the second curve $x_{01} = 2, x_{02} = x_{03} = 1$, for the third one $x_{02} = 2, x_{01} = x_{03} = 1$ and $x_{01} = x_{02} = 1, x_{03} = 2$ for the last curve.

References

[1] A. M. Bloch, *Nonholonomic Mechanics and Control*, With the collaboration of J. Baillieul, P. Crouch and J. Marsden. With scientific input from P. S. Krishnaprasad, R. M. Murray and D. Zenkov, Interdisciplinary Applied Mathematics, Vol. 24. Systems and Control (Springer-Verlag, New York, 2003), MR1978379 (2004e:37099).

[2] J. C. Monforte, *Geometric, Control and Numerical Aspects of Nonholonomic Systems*, Lecture Notes in Mathematics, Vol. 1793 (Springer-Verlag, Berlin, 2002), MR1942617 (2003k:70013).

[3] M. de León and D. M. de Diego, Symmetries and constants of motion for higher-order Lagrangian systems, *J. Math. Phys.* **36** (1995) 4138–4161.

[4] R. Miron, Sur la sphére non holonome et le plan non holonome, An. Ştinţ Univ. Al. I. Cuza Iasi, Sect. I. (N.S.) **1** (1955) 43–52, MR0087132 (19,306f).

[5] R. Miron, *The Geometry of Higher-Order Lagrange Spaces*: *Applications to Mechanics and Physics*, Fundamental Theories of Physics, Vol. 82 (Kluwer Academic Publishers, 1997).

[6] B. T. Hassan and R. Miron, Remarkable non-holonomic manifolds in the geometry of Finsler spaces, *Tensor* **70**(2) (2008) 135–145, MR2546910 (2011a:53148).