A type of first integrals for solenoidal vector fields

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Abstract

In this paper we obtain a type of conservation laws for free-divergence vector fields. An application to 2D isotropic harmonic oscillator is given.

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Introduction

The very well-known way to obtain conservation laws for a system of differential equations is Noether theorem([1]) which associates to every symmetry a conservation law. G. L. Jones gives in [3] another method based on a weaker generalization of notion of symmetry, namely *pseudosymmetries*. The advantages of this method are that it does not require any integration(if there are associated some natural invariants, see the free-divergence case below) and does not require, as Noether theorem, that the differential equations follow from a variational principle.

In this paper we present a generalization of Jones result. In a further paper, we extend these results to Hamiltonian systems.

1 From pseudosymmetries to conservation laws

Let M be a smooth, n-dimensional manifold, $C^\infty(M)$ the ring of real-valued smooth functions, $\mathcal{X}(M)$ the Lie algebra of vector fields and $\Omega^p(M)$ the $C^\infty(M)$-module of p-differential forms, $1 \leq p \leq n$.

For $X \in \mathcal{X}(M)$ with local expression $X = X^i(x)\frac{\partial}{\partial x^i}$ one consider the system of differential equations which give the flow of X:

$$(1.1) \quad \dot{x}^i(t) = \frac{dx^i}{dt}(t) = X^i\left(x^1(t), \ldots, x^n(t)\right), \qquad i = 1, \ldots, n.$$

A solution of (1.1) is called *integral curve* of X.

Definition 1.1 A function $f \in C^\infty(M)$ is called *conservation law*(or *first integral,* or *constant of motion,* or *invariant function*) for X or (1.1) if f is constant along the solutions of (1.1) that is $\frac{d(f\circ c)}{dt}(t) = 0$ for every integral curve $c(t)$ of X.

Because for $f \in C^\infty(M)$ its rate of change along (1.1) is: $\frac{df}{dt} = \frac{\partial f}{\partial x^i}\dot{x}^i = \frac{\partial f}{\partial x^i}X^i = \mathcal{L}_X f$ where the right-hand side means *the Lie derivative of f with respect to X* we get:

Proposition 1.2 *$f \in C^\infty(M)$ is conservation law for* (1.1) *if and only if:*

$$(1.2) \quad \mathcal{L}_X f = 0.$$

For our approach is necessary the following:

Definition 1.3 (i) $Y \in \mathcal{X}(M)$ is called *symmetry for X* if:

$$(1.3) \quad \mathcal{L}_X Y = 0.$$

(ii) If $Y \in \mathcal{X}(M)$ is fixed then $Z \in \mathcal{X}(M)$ is called *Y-pseudosymmetry for X* if there exists $f \in C^\infty(M)$ such that:

$$(1.4) \quad \mathcal{L}_X Z = fY.$$

(iii) $\omega \in \Omega^p(M)$ is called *invariant form for X* if:

$$(1.5) \quad \mathcal{L}_X \omega = 0.$$

Remark 1.4 (i) A 0-pseudosymmetry is obviously a symmetry.
(ii) A X-pseudosymmetry for X is called *pseudosymmetry for X* in [3, p.

1055] and *Lie point symmetry* in [5, p. 25].
(iii) If in (1.4) f is constant then $\mathcal{L}_X Z$ is symmetry for X.
(iv) If in (1.4) f is not constant then $\mathcal{L}_X Z$ is symmetry for X if and only if f is conservation law for X.

The result which give the association between pseudosymmetries and conservation laws is:

Theorem 1.5 *Let $X \in \mathcal{X}(M)$ be a fixed vector field and $\omega \in \Omega^p(M)$ be a p-form invariant for X. If $Y \in \mathcal{X}(M)$ is symmetry for X and $S_1, \ldots, S_{p-1} \in \mathcal{X}(M)$ are $(p-1)$ Y-pseudosymmetries for X then:*

$$(1.6) \qquad \phi = \omega\left(S_1, \ldots, S_{p-1}, Y\right)$$

or locally:

$$(1.7) \qquad \phi = S_1^{i_1} \ldots S_{p-1}^{i_{p-1}} Y^{i_p} \omega_{i_1 i_2 \ldots i_p}$$

is a conservation law for X. Particularly, if $Y, S_1, \ldots, S_{p-1}$ are symmetries for X then ϕ given by (1.6) *is conservation law.*

Proof Applying the properties of Lie derivatives one have:

$$\mathcal{L}_X \phi = \left(\mathcal{L}_X S_1\right)^{i_1} \ldots + S_1^{i_1}\left(\mathcal{L}_X S_2\right)^{i_2} \ldots + \ldots + \ldots \left(\mathcal{L}_X Y\right)^{i_p} \omega_{\ldots} + \ldots Y^{i_p}\left(\mathcal{L}_X \omega\right)_{i_1 \ldots i_p}.$$

In this relation each of the first $p-1$ terms has a factor of the form:

$$\mathcal{L}_X S_j = \lambda_j Y$$

so that ω is contracted with two factors of Y and then each term vanishes by the antisymmetry of ω. The p-th term and $p+1$-th term vanishes since (1.3) and (1.5). □

Remark 1.6 (i) If the pseudosymmetries are linearly dependent then $\phi = 0$ by the antisymmetry of ω.
(ii) For $Y = X$ one obtain the main result of G. L. Jones([3, p. 1056]).
(iii) If $p = 1$ one obtain theorem 2.5.10 of ten Eikelder([2, p. 48]).
(iv) The fact that the pseudosymmetries (1.4) with f =constant can be used to integrate planar($n = 2$) vector fields can be found in [6, p. 37-38, relation 8.13].

2 Vector fields with a special invariant 2-form

Let $M = \mathbf{R}^{2m} = \{(x,y) = (x^i, y^i)_{i=1,\dots,m}\}$ that is $n = 2m$, and let X with the form:

$$(2.1) \qquad X = X^i(x,y)\frac{\partial}{\partial x^i} + \tilde{X}^i(x,y)\frac{\partial}{\partial y^i}.$$

Let us consider the 2-form $\omega = (\omega_{ij})$ given by:

$$(2.2) \qquad \omega = \begin{pmatrix} 0_m & 1_m \\ -1_m & 0_m \end{pmatrix}$$

where 0_m is the null matrix and 1_m is the identity matrix of order m.

A straightforward computation give:

$$(2.3a) \qquad (\mathcal{L}_X\omega)\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \frac{\partial \tilde{X}^i}{\partial x^j} - \frac{\partial \tilde{X}^j}{\partial x^i}$$

$$(2.3b) \qquad (\mathcal{L}_X\omega)\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = \frac{\partial X^j}{\partial y^i} - \frac{\partial X^i}{\partial y^j}$$

$$(2.3c) \qquad (\mathcal{L}_X\omega)\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j}\right) = \frac{\partial X^j}{\partial x^i} + \frac{\partial \tilde{X}^i}{\partial y^j}$$

$$(2.3d) \qquad (\mathcal{L}_X\omega)\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial x^j}\right) = -\frac{\partial X^i}{\partial x^j} - \frac{\partial \tilde{X}^j}{\partial y^i}.$$

As consequence we obtain:

Proposition 2.1 *If X with expression* (2.1) *satisfy:*

$$(2.4a) \qquad \frac{\partial X^i}{\partial y^j} = \frac{\partial X^j}{\partial y^i}$$

$$(2.4b) \qquad \frac{\partial \tilde{X}^i}{\partial x^j} = \frac{\partial \tilde{X}^j}{\partial x^i}$$

$$(2.4c) \qquad \frac{\partial X^j}{\partial x^i} + \frac{\partial \tilde{X}^i}{\partial y^j} = 0$$

for all $i, j = 1, \dots, m$ then ω is 2-form invariant for X.

Applying the theorem 1.5 we have:

Proposition 2.2 *If X satisfy $(2.4a)-(2.4c)$, $Y\in\mathcal{X}(M)$ is symmetry for X and $S\in\mathcal{X}(M)$ is Y-pseudosymmetry for X then:*

$$(2.5)\quad \phi=\omega(S,Y)$$

is a conservation law for X where ω is given by (2.2). Particularly if Y,S are symmetries for X then ϕ given by (2.5) is conservation law.

If $Y=Y^i\frac{\partial}{\partial x^i}+\tilde{Y}^i\frac{\partial}{\partial y^i}$ and $S=S^i\frac{\partial}{\partial x^i}+\tilde{S}^i\frac{\partial}{\partial y^i}$ then:

$$\phi=\left(S,\tilde{S}\right)\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}Y\\\tilde{Y}\end{pmatrix}=\left(-\tilde{S},S\right)\begin{pmatrix}Y\\\tilde{Y}\end{pmatrix}=S\tilde{Y}-\tilde{S}Y$$

that is:

$$(2.6)\quad \phi=S^i\tilde{Y}^i-\tilde{S}^iY^i$$

with summation after $i=1,\ldots,m$.

Corollary 2.3 *If X satisfy $(2.4a)-(2.4c)$ and $S\in\mathcal{X}(M)$ is a pseudosymmetry for X then:*

$$(2.7)\quad \phi=\omega(S,X)=S^i\tilde{X}^i-\tilde{S}^iX^i$$

is a conservation law for X.

Remark 2.4 (i) If in relation $(2.4c)$ one makes $i=j$ and take the sum after $i=1,\ldots,m$ then *the divergence of* X vanishes:

$$(2.8)\quad divX:=\sum_{i=1}^{m}\left(\frac{\partial X^i}{\partial x^i}+\frac{\partial\tilde{X}^i}{\partial y^i}\right)=0$$

that is X is *divergence-free*(or *solenoidal* or *source-free*) vector field.
(ii) Every solenoidal vector field in dimension $n=2$, that is $m=1$, satisfy $(2.4a)-(2.4c)$. Applications for divergenceless vector fields in a odd dimension, namely $n=3$, are given in [3, p. 1056].

3 An example

Let the 2-dimensional istropic harmonic oscillator:

$$\begin{aligned}(3.1a)\quad \ddot{q}^1+\omega^2q^1&=0\\(3.1b)\quad \ddot{q}^2+\omega^2q^2&=0\end{aligned}$$

a toy model for many methods to finding conservation laws.

The Lagrangian is:

$$(3.2) \quad L = \frac{1}{2}\left[\left(\dot{q}^1\right)^2 + \left(\dot{q}^2\right)^2\right] - \frac{\omega^2}{2}\left[\left(q^1\right)^2 + \left(q^2\right)^2\right]$$

and then applying the conservation of energy H(L is time-independent) we have two conservation laws:

$$\begin{aligned} (3.3a) \quad \phi_1 &= \left(\dot{q}^1\right)^2 + \omega^2\left(q^1\right)^2 \\ (3.3b) \quad \phi_2 &= \left(\dot{q}^2\right)^2 + \omega^2\left(q^2\right)^2 . \end{aligned}$$

A straightforward computation give the Noetherian conservation law([1, p. 192]):

$$(3.4) \quad \phi_3 = q^2\dot{q}^1 - q^1\dot{q}^2.$$

But we can obtain a nonnoetherian conservation law with symmetries. The vector field of (3.1) is:

$$(3.5) \quad X = \dot{q}^1\frac{\partial}{\partial q^1} + \dot{q}^2\frac{\partial}{\partial q^2} - \omega^2 q^1\frac{\partial}{\partial \dot{q}^1} - \omega^2 q^2\frac{\partial}{\partial \dot{q}^2}$$

which is solenoidal, and another calculus give that:

$$(3.6) \quad Y = \dot{q}^2\frac{\partial}{\partial q^1} + \dot{q}^1\frac{\partial}{\partial q^2} - \omega^2 q^2\frac{\partial}{\partial \dot{q}^1} - \omega^2 q^1\frac{\partial}{\partial \dot{q}^2}$$

is a symmetry for X. Also, because X is total 1-homogeneous, i.e. with respect to all variables $(q, \dot{q})$ it result that:

$$(3.7) \quad Z = q^1\frac{\partial}{\partial q^1} + q^2\frac{\partial}{\partial q^2} + \dot{q}^1\frac{\partial}{\partial \dot{q}^1} + \dot{q}^2\frac{\partial}{\partial \dot{q}^2}$$

is symmetry for X. We have: $\phi = \omega(X, Y) = 0, \quad \phi = \omega(X, Z) = 2H$ i.e. we not obtain new conservation law. But:

$$(3.8) \quad \phi_4 = \omega(Y, Z) = \dot{q}^1\dot{q}^2 + \omega^2 q^1 q^2$$

is a new conservation law given by proposition 2.2. We remark that ϕ_4 represent the energy of a new Lagrangian of (3.1), that is:

$$(3.9) \quad L^* = \dot{q}^1\dot{q}^2 - \omega^2 q^1 q^2$$

a result very important from the point of view of Inverse Problem of Analytical Mechanics([4]). Our Lagrangian L^* appear in [4, p. 122].

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