Dedicated to Professor Radu Miron
on the occasion of his 85th birthday

ETA-RICCI SOLITONS ON HOPF HYPERSURFACES IN COMPLEX SPACE FORMS

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The Eisenhart problem of finding parallel symmetric covariant tensor fields is solved in the framework of Hopf hypersurfaces. The results are interpreted in terms of η-Ricci solitons, a natural generalization of Ricci solitons for the almost contact geometry.

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1. INTRODUCTION

In 1923, Eisenhart [9] proved that if a positive definite Riemannian manifold (M, g) admits a second order parallel symmetric covariant tensor other than a constant multiple of the metric tensor, then it is reducible. In 1926, Levy [15] proved that a second order parallel symmetric non-degenerated tensor α in a space form is proportional to the metric tensor. Note that this question can be considered as the dual to the problem of finding linear connections making parallel a given tensor field, a problem which was considered by Wong in [25]. Also, the former question implies topological restrictions namely if the (pseudo) Riemannian manifold M admits a parallel symmetric $(0, 2)$ tensor field then M is locally the direct product of a number of (pseudo) Riemannian manifolds, [26] (cited by [27]). Another situation where the parallelism of α is involved, appears in the theory of totally geodesic maps, namely, as is pointed out in [17, p. 114], $\nabla\alpha = 0$ is equivalent with the fact that the identity map $1 : (M, g) \to (M, \alpha)$ is a totally geodesic map.

While both Eisenhart and Levy worked locally, Ramesh Sharma gave in [18] a global approach based on Ricci identities. In addition to space form manifolds, Sharma considered this *Eisenhart problem* in contact geometry [19]–[21], for example for K-contact manifolds in [20]. Since then, several other studies appeared in various (almost) contact manifolds: nearly Sasakian [23], α-Sasakian [7], contact metrics with nonvanishing ξ-sectional curvature [10].

Returning to the general almost contact geometry, an important example of such manifolds is provided by real hypersurfaces in Kähler manifolds. Recently, there is an increasing number of papers in this geometry especially related to hypersurfaces in complex space forms, [16]. Motivated by this fact, we study Hopf hypersurfaces satisfying two special conditions called by us *regularity* and *surjectivity* and show that a symmetric parallel tensor field of second order must be a constant multiple of the Riemannian metric. There are two remarks regarding our result: i) it is in agreement with what happens in all previously recalled contact geometries for the symmetric case, ii) it is obtained in the same manner as in Sharma's paper [18].

Our main result is connected with the recent theory of Ricci solitons, a subject included in the Hamilton-Perelman approach (and proof) of Poincaré Conjecture. Ricci solitons in contact geometry were first studied by Ramesh Sharma (in [11] and [22]) and Jong Taek Cho (in [4], [2] and [3]); a paper authored by both is [6] and also the preprint [24] is available in arXiv. In Sharma's papers the K-contact and (k, μ)-contact (including Sasakian) cases are treated; thus our treatment for the Hopf hypersurfaces seems to be new.

Our work is structured as follows. The first section is a very brief review of Hopf hypersurfaces in complex space forms, Ricci solitons and η-Ricci solitons. The Section 2 is devoted to the (symmetric case of) Eisenhart problem in our framework which is applied for η-Ricci solitons in Hopf hypersurfaces. The last section is devoted to four examples, all computed for the η-umbilical case, in correspondence with a classification result of Cho and Kimura for η-Ricci solitons on Hopf hypersurfaces, [4]. For a Hopf hypersurface of type A0, i.e., horosphere, the η-Ricci soliton is unique (for a given dimension) and expanding while for type A1 Hopf hypersurfaces we derive that η-Ricci solitons appear as a 1-parameter family. A characterization for the Hopf hypersurfaces in $\mathbb{C}^{\frac{n+1}{2}}$ as well as a discussion of Ricci solitons in real hypersurfaces is included in [8].

2. REAL HYPERSURFACES IN COMPLEX SPACE FORMS AND RICCI SOLITONS

Let M be a real hypersurface of a complex space form $M_n(c)$ and fix N a unit normal vector field on M. Let g be the Riemannian metric of M induced from the Fubini-Study metric of $M_n(c)$ and denotes by J the almost complex structure of the ambient manifold and by A the shape operator of M. Define $\xi = -JN$ and then for any vector field $X \in \mathfrak{X}(M)$ the decomposition holds: $JX = \varphi X + \eta(X)N$. The data (φ, ξ, η, g) is an almost contact metric structure on M

$$\begin{cases} \varphi^2 = -I + \eta \otimes \xi, \ \eta(\xi) = 1, \ \eta \circ \varphi = 0, \\ \varphi(\xi) = 0, \ \eta(X) = g(X, \xi), \ g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \end{cases} \tag{2.1}$$

for $X, Y \in \mathfrak{X}(M)$, where I is the identity of the tangent bundle TM. All objects are differentiable of class C^∞. Then $\xi^\perp = \ker \eta$ is *the contact subbundle* or *the holomorphic distribution*, [1, p. 439].

From the Kählerian nature of $M_n(c)$ and making use of the Gauss and Weingarten formulas, we obtain for the Levi-Civita connection ∇ of g

$$(\nabla_X \varphi)(Y) = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X \xi = \varphi AX, \tag{2.2}$$

and the curvature tensor field, [4],

$$R(X,Y)Z = \frac{c}{4}\big[g(Y,Z)X - g(X,Z)Y + g(\varphi Y, Z)\varphi X - \tag{2.3}$$

$$-g(\varphi X, Z)\varphi Y - 2g(\varphi X, Y)\varphi Z\big] + g(AY,Z)AX - g(AX,Z)AY.$$

Then

$$R(X,Y)\xi = \frac{c}{4}[\eta(Y)X - \eta(X)Y] + g(Y, A\xi)AX - g(X, A\xi)AY \tag{2.4}$$

since A is self-adjoint with respect to g

$$g(AX, Y) = g(X, AY). \tag{2.5}$$

Definition 2.1. Let ρ be a real scalar. Then M is called a *ρ-Hopf hypersurface* if ρ is an eigenvalue of A with respect to ξ, i.e., $A\xi = \rho\xi$ holds.

From the second equation of (2.2) we find that M is a Hopf hypersurface if and only if ξ is a geodesic vector field, i.e., $\nabla_\xi \xi = 0$. In the following we restrict to Hopf hypersurfaces (it follows $\rho = \eta(A\xi)$) and then

$$R(X,Y)\xi = \frac{c}{4}\left[\eta(Y)X - \eta(X)Y\right] + \rho[\eta(Y)AX - \eta(X)AY]\,. \tag{2.6}$$

while the Ricci curvature is, [4],

$$S(X,Y) = \frac{c}{4}\left[(2n+1)g(X,Y) - 3\eta(X)\eta(Y)\right] + hg(AX,Y) - g(AX,AY), \tag{2.7}$$

where h (= trace of A) denotes the mean curvature. A formula for S in terms of CR-structures appears in [1, p. 442].

In the last part of this section we recall the notion of Ricci soliton, according to [22, p. 139]. On the manifold M, a *Ricci soliton* is a triple (g, V, λ) with g a Riemannian metric, V a vector field and λ a real scalar such that

$$\mathcal{L}_V g + 2S + 2\lambda g = 0 \tag{2.8}$$

where S is the Ricci tensor of g. In [4] (and in [5] for the compact case) it is proved that a real hypersurface in a non-flat ($c \neq 0$) complex space form $M_n(c)$ does not admit a Ricci soliton with ξ as soliton vector field and then we adopt the notion of *η-Ricci soliton* introduced in the same paper as a data (g, ξ, λ, μ) with

$$\mathcal{L}_\xi g + 2S + 2\lambda g - 2\mu\eta \otimes \eta = 0. \tag{2.9}$$

Another generalization appears in [14].

3. PARALLEL SYMMETRIC COVARIANT TENSORS AND ETA-RICCI SOLITONS

Fix α a symmetric tensor field of $(0, 2)$-type which we suppose to be parallel with respect to the Levi-Civita connection ∇, i.e., $\nabla\alpha = 0$. Applying the Ricci identity $\nabla^2\alpha(X, Y; Z, W) - \nabla^2\alpha(X, Y; W, Z) = 0$ we obtain the relation (1.1) of [18, p. 787]

$$\alpha(R(X, Y)Z, W) + \alpha(Z, R(X, Y)W) = 0, \tag{3.1}$$

which is fundamental in all papers treating this subject. Replacing $Z = W = \xi$ and using (2.6) one obtains

$$\frac{c}{4}[\eta(Y)\alpha(X, \xi) - \eta(X)\alpha(Y, \xi)] + \rho[\eta(Y)\alpha(AX, \xi) - \eta(X)\alpha(AY, \xi)] = 0, \tag{3.2}$$

by the symmetry of α. With $X = \xi$ we derive

$$\frac{c}{4}[\eta(Y)\alpha(\xi, \xi) - \alpha(Y, \xi)] + \rho[\rho\eta(Y)\alpha(\xi, \xi) - \alpha(AY, \xi)] = 0$$

and supposing, similar to (2.5),

$$\alpha(AX, \xi) = \alpha(X, A\xi) = \rho\alpha(X, \xi), \tag{3.3}$$

it results

$$\left(\frac{c}{4} + \rho^2\right)[\eta(Y)\alpha(\xi, \xi) - \alpha(Y, \xi)] = 0. \tag{3.4}$$

Definition 3.1. A *regular ρ-Hopf hypersurface* is one satisfying the condition: $c + 4\rho^2 \neq 0$.

For example, every ρ-Hopf hypersurface of the complex projective space $\mathbb{C}P^n$ is regular since $c > 0$.

Now, we are ready for the first main result of this paper:

Theorem 3.2. *A symmetric parallel second order covariant tensor α in a regular ρ-Hopf hypersurface for which:*

– the shape operator is α-ξ-symmetric which means that (3.3) *holds;*

– 0 *is not a principal curvature* (*equivalently, the shape operator is surjective*) *is a constant multiple of the metric tensor.*

Proof. From Definition 3.1 and (3.4), it results

$$\alpha(Y, \xi) = \eta(Y)\alpha(\xi, \xi). \tag{3.5}$$

Differentiating the last equation covariantly with respect to X, we have

$$\alpha(\nabla_X Y, \xi) + \alpha(\varphi AX, Y) = \alpha(\xi, \xi)[g(\nabla_X Y, \xi) + g(\varphi AX, Y)], \tag{3.6}$$

which means via (3.5) with $Y \to \nabla_X Y$,

$$\alpha(\varphi AX, Y) = \alpha(\xi, \xi) g(\varphi AX, Y). \tag{3.7}$$

Since A is surjective we can make the change $AX \to X$ and then

$$\alpha(\varphi X, Y) = \alpha(\xi, \xi) g(\varphi X, Y). \tag{3.8}$$

Now, we make $X \to \varphi X$, use (2.1a) and (3.5) and get

$$\alpha(X, Y) = \alpha(\xi, \xi) g(X, Y). \tag{3.9}$$

This relation which together with the standard fact that the parallelism of α implies that $\alpha(\xi, \xi)$ is a constant, yields the Conclusion. □

In [12] it was proved that there are no real hypersurfaces with parallel Ricci tensor in a non-flat complex space form $M_n(c)$ with $c \neq 0$ when $n \geq 3$. Furthermore, Kim [13] proved that this is also true when $n = 2$. These results imply, in particular, that there do not exist Einstein real hypersurfaces in a non-flat complex space form.

We continue this section with applications of our Theorem to Ricci solitons:

Corollary 3.3. *Suppose that on a regular ρ-Hopf hypersurface with surjective shape operator the $(0, 2)$-type field $\alpha := \mathcal{L}_V g + 2S$ is parallel, where V is a given vector field. If, for any vector field X,*

$$\eta([V, AX]) + g(AX, [V, \xi]) = \rho\eta([V, X]) + \rho g(X, [V, \xi]), \tag{3.10}$$

then (g, V) yield a Ricci soliton.

Proof. The Equation (3.3) becomes for the present α the relation (3.10). □

Naturally, two situations appear regarding the vector field $V : V \in \operatorname{span} \xi$ and $V \perp \xi$ but the second class seems far too complex to analyze in practice. For this reason we investigate only the case $V = \xi$ which can yields only η-Ricci solitons.

We are interested in expressions for $\alpha := \mathcal{L}_\xi g + 2S - 2\mu\eta \otimes \eta$. A straightforward computation gives

$$\mathcal{L}_\xi g(X, Y) = g((\varphi A - A\varphi)X, Y), \tag{3.11}$$

and then we get

$$\alpha = \frac{c}{2}[(2n+1)g - 3\eta \otimes \eta] + g((2hA + \varphi A - A\varphi - 2A^2)\cdot, \cdot) - 2\mu\eta \otimes \eta, \tag{3.12}$$

which gives

$$\alpha(\xi,\xi) = (n-1)c + 2h\rho - 2\rho^2 - 2\mu \tag{3.13}$$

and from $\lambda = -\frac{1}{2}\alpha(\xi,\xi)$ we derive that an η-Ricci soliton (g,ξ,λ,μ) on a regular ρ-Hopf hypersurface with surjective shape operator is given by

$$\mu = \lambda + h\rho - \rho^2 + \frac{n-1}{2}c. \tag{3.14}$$

4. EXAMPLES

Returning to the general equation of η-Ricci solitons

$$\begin{aligned}[(2n+1)c + 4\lambda]g(X,Y) + 2g(AX,\varphi Y) + 2g(\varphi X, AY) = \\ = 4g(A^2X - hAX, Y) + (4\mu + 3c)\eta(X)\eta(Y)\end{aligned} \tag{4.1}$$

the same relation (3.14) is derived for $X = Y = \xi$.

In the following we restrict to the case of η-umbilical framework which means that A is also umbilical on $\xi^\perp$, [16, p. 234], i.e., $A = \sigma I + (\rho - \sigma)\eta \otimes \xi$. Remark that a η-umbilical hypersurface with $\rho \neq 0$ and $\sigma \neq 0$ is surjective and then our aim is to compute explicitly the scalars λ and σ. Beside the above situation $X = Y = \xi$ we have other two for (4.1):

– if $X = \xi$ and $Y \in \xi^\perp$ is a principal vector field ($AY = \sigma Y$) it results $0 = 0$;

– if $X = Y \in \xi^\perp$ is as above and unitary then

$$\lambda = \sigma(\sigma - h) - (2n+1)\frac{c}{4}. \tag{4.2}$$

Therefore, (4.2) and (3.14) describe the scalars (λ,μ) of η-Ricci solitons. Since $h = \rho + (2n-2)\sigma$ this formulae yields

$$\lambda = (3-2n)\sigma^2 - \rho\sigma - \frac{2n+1}{4}c, \quad \mu = (3-2n)\sigma(\rho+\sigma) - \frac{3}{4}c. \tag{4.3}$$

Example 4.1 ($c > 0$). We shall consider $u \in (0, \frac{\pi}{2})$ and r a positive constant given by $\frac{1}{r^2} = \frac{c}{4}$. A hypersurface of Type A1 in $\mathbb{C}P^n(c)$ is a geodesic sphere and it has two distinct principal curvatures, [16, p. 260]: $\sigma = \frac{1}{r}\cot u$ of multiplicity $2n-2$ and $\rho = \frac{2}{r}\cot 2u$ of multiplicity 1. Then

$$\begin{cases} \lambda = \frac{c}{4}\{\cot u[(3-2n)\cot u - 2\cot 2u] - 2n - 1\}, \\ \mu = \frac{c}{4}\left[(3-2n)\cot u(\cot u + 2\cot 2u) - 3\right]. \end{cases} \tag{4.4}$$

Example 4.2 ($c < 0$). Let r be a positive number such that $4c = \frac{-4}{r^2}$. The geodesic spheres (Type A1) in complex hyperbolic space have two distinct principal curvatures [16, p. 257]: $\sigma = \frac{1}{r}\coth u$ of multiplicity $2n-2$ and $\rho = \frac{2}{r}\coth 2u$ of multiplicity 1. It follows $c+4\rho^2 = c(1-4\coth^2 2u) < 0$ which gives

$$(4.5)\qquad \begin{cases} \lambda = \dfrac{c}{4}\left\{4\coth u[(2n-3)\coth u + 2\coth 2u] - 2n - 1\right\}, \\ \mu = \dfrac{c}{4}\left[4(2n-3)\coth u(\coth u + 2\coth 2u) - 3\right]. \end{cases}$$

Example 4.3 ($c<0$). The horospheres (Type A0) are hypersurfaces of complex hyperbolic space $\mathbb{C}H^n(c)$ that have two distinct principal curvatures [16, p. 255]: $\sigma = \frac{1}{r}$ of multiplicity $2n-2$ and $\rho = \frac{2}{r}$ of multiplicity 1. Then $c+4\rho^2 = -15c > 0$ and therefore

$$(4.6)\qquad \begin{cases} \lambda = c\left(2n-1-\dfrac{2n+1}{4}\right) = \dfrac{(6n-5)c}{4} < 0, \\ \mu = \dfrac{(24n-39)c}{4} < 0. \end{cases}$$

For example, if $n=2$ then $\lambda = \frac{7c}{4}$ and $\mu = \frac{9c}{4}$.

Example 4.4 ($c<0$). The tubes around complex hyperbolic hyperplanes (Type A1) in complex hyperbolic space have two distinct principal curvatures [16, p. 257]: $\sigma = \frac{1}{r}\tanh u$ of multiplicity $2n-2$ and $\rho = \frac{2}{r}\coth 2u$ of multiplicity 1. Then $c+4\rho^2 = c(1-4\coth^2 2u) < 0$ and then

$$(4.7)\qquad \begin{cases} \lambda = \dfrac{c}{4}\left\{4\tanh u[(2n-3)\tanh u + 2\coth 2u)] - 2n - 1\right\}, \\ \mu = \dfrac{c}{4}\left[4(2n-3)\tanh u(\tanh u + 2\coth 2u) - 3\right]. \end{cases}$$

Remark 4.5. The Theorem 6 of [4] cover all Hopf hypersurfaces with η-Ricci solitons. Our Examples 4.1–4.3 correspond to item (i) while the last example is about the item (ii) of this theorem.

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REFERENCES

[1] A. Bejancu, *Real hypersurfaces of a complex projective space.* Rend. Mat. Appl. (6) **12** (1979), *3–4*, 439–445.

[2] J.T. Cho, *Ricci solitons and odd-dimensional spheres.* Monatsh. Math. **160** (2010), *4*, 347–357 (Erratum: **160** (2010), *4*, 359–360).

[3] J.T. Cho, *Notes on contact Ricci solitons.* Proc. Edinb. Math. Soc. (2) **54** (2011), *1*, 47–53.
[4] J.T. Cho and M. Kimura, *Ricci solitons and real hypersurfaces in a complex space form.* Tohoku Math. J. **61** (2009), *2*, 205–212.
[5] J.T. Cho and M. Kimura, *Ricci solitons of compact real hypersurfaces in Kähler manifolds.* Math. Nachr. **284** (2011), *11–12*, 1385–1393.
[6] J.T. Cho and R. Sharma, *Contact geometry and Ricci solitons.* Int. J. Geom. Methods Mod. Phys. **7** (2010), *6*, 951–960.
[7] L.S. Das, *Second order parallel tensors on α-Sasakian manifold.* Acta Math. Acad. Paedagog. Nyházi. **23** (2007), *1*, 65–69.
[8] S. Deshmukh, *Real hypersurfaces of a complex space form.* Monatsh. Math., DOI 10.1007/s00605-010-0269-x.
[9] L.P. Eisenhart, *Symmetric tensors of the second order whose first covariant derivatives are zero.* Trans. Amer. Math. Soc. **25** (1923), *2*, 297–306.
[10] A. Ghosh and R. Sharma, *Some results on contact metric manifolds.* Ann. Global Anal. Geom. **15** (1997), *6*, 497–507.
[11] A. Ghosh, R. Sharma and J.T. Cho, *Contact metric manifolds with η-parallel torsion tensor.* Ann. Global Anal. Geom. **34** (2008), *3*, 287–299.
[12] U-Hang Ki, *Real hypersurfaces with parallel Ricci tensor of a complex space form.* Tsukuba J. Math. **13** (1989), 73–81.
[13] U.K. Kim, *Nonexistence of Ricci-parallel real hypersurfaces in $P_2(\mathbb{C})$ or $H_2(\mathbb{C})$.* Bull. Korean Math. Soc. **41** (2004), 699–708.
[14] M. Kon and M. Kon, *Ricci solutions and real hypersurfaces.* Bull. Fac. Educ. Hirosaki Univ. **105** (2011), 65–68. http://repository.ul.hirosaki-u.ac.jp/dspace/bitstream/10129/4449/1/BFEduHirosaki_105_65.pdf
[15] H. Levy, *Symmetric tensors of the second order whose covariant derivatives vanish.* Ann. of Math. (2) **27** (1925), *2*, 91–98.
[16] R. Niebergall and P.J. Ryan, *Real hypersurfaces in complex space forms.* In: *Tight and taut submanifolds* (Berkeley, CA, 1994), pp. 233–305. Math. Sci. Res. Inst. Publ. **32**, Cambridge Univ. Press, Cambridge, 1997.
[17] C. Oniciuc, *Nonlinear connections on tangent bundle and harmonicity.* Ital. J. Pure Appl. Math. **6** (1999), 109–122 (2000).
[18] R. Sharma, *Second order parallel tensor in real and complex space forms.* Internat. J. Math. Math. Sci. **12** (1989), *4*, 787–790.
[19] R. Sharma, *Second order parallel tensors on contact manifolds.* I. Algebras Groups Geom. **7**(1990), *2*, 145–152.
[20] R. Sharma, *Second order parallel tensors on contact manifolds.* II. C. R. Math. Rep. Acad. Sci. Canada **13** (1991), *6*, 259–264.
[21] R. Sharma, *On the curvature of contact metric manifolds.* J. Geom. **53** (1995), *1-2*, 179–190.
[22] R. Sharma, *Certain results on K-contact and (k,μ)-contact manifolds.* J. Geom. **89** (2008), 138–147.
[23] M. Tarafdara and A. Mayra, *On nearly Sasakian manifold.* An. Stiint. Univ. “Al.I. Cuza” Iaşi **45** (1999), *2*, 291–294.
[24] M.M. Tripathi, *Ricci solitons in contact metric manifolds.* arXiv:0801.4222.
[25] Y.-C. Wong, *Existence of linear connections with respect to which given tensor fields are parallel or recurrent.* Nagoya Math. J. **24** (1964), 67–108.

[26] H. Wu, *Holonomy groups of indefinite metrics*. Pacific J. Math. **20** (1967), 351–392.
[27] G. Zhao, *Symmetric covariant tensor fields of order* 2 *on pseudo-Riemannian manifolds*. Viena Preprint ESI **479** (1997). Available at: `http://www.esi.ac.at/preprints/esi479.ps`

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