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PARALLEL TENSORS AND RICCI SOLITONS IN $N(k)$-QUASI EINSTEIN MANIFOLDS

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The Eisenhart problem of finding parallel and symmetric tensors is considered in the framework of $N(k)$-quasi Einstein manifolds and the result is connected with Ricci solitons. If the generator of the manifold provides a Ricci soliton then this is: i) shrinking on a class of conformally flat perfect fluid space-times and on quasi-umbilical hypersurfaces, in particular unit spheres; ii) expanding if the generator is of torse-forming type.

Key words : Parallel second order covariant tensor field; $N(k)$-quasi Einstein manifold; Torse-forming vector field; Ricci soliton; Perfect fluid

1. INTRODUCTION

In 1923, Eisenhart [6] proved that if a Riemannian manifold (M, g) admits a parallel symmetric second order covariant tensor other than a constant multiple of the metric tensor, then it is reducible. In 1925, Levy [7] proved that a parallel and symmetric second order non-degenerated tensor α in a space form is proportional to the metric tensor. Note that this question can be considered as the dual to the the problem of finding linear connections making parallel a given tensor field; a problem which was considered by Wong in [21]. Also, the former question implies topological restrictions namely if the (semi) Riemannian manifold M admits a parallel symmetric $(0, 2)$ tensor field then M is locally the direct product of a number of (semi) Riemannian manifolds, [22] (cited by [23]). Another situation where the parallelism of α is involved appears in the theory of totally geodesic maps, namely, as is point out in [8, p. 114], $\nabla\alpha = 0$ is equivalent with the fact that the identity map $1 : (M, g) \rightarrow (M, \alpha)$ is a totally geodesic map.

While both Eisenhart and Levy work locally, Ramesh Sharma gave in [14] a global approach based on Ricci identities. In addition to space-forms, Sharma considered this *Eisenhart problem* in contact geometry [15-17], for example for K-contact manifolds in [16]. Since then, several other studies appeared in various almost contact manifolds, see for example, the bibliography of [2].

The present paper is devoted to another framework, namely $N(k)$-quasi Einstein manifolds introduced by Tripathi and Kim in [20]. In the last years these manifolds are the subject of several studies: [9, 10, 11, 18, 19]. Our main result, namely Theorem 2.4, is connected with the recent theory of Ricci solitons, a subject included in the Hamilton-Perelman approach (and proof) of Poincaré Conjecture, [4].

The work is structured as follows. The first section is a very brief review of $N(k)$-quasi Einstein manifolds and Ricci solitons. The next section is devoted to the (symmetric case of) Eisenhart problem and the relationship with the Ricci solitons is pointed out. A technical condition appears, which is called *regularity*, namely the non-vanishing of the Ricci curvature with respect to the generator of the given manifold. A characterization of these manifolds as well as some remarkable cases which are out of this condition are presented. Three examples are discussed from the point of view of Ricci solitons. Since the regularity means non-steady Ricci solitons we give examples for both shrinking and expanding cases. So, an important consequence of our study is that a Ricci soliton on a unit sphere $\mathbb{S}^n$ with the induced metric from the Euclidean space $\mathbb{E}^{n+1}$ must be shrinking.

2. $N(k)$-QUASI EINSTEIN MANIFOLDS. RICCI SOLITONS

Fix a triple (M, g, ξ) with M^n a smooth $n(> 2)$-dimensional manifold, g a semi-Riemannian metric on M and ξ an unitary vector field on M. Let η be the 1-form dual to ξ with respect to g. Fix also three smooth functions on M namely a, b and k. Let $\mathcal{X}(M)$ be the Lie algebra of vector fields on M and R the curvature tensor field of g.

Definition 1.1 — The data (M^n, g, ξ, a, b, k) is called $N(k)$-*quasi Einstein manifold* if:

i) it is *quasi-Einstein* with respect to a and b i.e. the Ricci tensor S of g is:

$$S = ag + b\eta \otimes \eta. \tag{1.1}$$

(ii) ξ belongs to the k-*nullity distribution*:

$$p \in M \rightarrow N_p(k) = \{U \in T_pM; R(X,Y)U = k[g(Y,U)X - g(X,U)Y]\}.$$

ξ is called *the generator*. If a and b are constants then M is *eta-Einstein*.

Remark 1.2 : There are some remarkable consequences of the Definition:

(a) The given functions are not independent, [11]:

$$(S(\xi,\xi) =)a + b = k(n-1) \tag{1.2}$$

and then we denote $M^n_{a,b}(\xi)$ this manifold. In the eta-Einstein case it results that k is also a constant. Well-known examples of quasi-Einstein manifolds are the Robertson-Walker spacetimes; for other physical examples see [13].

(b) From ii) we get:

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] \tag{1.3}$$

as well as:

$$R(X,\xi)Z = k[\eta(Z)X - g(X,Z)\xi]. \tag{1.4}$$

(c) The scalar curvature is:

$$r = na + b \tag{1.5}$$

and then an eta-Einstein manifold has constant scalar curvature.

(d) Let us consider also the Ricci $(1,1)$ tensor field Q given by: $S(X,Y) = g(QX,Y)$. From the quasi Einstein condition we get:

$$Q = aI + b\eta \otimes \xi \tag{1.6}$$

which yields that ξ is an eigenvector of Q:

$$Q(\xi) = (a+b)\xi. \tag{1.7}$$

In the last part of this section we recall the notion of Ricci solitons. On the manifold M, a *Ricci soliton* is a triple (g,V,λ) with g a (semi) Riemannian metric, V a vector field and λ a real scalar such that the tensor field $Ric_{(\xi,\lambda)}$ of $(0,2)$-type:

$$Ric_{(\xi,\lambda)} := \mathcal{L}_V g + 2S + 2\lambda g = 0 \tag{1.8}$$

where $\mathcal{L}_V$ is the Lie derivative with respect to V. The Ricci soliton is said to be *shrinking*, *steady* or *expanding* according as λ is negative, zero or positive.

3. Parallel Symmetric Second Order Tensors on $N(k)$-Quasi-Einstein

Fix α a symmetric tensor field of $(0,2)$ type which we suppose to be parallel with respect to the Levi-Civita connection ∇ of g: $\nabla\alpha = 0$. Applying the Ricci commutation identity [4, p. 14] and $\nabla^2_{X,Y}\alpha(Z,W) - \nabla^2_{X,Y}\alpha(W,Z) = 0$ we obtain the relation (1.1) of [14, p. 787]:

$$\alpha(R(X,Y)Z,W) + \alpha(Z,R(X,Y)W) = 0 \tag{2.1}$$

which is fundamental in all papers treating this subject. Replacing $Z = W = \xi$ and using (1.3) it results, by the symmetry of α:

$$k[\eta(Y)\alpha(X,\xi) - \eta(X)\alpha(Y,\xi)] = 0. \tag{2.2}$$

Definition 2.1 — $M^n_{a,b}(\xi)$ is called *regular* if $k \neq 0$, equivalently $a + b \neq 0$.

In order to obtain a characterization of such manifolds from the point of view of ξ we consider:

Definition 2.2([12]) — ξ is called *semi-torse forming vector field* for (M, g) if, for all vector fields X:

$$R(X,\xi)\xi = 0. \tag{2.3}$$

From (1.3) we get: $R(X,\xi)\xi = \frac{a+b}{n-1}(X - \eta(X)\xi)$ and therefore, if $X \in \mathrm{Ker}\{\eta\} = \xi^{\perp}$ then $R(X,\xi)\xi = \frac{a+b}{n-1}X$ and we obtain that for $M^n_{a,b}(\xi)$ the following are equivalent:

(i) is regular,

(ii) ξ is not semi-torse forming,

(iii) $S(\xi,\xi) \neq 0$ i.e. ξ is isotropic with respect to S,

(iv) $Q(\xi) \neq 0$ i.e. ξ does not belong to the kernel of Q.

In particular, if ξ is parallel ($\nabla\xi = 0$) then M is not regular.

Remark 2.3 : In a regular $M^n_{a,b}(\xi)$ there are some non-vanishing curvature tensors related to ξ:

(i) From Theorem 3.2 of [20, p. 414] a regular $M^n_{a,b}(\xi)$ is not semi-symmetric since $R(\xi, X) \cdot R \neq 0$. From the same paper in a regular $M^n_{a,b}(\xi)$ we have also $R(\xi, X) \cdot S \neq 0$.

(ii) From Corollary 4.4. of [9, p. 1376] in a regular $M^n_{a,b}(\xi)$ we have $R(\xi, X) \cdot P \neq 0$ and $P(\xi, X) \cdot S \neq 0$ where P is the projective curvature tensor.

(iii) From Theorem 4.1. of [10, p. 76] in a regular $M^n_{a,b}(\xi)$ with $n \geq 4$ which is not conformally flat we have $R(\xi, X) \cdot C \neq 0$ where C is the Weyl conformal curvature tensor.

In the following we restrict to the regular case.

***Theorem* 2.4** — *A parallel and symmetric second order covariant tensor field in a regular $M^n_{a,b}(\xi)$ is a constant multiple of the metric tensor.*

PROOF : From (2.2), with $X = \xi$ in:

$$\eta(Y)\alpha(X, \xi) = \eta(X)\alpha(Y, \xi) \tag{2.4}$$

we derive:

$$\alpha(Y, \xi) = \eta(Y)\alpha(\xi, \xi) = \alpha(\xi, \xi)g(Y, \xi). \tag{2.5}$$

The parallelism of α and (2.5) imply also that $\alpha(\xi, \xi)$ is a constant:

$$X(\alpha(\xi, \xi)) = 2\alpha(\nabla_X \xi, \xi) = 2\alpha(\xi, \xi)g(\nabla_X \xi, \xi) = 2\alpha(\xi, \xi) \cdot 0 = 0. \tag{2.6}$$

Making $Y = \xi$ in (2.1) and using (1.4) we get:

$$\eta(Z)\alpha(X, W) - g(X, Z)\alpha(\xi, W) + \eta(W)\alpha(X, Z) - g(X, W)\alpha(\xi, Z) = 0$$

which yields, via (2.5) and $W = \xi$:

$$\alpha(X, Z) = \alpha(\xi, \xi)g(X, Z). \tag{2.7}$$

which is our Conclusion. □

4. RICCI SOLITONS ON $N(k)$-QUASI-EINSTEIN MANIFOLDS

Let us remark that in fact, for the above result we use only the second part of the Definition 1.1. Now we get the expression (1.1) of the Ricci tensor in order to include applications of the above Theorem to Ricci solitons:

(1) the vector field V gives a Ricci soliton on a regular $N(k)$-quasi Einstein manifold if and only if $\alpha := \mathcal{L}_V g + 2S = \mathcal{L}_V g + 2ag + 2b\eta \otimes \eta$ is parallel with respect to the Levi-Civita connection ∇ of g,

(2) the pair (g, V) on a regular $N(k)$ eta-Einstein manifold is a Ricci soliton if and only if $\alpha := \mathcal{L}_V g + 2b\eta \otimes \eta$ is parallel with respect to ∇.

Naturally, two remarkable situations appear regarding the vector field V: $V \in \mathrm{Span}\{\xi\}$ or $V \perp \xi$ but the second class seems far too complex to analyse in practice. For this reason it is appropriate to investigate only the case $V = \xi$.

***Theorem* 3.1** — *Fix a $M^n_{a,b}(\xi)$ not necessary regular. Then (g, ξ, λ) is a Ricci soliton on $M^n_{a,b}(\xi)$ if and only if $k = -\lambda$ is a constant and*:

$$\mathcal{L}_\xi g = 2b(g - \eta \otimes \eta). \tag{3.1}$$

PROOF : We apply the tensor field $Ric_{(\xi,\lambda)}$ on three pairs of vector fields:

(i) $Y = Z = \xi$; then $a + b + \lambda = 0$ which gives that k is a constant,

(ii) $Y = \xi$ and $Z \in \mathrm{Ker}\eta$; then $0 = 0$,

(iii) $Y, Z \in \mathrm{Ker}\eta$; then $\mathcal{L}_\xi(Y, Z) + 2(a + \lambda)g(Y, Z) = 0$. Using I) we get that $\mathcal{L}_\xi g = 2bg$ on $\mathrm{Ker}\eta$ which means (3.1). □

Remark 3.2 —

(i) In the eta-Einstein case the single condition for Ricci solitons provided by ξ is (3.1). The main result of [18] is that the generator ξ of a $N(k)$-eta Einstein manifold M with $b \neq 0$ is a Killing vector field if M is *Ricci symmetric* i.e. $\nabla S = 0$. But from, (3.1) it results that $\eta \otimes \eta = g$ and then returning to (1.1) we get that $S = (a + b)g$. So, a Ricci symmetric eta-Einstein manifold reduces to an Einstein manifold.

(ii) In the regular case a Ricci soliton given by ξ is shrinking (if $a + b > 0$) or expanding (when $a + b < 0$).

(iii) The addition of a topological property, namely compactness, gives new informations about the possible types of Ricci solitons on a (not necessarily regular) $M^n_{a,b}(\xi)$. More precisely, from [1, p. 3]c:hd, a compact steady or expanding Ricci soliton must be Einstein and therefore a compact $M^n_{a,b}(\xi)$ with $a+b \leq 0$ with ξ as Ricci soliton is Einstein with ξ a conformal Killing vector field.

(iv) An important consequence of (3.1) is as follows; applying $\mathcal{L}_\xi g$ on the pair $(Y \in \mathrm{Ker}\eta, \xi)$ gives $g(Y, \nabla_\xi \xi) = 0$. Then ξ is a *geodesic vector field* namely $\nabla_\xi \xi = 0$.

Example 3.3 : (i) Let (M^4, g) be a conformally flat perfect fluid space-time satisfying the Einstein equations without cosmological constant. This manifold is an $M^4_{a,b}(\xi)$ with ξ the unit time-like velocity vector of the fluid and [9, p. 1375]:

$$k = \frac{\kappa(3\sigma + p)}{6} \tag{3.2}$$

where: κ is the gravitational constant, σ is the energy density and p is the isotropic pressure of the fluid. So, if (g, ξ) is a Ricci soliton on this $M^4_{a,b}(\xi)$ then k is a constant and this soliton is shrinking (if κ, σ and p are strictly positive).

(ii) Suppose now that (M^4, g) is as above but satisfying the Einstein equations with the cosmological constant λ. It follows again the manifold $M^4_{a,b}(\xi)$ with [9, p. 1375]:

$$k = \frac{\lambda}{3} + \frac{\kappa(3\sigma + p)}{6}. \tag{3.3}$$

Therefore, if (g, ξ) is a Ricci soliton then the above k is constant and this soliton is shrinking (if in addition $\lambda > 0$).

Example 3.4 : Let $\bar{M}^{n+1}_{\bar{a},\bar{b}}(\xi)$ be a $N(\bar{k})$-quasi Einstein manifold and M^n an orientable *quasi-umbilical hypersurface* in $\bar{M}$, [3], i.e. there exist two smooth functions α, β on M and a 1-form u of norm 1 such that the second fundamental form of M is:

$$h = \alpha g + \beta u \otimes u. \tag{3.4}$$

According to the [20, p. 416] if M is normal to ξ then M is an $M^n_{a,b}(U)$ with:

$$a = \bar{a} - \bar{k} + (n-1)\alpha^2 + \alpha\beta, \quad b = (n-2)\alpha\beta \tag{3.5}$$

where U the g-dual of u. The associated k is then:

$$k = \frac{(n-1)\bar{a} - \bar{b}}{n(n-1)} + \alpha^2 + \alpha\beta \tag{3.6}$$

which means that M is regular if and only if $\bar{b} - (n-1)\bar{a} \neq n(n-1)(\alpha^2 + \alpha\beta)$. If (g, U) is a Ricci soliton on M then this is:

(a) shrinking if $\bar{b} - (n-1)\bar{a} < n(n-1)(\alpha^2 + \alpha\beta)$,

(b) expanding if $\bar{b} - (n-1)\bar{a} > n(n-1)(\alpha^2 + \alpha\beta)$.

In particular, let $\bar{M} = \bar{M}^{n+1}(c)$ be a space form; then $\bar{a} = nc$, $\bar{b} = 0$ and $\bar{k} = c$. If we choose:

(1) $c > 0$, namely the elliptic case, and M quasi-umbilical with $\alpha\beta \geq 0$ then the Ricci soliton (g, U) of M, if it exists, is shrinking; if in addition M is compact then applying Theorem 3 of [5, p. 185] then M is a real homology sphere i.e., all Betti numbers vanish,

(2) $c = 0$, namely the flat case with ξ the radial (identity) vector field, and M umbilical ($\beta = 0$), then the Ricci soliton (g, U) of M, if it exists, is shrinking. Recall that the umbilical hypersurfaces of $E^{n+1} = \mathbb{R}^{\ltimes+\nVdash}$ are spheres, which are normal to ξ if there are centered in the origin of E^{n+1}; also from $\|\xi\| = 1$ we must restrict to $\mathbb{S}^n$. In conclusion, a Ricci soliton on a round sphere $\mathbb{S}^n$ must be shrinking.

Example 3.5 : Suppose that ξ is a *torse-forming vector field* i.e. there exist a smooth function f and a 1-form ω such that:

$$\nabla_X \xi = fX + \omega(X)\xi. \tag{3.7}$$

From the fact that ξ has unitary length and $g(\nabla_X \xi, \xi) = 0$ it results $f\eta + \omega = 0$ which means that $\omega = -f\eta$. In particular, it results ξ is a geodesic vector field. From:

$$\nabla_X \xi = f(X - \eta(X)\xi) \tag{3.8}$$

a straightforward computation gives:

$$\begin{aligned} R(X,Y)\xi &= X(f)[Y - \eta(Y)\xi] - Y(f) \\ &\quad [X - \eta(X)\xi] + f^2[\eta(X)Y - \eta(Y)X] \end{aligned} \tag{1}$$

and a comparison with (1.3) yields $k = -f^2$ and f must be a constant, different from zero from regularity of the manifold. So, a Ricci soliton (g, ξ) in a $N(-f^2)$-quasi Einstein manifold with ξ of torse-forming type must be expanding, a result similar to Proposition 3.4 of [2, p. 366].

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