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Full Length Research Paper

A note on dynamical systems satisfying the Wünschmann-type condition

Mircea Crăşmăreanu[1] and Vlad Enache[2]*

[1]Faculty of Mathematics,"Al. I. Cuza" University, Iasi, 700506, Romania
[2]Department of Sciences, "Al. I. Cuza" University, Iasi, 700107, Romania

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Dynamical systems of second order ordinary differential equations (SODEs) satisfying the Wünschmann-type condition are presented, with a special emphasis on SODEs provided by Euler-Lagrange equations, particularly, geodesics on surfaces. Some remarkable facts are to be pointed: three of our examples are of quasi Euler-Lagrange nature; also, except one example that leads to classical Euclidean 2D metric, all yields a semi-Riemannian metric.

Key words: Second order ordinary differential equations (SODE), semispray-metric, Hamilton-Jacobi equation, dynamical systems, Wünschmann-type condition, (quasi) Euler-Lagrange equation, geodesic equation.

INTRODUCTION

It is well-known that on the space of solutions given to some classes of second order ordinary differential equations (SODEs), there is a rich geometry. For example, Ferrand (1997) proved that the space of geodesics of a Hadamard n-manifold is symplecto-morphic to the cotangent bundle of the sphere (S^{n-1}). Since the Riemannian geometry is the most used framework, it is natural to ask under what conditions, on the space of the solutions given to SODEs does a metric exist?

An important step towards a complete solution of this problem is the paper (Garcia-Godinez et al., 2004) where a Riemannian or semi-Riemannian metric on the space of solutions for a type of scalar SODEs is determined via the Hamilton-Jacobi equation. The type of SODE for which an associated Hamilton-Jacobi relation holds is determined by the so-called Wünschmann-type condition. Invariants of Wünschmann-type were discussed in several papers (Crampin and Saunders, 2007; Frittelli et al., 2001; Gallo, 2004; Garcia-Godinez et al., 2004; Montiel-Piña et al., 2005; Newman and Nurowski, 2003; Nurowski, 2005). Also, the last two years show a flow of papers towards this subject (Bucataru et al., 2011; Crampin, 2010; Holland-Sparling 2011).

As pointed in McKay (2001) and recognized in the bibliography of Frittelli et al. (2003), the Wünschmann reference (Wünschmann, 1905) is "almost impossible" to find. The present paper is devoted searching for examples of SODEs satisfying this condition. Our examples of dynamical systems are of mechanical or geometrical nature, but it could be of chemical or any other nature too as long as these systems are described by SODEs equations.From a unified point of view, both classes may be considered as originating from a variational principle, because the geodesics, from the geometrical side, are also solutions to variational equations (namely, Euler-Lagrange) as the trajectories of physical examples.

The content of this paper is as follows. Firstly, the revision of Garcia-Godinez et al. (2004) in order to fix the terminology and notation and at the end, an example of SODE which coincides with associated Wünschmann-type condition is included. This example has a special feature of appearing as quasi Euler-Lagrange equation for a quasi Lagrangian after the approach of Rauch-Wojciechowski et al. (1999). Secondly, the physical oriented examples, derived from Lagrangian and Hamiltonian approaches of analytical mechanics. The main tools here are the natural Lagrangians and various types of harmonic oscillators. The classical harmonic oscillator yields the usual Euclidean 2D metric, while a second quadratic Lagrangian generates the Minkowski

*Corresponding author. E-mail: vladanea@yahoo.com.

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2D metric. The Wünschmann-type condition applied to the time-dependent damped oscillator produces a conformal equivalence with the usual harmonic oscillator. Thirdly, the separation of variables in the Wünschmann-type condition. Let us point out that for Hamilton-Jacobi equations, there exists a proper notion of separation of variables cf. (Benenti, 2002). There are three cases, but of course the autonomous case, namely, independence of independent variable, is of most significance. In this case, we treat three types of separation, namely, additive, multiplicative, and Lienard-type. The classical Lienard equation is $\ddot{u} = f(u)\dot{u} + g(u)$, but we extend to the form $\ddot{u} = f(u)(\dot{u})^{\alpha} + g(u)$. On this way, we obtain a generalization of previous case, namely, multiplicative separation. All resulting Wünschmann SODEs are quasi-Lagrange equations. Fourthly, the study of the Wünschmann-type condition for SODEs with the right-hand side of third order in u'. As main examples, the equation of geodesics for metrics on surfaces are considered. Connections with the multiplicative separation case of the previous explaned appear, because the solutions in this case of separation depend on a real constant C. The case $C = 0$ corresponds to the geodesics of metric $g_S = ds^2 + u^{-2}du^2$, while the case $C = -1$ is associated to the Liouville metric $g_S = u^2(ds^2 + du^2)$. The case $C = 1$ is the equation yielding scalar flat rotationally symmetric metrics on the 4-manifold $I \times S^3$. Sharpe (2000) worked on the Cartan's geometrization of an SODE of the form $y'' = A(x,y) + B(x,y)y' + C(x,y)(y')^2 + D(x,y)(y')^3$. Finally, a step towards the next class of differential equation, namely, third order was made. For this type of equations, there exists a well-known Chern invariant of Wünschmann-type. We present two examples. First, is a SODE satisfying the Wünschmann-type condition such that the derived third order differential equation (ODE) has a vanishing Chern-Wünschmann invariant. Our last example is related to the classical Halphen equation with a remarkable geometry.

Some amazing facts are to be pointed. Except the Euclidean 2D metric, all resulting metrics are semi-Riemannian. Unfortunately, the final expression of some of these metrics remains an open problem. Also, some SODEs remarkable from various reasons (physics, geometry, and analysis) do not satisfy this Wünschmann-type condition or satisfy a degenerate condition $0 = 0$; therefore, without any meaning for our study. An example from this class is provided by Painlevé equations.

THE WÜNSCHMANN-TYPE CONDITION REVISITED

Let us consider the following SODE:

$$u'' = \Lambda(u', u, s) \tag{1}$$

where the prime denotes the derivative with respect to the independent parameter s. Using two different procedures, in Garcia-Godinez et al. (2004), it is proved that on the space of solutions to Equation 1 which is a 2D manifold parametrized by two-independent constants of integration $x^a = (x^1, x^2) = (x, y)$, there exists a metric g with the expression:

$$g = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{\Lambda_u} \end{pmatrix} \tag{2}$$

satisfying the Hamilton-Jacobi (HJ) equation:

$$(HJ): g^{ab}u_{,a}u_{,b} = 1 \tag{3}$$

if and only if a so-called Wünschmann-like condition holds:

$$D(\Lambda_u) = 2\Lambda_{u'}\Lambda_u. \tag{4}$$

The following indices denote the partial derivatives, and D is the total derivative with respect to s. Thus, the condition of equation 4 reads:

$$\Lambda_{us} + \Lambda_{uu}u' + \Lambda_{uu'}\Lambda = 2\Lambda_{u'}\Lambda_u \tag{5}$$

and, because:

$$g^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -\Lambda_u \end{pmatrix},$$

the HJ equation is:

$$u_x^2 - \Lambda_u u_y^2 = 1. \tag{6}$$

Using a more geometrical language, a SODE is called a semispray and then, the metric g given by equation 2 shall be called a semispray-metric.

Let us point out that relation equation 4 looks like a homogeneity of Λ_u with homogeneous factor $2\Lambda_{u'}$. Also, Equation 4 is a SODE with a first integral:

$$\Lambda_u = Ce^{2\int \Lambda_{u'}ds}, \quad C \neq 0. \tag{7}$$

C is non-null since Λ_u appears at the denominator in Equation 2. For instance, if Equation 1 is a linear SODE $\ddot{u} = a(s)\dot{u} + b(s)u$, then Equation 7 gives:

$$\ddot{u} = a(s)\dot{u} + Ce^{2A(s)}u \tag{8}$$

where $A(s)$ is a primitive of $a(s)$.

At the end of Garcia-Godinez et al. (2004) two examples are included:

I) $u'' = -u$ yields the usual Euclidean metric $g = dx^2 + dy^2$,
II) $u'' = u$ yields the 2D Minkowski metric $g = dx^2 - dy^2$.

These examples follow from Equation 8 with $a(s) \equiv 0$ and $C = \begin{cases} -1, & I \\ +1, & II \end{cases}$. Moreover, every SODE (Equation 8) can be transformed into $\frac{d^2u}{dt^2}(t) = \varepsilon u(t)$ with the change of independent variable $t = \sqrt{\varepsilon C} \int e^{A(s)} ds$ and the choice $\varepsilon = sgnC$.

Subsequently, these examples are obtained from another unified (namely, variational) point of view. But, we end the Wünschmann-type condition revisited with another example:

$$u'' = -\frac{(u')^2}{u} + Cu', \quad C \neq 0 \tag{9}$$

which has the remarkable property that the associated Wünschmann-type condition coincides with the given SODE. A straightforward computation proves this fact. The case $C = 0$ in Equation 9 will be recast in Equation 43 with the same choice $C = 0$ and it was analysed at the end of Example 7. The solutions of Equation 9 are:

$$u(s,x,y) = \pm\sqrt{2\left(y + \frac{1}{C}e^{x+Cs}\right)} \tag{10}$$

and the associated semi-spray metric:

$$g = dx^2 - \frac{4\left(y + \frac{1}{C}e^{x+Cs}\right)^2}{e^{2(x+Cs)}} dy^2 \tag{11}$$

is semi-Riemannian.

In Rauch-Wojciechowski et al. (1999), the quasi-Lagrangian systems, that is, dynamical systems generated by a function $E(u,u')$ are treated via the quasi Euler-Lagrange equations:

$$\delta^+ E := D(E_{u'}) + E_u = 0. \tag{12}$$

In the following, we study if Equation 9 is of Equation 11-type with:

$$E = \alpha(u)(u')^2 + \beta(u)u'. \tag{13}$$

Because:

$$\begin{cases} E_{u'} = 2\alpha u' + \beta \\ E_u = \alpha'(u')^2 + \beta' u' \end{cases}$$

we obtain:

$$\delta^+ E: 2\alpha u'' + 3\alpha'(u')^2 + 2\beta' u' = 0 \tag{14}$$

and a comparison with equation gives:

$$\begin{cases} -\frac{3\alpha'}{2\alpha} = -\frac{1}{u} \\ -\frac{\beta'}{\alpha} = C \end{cases}$$

with solution:

$$\begin{cases} \alpha(u) = u^{\frac{2}{3}} \\ \beta(u) = -\frac{3C}{5} u^{\frac{5}{3}}. \end{cases}$$

In conclusion, Equation 9 is exactly the quasi Euler-Lagrange equation for the quasi-Lagrangian:

$$E = u^{\frac{2}{3}}(u')^2 - \frac{3C}{5} u^{\frac{5}{3}} u'. \tag{15}$$

SEMISPRAY-METRICS FROM LAGRANGIANS

A remarkable class of SODEs is provided by the Euler-Lagrange equation:

$$(EL): D(L_{u'}) = L_u \tag{16}$$

for a Lagrangian $L = L(u', u, s)$. A useful type of Lagrangian is given by natural Lagrangian L = kinetic energy - potential energy:

$$L = \frac{u'^2}{2} - V(u,s) \tag{17}$$

for which:

$$(EL): u'' = -V_u. \tag{18}$$

Searching for $\Lambda = -V_u(u,s)$ in equation 4, we get:

$$D(V_{uu}) = 0 \tag{19}$$

with solution, a quadratic potential:

$$V = \frac{A}{2}u^2 + B(s)u + C(s). \tag{20}$$

Here, A is a constant and from the presence of $\Lambda_u = -V_{uu} = -A$ in Equation 2 it follows that $A \neq 0$.

Also, because $B(s)$ and $C(s)$ do not appear in the expression of metric (Equation 2) in what follows, we will consider $B(s) = C(s) \equiv 0$. The corresponding (EL) and (HJ) equations are:

$$\begin{cases}(EL): u'' = -Au \\ (HJ): u_x^2 + Au_y^2 = 1\end{cases} \tag{21}$$

which implies two situations:

Example 1: $A > 0$ **with solution to** (HJ)**:** $u(x,y,s) = x\cos s + \frac{1}{\sqrt{A}} y\sin$

Comparing the second derivative of this expression with (EL) results that $A = 1$. In conclusion, we have the solutions:

$$u = x\cos s + y\sin s \tag{22}$$

of,

$$\begin{cases}(EL): u'' = -u \\ (HJ): u_x^2 + u_y^2 = 1\end{cases} \tag{23}$$

for the natural Lagrangian with potential:

$$V = \frac{1}{2}u^2 \tag{24}$$

which is the Lagrangian of the harmonic oscillator.

Example 2: $A < 0$ **with solution to** $(HJ): u(x,y,s) = x\cosh s + \frac{1}{\sqrt{-A}} y\sinh s$

A similar analysis yields $A = -1$ and solutions:

$$u = x\cosh s + y\sinh s \tag{25}$$

for,

$$\begin{cases}(EL): u'' = u \\ (HJ): u_x^2 - u_y^2 = 1\end{cases} \tag{26}$$

The aforementioned cases I) and II) are exactly the examples from the Wünschmann-type condition revisited. The dual point of view of the Lagrangian approach is the Hamiltonian framework in which the Lagrangian is replaced with a Hamiltonian $H = H(u,p,s)$ and the (EL) equation with Hamilton equations:

$$(H): D(u) = H_p, \quad D(p) = -H_u. \tag{27}$$

Example 3: Time-dependent damped harmonic oscillator

The Hamiltonian of the one-dimensional time-dependent harmonic oscillator with damping forces linear in velocity is (Struckmeier and Reidel, 2002):

$$H = \frac{1}{2}p^2 e^{-F(s)} + \frac{1}{2}\omega^2(s)u^2 e^{F(2)} \tag{28}$$

and let $f = F'$. The associated (H) equations are:

$$D(u) = pe^{-F(s)}, \quad D(p) = -u\omega^2(s)e^{F(s)} \tag{29}$$

which yields (Struckmeier and Reidel, 2002):

$$u'' = -f(s)u' - \omega^2(s)u. \tag{30}$$

Searching in Wünschmann-like condition for $\Lambda_{u'} = -f(s), \Lambda_u = -\omega^2(s)$ we get:

$$\omega' = -f\omega \tag{31}$$

with solution:

$$\omega(s) = e^{-F(s)} \tag{32}$$

and then, the Hamiltonian becomes:

$$H = e^{-F(s)}\left(\frac{1}{2}p^2 + \frac{1}{2}u^2\right) \tag{33}$$

which is a conformal deformation, with time-dependent conformal factor $e^{-F(s)}$ of the Hamiltonian for the harmonic oscillator of Example 1.

Example 4: Time-dependent anharmonic undamped oscillator

The Hamiltonian for one-dimensional time-dependent anharmonic oscillator without damping is (Struckmeier and Reidel, 2002):

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2(s)u^2 + a(s)u^3 + b(s)u^4. \tag{34}$$

The Hamilton equations:

$$(H): \begin{cases}D(u) = p \\ -D(p) = \omega^2(s) + 3a(s)u^2 + 4b(s)u^3\end{cases} \tag{35}$$

yields the SODE:

$$u'' = -4b(s)u^3 - 3a(s)u^2 - \omega^2(s)u. \qquad (36)$$

Because $\Lambda_{u'} = 0$, an immediate analysis gives that the Wünschmann-type condition is not satisfied; so, it is explained further in case II of separation of variables in the Wünschmann-like condition.

SEPARATION OF VARIABLES IN THE WÜNSCHMANN-LIKE CONDITION

The analysis of semispray-metrics from Lagrangians has as a special feature the separation of variables u' and (u, s) in the expression of Λ. Inspired by this remark, we discuss, returning to general SODE (Equation 1), three particular variants for Λ:

Case I: $\Lambda = \Lambda(u', s)$ is not important from the point of view of semispray-metrics, because the expression of Equation 2 has no meaning. Also, the relation of Equation 4 is a degenerate identity.

Case II: $\Lambda = \Lambda(u, s)$, the condition of Equation 4 reads $D(\Lambda_u) = 0$ with general solution $\Lambda = Au + B(s)$ which yields exactly the results of Wünschmann-type condition revisited.

Example 5: Painlevé equations

In Bordag's (1997) Theorem 2, it is proved that all six well-known Painlevé equations can be reduced to the form $u'' = \Lambda(u, s)$; obviously, for us, more interest are the real versions of these equations. These six expressions are different from the previous linear one and in conclusion, the Painlevé equations do not satisfy the Wünschmann-type condition.

Case III: $\Lambda = \Lambda(u', u)$, that is, the SODE (Equation 1) is time-independent or autonomous. We study three particular cases of separated variables, namely: additive separation, multiplicative separation, and Lienard-type expression.

Additive separation: $\Lambda = \varphi(u) + \psi(u')$

The Wünschamnn-type condition (Equation 5) is:

$$\varphi''(u)u' = 2\varphi'(u)\psi'(u') \qquad (37)$$

and again there are three situations:

IIIi) $\varphi'(u) = 0$ and $\psi(u')$ is arbitrary. Then, $\Lambda_u = 0$ and we obtain the case I.

IIIii) $\varphi'(u) \neq 0$, but $\varphi''(u) = 0$ which implies $\psi'(u') = 0$. Then, $\psi(u') = C, \varphi(u) = Au$ and $\Lambda_u = A$ implies $A \neq 0$. From the point of view of semispray-metrics, we may suppose $C = 0$ and the case II is obtained.

IIIiii) $\varphi''(u) \neq 0$ which requires $\varphi'(u) \neq 0$. The relation of Equation 37 reads:

$$\frac{\varphi''(u)}{\varphi'(u)} = \frac{2\psi'(u')}{u'} \qquad (38)$$

which implies the existence of a constant $A \neq 0$ such that: $\psi'(u') = \frac{A}{2}u'$ and $\frac{\varphi''(u)}{\varphi'(u)} = A$. Integration gives: $\psi(u') = \frac{A}{4}u'^2$ and $\varphi(u) = \frac{1}{A}e^B e^{Au} + C$ with constants B, C. Since e^B has a multiplicative rôle, we suppose $e^B = 1$ and $C = 0$ which gives the final form:

$$u'' = \frac{A}{4}(u')^2 + \frac{1}{A}e^{Au}. \qquad (39)$$

The associated semispray-metric is:

$$g = \begin{pmatrix} 1 & 0 \\ 0 & -e^{-Au} \end{pmatrix} \qquad (40)$$

and from $A \neq 0$ it results that g is semi-Riemannian.

In order to solve equation 39, let $u' = p = p(u)$ as a new unknown function; then:

$$p\dot{p} = \frac{A}{4}p^2 + \frac{1}{A}e^{Au} \qquad (41)$$

which, using notation $w = \frac{p^2}{2}$, becomes:

$$\dot{w} = \frac{A}{2}w + \frac{1}{A}e^{Au}. \qquad (42)$$

The last equation is a linear one with solution:

$$w(u) = \frac{(u')^2}{2} = e^{\frac{A}{2}u}\left(x + \frac{2}{A^2}e^{\frac{A}{2}u}\right) \qquad (43)$$

with x the first constant of integration. We get:

$$u' = \pm\sqrt{2}e^{\frac{A}{4}u}\left(x + \frac{2}{A^2}e^{\frac{A}{2}u}\right)^{\frac{1}{2}} \qquad (44)$$

and we have two subcases:

1) $x = 0$; then Equation 44 becomes:

$$u' = \pm\frac{2}{|A|}e^{\frac{A}{2}u} \tag{45}$$

with the solution:

$$u = u(s, y) = -\frac{2}{A}\ln\left(\frac{-A}{2}y - \varepsilon_A s\right) \tag{46}$$

where $\varepsilon_A = \frac{A}{|A|}$ and y is the second constant of integration. Let us remark that in the survey (Hardt, 1997) there are some pointed connections between the function $\frac{x}{|x|}$ and the harmonic maps, a class of functions satisfying remarkable SODEs.

2) $x \neq 0$. Equation 44 is:

$$\frac{u'}{\left(\frac{x}{2}e^{Au/2}+\frac{1}{A^2}e^{Au}\right)^{\frac{1}{2}}} = \pm 2. \tag{47}$$

A long but straightforward computation gives the solution:

$$u = u(s, x, y) = -\frac{2}{A}\ln\frac{A^4x^2(y\pm s)^2-16}{8A^2x}. \tag{48}$$

We study if Equation 29 is of quasi Euler-Lagrange type for the quasi-Lagrangian:

$$E = \alpha(u)(u')^2 + \beta(u). \tag{49}$$

From:

$$\begin{cases} E_{u'} = 2\alpha u' \\ E_u = \alpha'(u')^2 + \beta' \end{cases}$$

it results that:

$$\delta^+E: 2\alpha u'' + 3\alpha'(u')^2 + \beta' = 0$$

and a comparison with Equation 39 gives:

$$\begin{cases} -\frac{3\alpha'}{2\alpha} = \frac{A}{4} \\ -\frac{\beta'}{2\alpha} = \frac{e^{Au}}{A} \end{cases} \tag{50}$$

with the solution:

$$\begin{cases} \alpha(u) = e^{-\frac{Au}{6}} \\ \beta(u) = -\frac{12}{5A^2}e^{\frac{5Au}{6}}. \end{cases}$$

In conclusion, Equation 39 is exactly the quasi Euler-Lagrange equation for the quasi-Lagrangian:

$$E = e^{-Au/6}(u')^2 - \frac{12}{5A^2}e^{5Au/6}.$$

Multiplicative separation

$\Lambda = \varphi(u)\psi(u')$, and the Wünschmann-type condition becomes:

$$\varphi''u' = \varphi\varphi'\psi'$$

which can be put in the form:

$$\frac{\varphi''(u)}{\varphi(u)\varphi'(u)} = \frac{\psi'(u')}{u'}.$$

Again, it results that there exists a constant $A \neq 0$ such that:

$$\begin{cases} \psi'(u') = Au' \\ \varphi'' = A\varphi\varphi' \end{cases}. \tag{51}$$

Searching Equation 51 (2) for a solution $\varphi(u) = u^{\alpha}$, we get: $\alpha = -1$ and $A = -2$. In conclusion, the solution of Equation 49 is:

$$\begin{cases} \varphi(u) = \frac{1}{u} \\ \psi(u') = C - (u')^2 \end{cases} \tag{52}$$

with C a real constant. The SODE (Equation 1) is:

$$u'' = \frac{C-(u')^2}{u} \tag{53}$$

with general solution:

$$u(s, x, y) = \pm\sqrt{2\left(\frac{C}{2}s^2 + xs + y\right)}. \tag{54}$$

From Equations 53 and 54:

$$\Lambda_u = \left(\frac{u'}{u}\right)^2 = \frac{(Cs+x)^2}{(Cs^2+2xs+2y)^2} \tag{55}$$

and then the semispray-metric is:

$$g = dx^2 - \frac{(Cs^2+2xs+2y)^2}{(Cs+x)^2}dy^2 \tag{56}$$

which is again a semi-Riemann metric.

Let us remark that Equations 9 and 53 can be treated in an unitary way, namely, for the SODE:

$$\frac{d}{ds}(u\dot{u}) = C(u\dot{u})^{\alpha} \quad (57)$$

the Wünschmann-type condition reads:

$$\frac{C^2}{2}\alpha(\alpha-1)(\dot{u})^{2\alpha-1}u^{2\alpha-3} = 0$$

with solutions $\alpha = 0, \alpha = 1$. But Equation 57 for $\alpha = 0$ is Equation 9 and for $\alpha = 1$ is Equation 53.
Let I be a real open interval and the rotationally symmetric metric $ds^2 + u^2(s)dr_{n-1}^2$ on $I \times S^{n-1}$. In Petersen (1998), the equation:

$$2(n-1)\left[-\frac{\ddot{u}}{u} + \frac{n-2}{2}\cdot\frac{1-\dot{u}^2}{u^2}\right] = 0$$

appears as necessary and sufficient condition in order to have a scalar flat metric for $n \geq 3$. For $n = 4$ we obtain exactly Equation 53 with $C = 1$.

Lienard-type expression

The Wünschmann-type condition for $\Lambda = \varphi(u)(\dot{u})^{\alpha} + \psi(u)$, is:

$$\ddot{\varphi}(\dot{u})^{\alpha+1} + \ddot{\psi}\dot{u} + \alpha\dot{\varphi}(\dot{u})^{\alpha-1} = \alpha\varphi\dot{\varphi}(\dot{u})^{2\alpha-1} + 2\alpha\varphi\dot{\psi}(\dot{u})^{\alpha-1}$$

and identification:

$$\begin{cases}\alpha+1 = 2\alpha-1\\ \ddot{\varphi} = 2\varphi\dot{\varphi}\\ \ddot{\psi} + \alpha\dot{\varphi}\psi = 2\alpha\varphi\dot{\psi}\end{cases}$$

yields $\alpha = 2, \varphi = -\frac{1}{u}, \psi = \frac{A}{u^2} + \frac{B}{u}$ and then Equation 1 is:

$$\ddot{u} = -\frac{\dot{u}^2}{u} + \frac{A}{u^2} + \frac{B}{u} \quad (58)$$

which gives Equation 53 for $A = 0, B = C$. This SODE is a quasi Euler-Lagrange equation for Equation 49 since a comparison with equation 50:

$$\begin{cases}-\frac{3\alpha'}{2\alpha} = -\frac{1}{u}\\ -\frac{\beta}{\alpha} = \frac{A}{u^2} + \frac{B}{u}\end{cases}$$

gives $\alpha(u) = u^{2/3}, \beta(u) = -Ae^{-4/3} - Bu^{-1/3}$.

The open problem is to connect the aforementioned solutions from additive and multiplicative separation and Equation 10 with (HJ) equation!

WÜNSCHMANN-TYPE CONDITION FOR $u'' = Au'^3 + Bu'^2 + Cu' + D$

It is obvious that the simplest SODE has the form $u'' = 0$. After a general transformation belonging to the pseudogroup of point transformation:

$$\tilde{s} = \varphi(s,u), \quad \tilde{u} = \psi(s,u) \quad (59)$$

we get the SODE (Bordag, 1997):

$$\frac{d^2\tilde{u}}{rd\tilde{s}^2} = a_1(\tilde{s},\tilde{u})\left(\frac{d\tilde{u}}{d\tilde{s}}\right)^3 + 3a_2(\tilde{s},\tilde{u})\left(\frac{d\tilde{u}}{d\tilde{s}}\right)^2 + 3a_2(\tilde{s},\tilde{u})\frac{d\tilde{u}}{d\tilde{s}} + a_4(\tilde{s},\tilde{u}). \quad (60)$$

Then, we search the Wünschmann-like condition for the equation:

$$u'' = A(s,u)(u')^3 + B(s,u)(u')^2 + C(s,u)u' + D(s,u) \quad (61)$$

for which:

$$\begin{cases}\Lambda_{u'} = 3A(u')^2 + 2Bu' + C\\ \Lambda_u = A_u(u')^3 + B_u(u')^2 + C_u u' + D_u\end{cases} \quad (62)$$

Example 6: Geodesic curves on surfaces

Consider the surface S with coordinates $(s,u) = (x^1,x^2)$ and metric g_S. The system of differential equations for the geodesics of g_S can be

transformed into the SODE:

$$\frac{d^2u}{ds^2} = A\left(\frac{du}{ds}\right)^3 + B\left(\frac{du}{ds}\right)^2 + C\frac{du}{ds} + D \quad (63)$$

where:

$$\begin{cases}A = -\Gamma_{22}^1, & B = -2\Gamma_{12}^1 + \Gamma_{22}^2\\ C = -\Gamma_{11}^1 + 2\Gamma_{12}^2, & D = \Gamma_{11}^2.\end{cases} \quad (64)$$

with Γ_{jk}^i the Christoffel symbols.

The case $A = B = C = D = 0$ corresponds to the metric:

$$g_S = 2\left[\frac{ds^2+du^2}{1+s^2+u^2} - \frac{(sds+udu)^2}{(1+s^2+u^2)^2}\right] \quad (65)$$

with:

$$\begin{cases}\Gamma_{11}^1 = 2\Gamma_{12}^2 = \frac{2s}{1+s^2+u^2} \\ \Gamma_{22}^2 = 2\Gamma_{12}^1 = \frac{2u}{1+s^2+u^2} \\ \Gamma_{22}^1 = \Gamma_{11}^2 = 0.\end{cases} \quad (66)$$

Recall that two metrics with the same geodesics are said to be projectively equivalent. Then, because the geodesics of Equation 65 are straight lines, this metric is called the projective Euclidean metric.

Another case which is out of our interest is when $g_S = g_S(s)$, because $\Lambda_u = 0$ from Equation 62 (2). For example, if S is the usual plane, from $(x^1, x^2) = (s\cos u, s\sin u)$ we arrive at $g_S = ds^2 + s^2du^2$. This example leads to the following.

Example 7: Metrics in geodesic polar coordinates

Let us suppose:

$$g_S = ds^2 + G(s,u)du^2. \quad (67)$$

The Christoffel symbols are:

$$\begin{cases}\Gamma_{11}^1 = \Gamma_{11}^2 = \Gamma_{12}^1 = 0 \\ \Gamma_{12}^2 = \frac{G_s}{2G}, \Gamma_{22}^1 = -\frac{G_s}{2}, \Gamma_{22}^2 = \frac{G_u}{2G}\end{cases} \quad (68)$$

and then:

$$A = \frac{G_s}{2}, B = \frac{G_u}{2G}, C = \frac{G_s}{G}, D = 0. \quad (69)$$

Let us notice that the semispray-metric (Equation 2) is exactly of Equation 67-type. Supposing that in Equation 67 we have $G = G(u)$, the only non-vanishing coefficient is $B = \frac{G'(u)}{2G(u)}$ and then the SODE of Equation 63 reads:

$$u'' = \frac{G'(u)}{2G(u)}(u')^2 \quad (70)$$

or

which can be integrated:

$$\ln u' = \frac{1}{2}\ln G(u) + x.$$

From $u' = e^x\sqrt{G(u)}$, we get an implicit solution:

$$\int \frac{1}{\sqrt{G(u)}}du = e^x s + y. \quad (71)$$

For example, the SODE oe Equation 53 with $C = 0$ which is Equation 68 from the work of Garcia-Godinez et al. (2004) is of this type with $G(u) = u^{-2}$. Therefore, the equation of geodesics for the metric $g_S = ds^2 + u^{-2}du^2$ satisfies the Wünschmann-type condition.

Example 8: Liouville metrics

Let us consider a Liouville metric $g_S = \varphi(s,u)(ds^2 + du^2)$, hence:

$$\begin{cases}\Gamma_{11}^1 = \Gamma_{12}^2 = -\Gamma_{22}^1 = \frac{\varphi_s}{2\varphi} \\ -\Gamma_{11}^2 = \Gamma_{12}^1 = \Gamma_{22}^2 = \frac{\varphi_u}{2\varphi}\end{cases}$$

$$A = -C = -\frac{\varphi_s}{2\varphi}, B = D = -\frac{\varphi_u}{2\varphi}.$$

Let us study if the SODE (Equation 53) with $C \neq 0$ is of this type. From $A = 0$, we get that $\varphi = \varphi(u)$ and from:

$$\begin{cases}B = -\frac{\varphi'(u)}{2\varphi(u)} = -\frac{1}{u} \\ D = -\frac{\varphi'(u)}{2\varphi(u)} = \frac{C}{u}\end{cases}$$

we must have $C = -1$ and $\varphi(u) = u^2$. It results in the Liouville metric $g_S = u^2(ds^2 + du^2)$.

CHERN-WÜNSCHMANN INVARIANT FOR THIRD-ODEs

For the general third-order ODE:

$$u''' = F(s,u,u',u'') \quad (72)$$

Chern (1978) introduced a Wünschmann-type invariant:

$$I = D^2F_{u''} - 2F_{u''}DF_{u''} - 3DF_{u'} + 6F_u + \frac{4}{9}F_{u''}^3 + 2F_{u''}F_{u'}. \quad (73)$$

For the geometrical significance of relation $I = 0$, Theorem 1 from the work of Frittelli et al. (2003) and also

Crampin and Saunders (2005).

Firstly, we give an example of SODE satisfying the Wünschmann-type condition such that the derived third-ODE also satisfies $I = 0$. Namely, a direct computation shows these facts for:

$$u'' = -\frac{u'}{s} + \frac{u}{s^2} \quad (74)$$

with derived third-ODE:

$$u''' = \frac{3u'}{s^2} - \frac{3u}{s^3}. \quad (75)$$

Equation 74 is of Euler type and the well-known change $s = e^t$ gives $\frac{d^2u}{dt^2} = u$ which is example II from the Wünschmann-type condition revisited. Moreover, the solutions of Equation 74 are:

$$u(s) = C_1 s + \frac{C_2}{s}, \quad (76)$$

while the solutions to Equation 75 are:

$$u(s) = C_1 s + \frac{C_2}{s} + C_3 s^3. \quad (77)$$

Since in the cited papers (Chern, 1978; Frittelli et al., 2003) there are no examples of equations satisfying $I = 0$, we end this paper with such an example. In Banaru (1996) the geometry associated to the following equation of fifth-order is considered:

$$y^{(5)} = \frac{5y^{(3)}y^{(4)}}{y''} - \frac{40\left(y^{(3)}\right)^3}{9(y'')^2}.$$

With notation $y'' = u$, we obtain:

$$u''' = \frac{5u'u''}{u} - \frac{40(u')^3}{9u^2} \quad (78)$$

for which a straightforward verification gives $I = \frac{5u'u''}{u} - \frac{40(u')^3}{9u^2} - u'''$ yielding the vanishing of I.

The solutions of Equation 78 are:

$$u(s) = C_3[-9 + C_1 s^2 + 2C_1C_2 s + C_1 C_2^2]^{-\frac{3}{2}}. \quad (79)$$

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