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THE GEOMETRY OF PRODUCT CONJUGATE CONNECTIONS

BY

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Dedicated to Professor Vasile Cruceanu on the occasion of his 80th birthday

Abstract. Properties of pairs of product conjugate connections are stated with a special view towards the integrability of the given almost product structure. We define the analogous in product geometry of the structural and the virtual tensors from the Hermitian geometry and express the product conjugate connections in terms of these tensors. Some examples from the geometry of a pair of complementary distributions are discussed and for this case the above structural and virtual tensors are expressed in terms of O'Neill-Gray tensor fields.

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Introduction

Fix M a smooth, n-dimensional manifold for which we denote: $C^{\infty}(M)$ – the algebra of smooth real functions on M, $\mathfrak{X}(M)$ – the Lie algebra of vector fields on M, $T^r_s(M)$ – the $C^{\infty}(M)$-module of tensor fields of (r,s)-type on M. Usually $X, Y, Z, ...$ will be vector fields on M and if $T \to M$ is a vector bundle over M, then $\Gamma(T)$ denotes the C^{∞}-module of sections of T [e.g. $\Gamma(TM) = \mathfrak{X}(M)$].

Let $\mathcal{C}(M)$ be the set of linear connections on M. Since the difference of two linear connections is a tensor field of $(1,2)$-type, it results that $\mathcal{C}(M)$ is a $C^{\infty}(M)$-affine module associated to the $C^{\infty}(M)$-linear module $T^1_2(M)$.

Fix now E an almost product structure on M, i.e. an endomorphism of the tangent bundle such that $E^2 = I_{\mathfrak{X}(M)}$. Then the associated linear connections are:

Definition 0.1. $\nabla \in \mathcal{C}(M)$ is an *E-connection* if E is covariant constant with respect to ∇, namely $\nabla E = 0$. Let $\mathcal{C}_E(M)$ be the set of these connections.

In order to find the above set, let us consider after [5, p. 342] the maps

$$\psi_E : \mathcal{C}(M) \to \mathcal{C}(M), \quad \chi_E : T^1_2(M) \to T^1_2(M) \tag{0.1}$$

given by

$$\psi_E(\nabla) := \frac{1}{2}\left(\nabla + E \circ \nabla \circ E\right), \quad \chi_E(\tau) := \frac{1}{2}(\tau + E \circ \tau \circ E). \tag{0.2}$$

So

$$\begin{cases} \psi_E(\nabla)_X Y = \frac{1}{2}\left[\nabla_X Y + E(\nabla_X EY)\right] \\ \chi_E(\tau)(X,Y) = \frac{1}{2}\left[\tau(X,Y) + E(\tau(X,EY))\right]. \end{cases} \tag{0.3}$$

Then, ψ_E is a $C^\infty(M)$-projector on $\mathcal{C}(M)$ associated to the $C^\infty(M)$-linear projector χ_E:

$$\psi_E^2 = \psi_E, \quad \chi_E^2 = \chi_E, \quad \psi_E(\nabla + \tau) = \psi_E(\nabla) + \chi_E(\tau). \tag{0.4}$$

It follows that $\nabla E = 0$ means $\psi_E(\nabla) = \nabla$ which gives that $\mathcal{C}_E(M) = Im\psi_E$. This determines completely $\mathcal{C}_E(M)$. Fix ∇_0 arbitrary in $\mathcal{C}(M)$ and ∇ in $\mathcal{C}_E(M)$. So, $\nabla = \psi_E(\nabla')$ with $\nabla' = \nabla_0 + \tau$. In conclusion, $\nabla = \psi_E(\nabla_0) + \chi_E(\tau)$; in other words, $\mathcal{C}_E(M)$ is the affine submodule of $\mathcal{C}(M)$ passing through the E-connection $\psi_E(\nabla_0)$ and having the direction given by the linear submodule $Im\chi_E$ of $T^1_2(M)$.

Let us remark a decomposition (of arithmetic mean type) of it [5, p. 343]:

$$\psi_E(\nabla) = \frac{1}{2}(\nabla + C_E(\nabla)) \tag{0.5}$$

with *the conjugation map* $C_E : \mathcal{C}(M) \to \mathcal{C}(M)$:

$$C_E(\nabla)_X := E \circ \nabla_X \circ E. \tag{0.6}$$

Then *the product conjugate connection* $C_E(\nabla)$ measures how far the connection ∇ is from being an E-connection and as it is pointed out in [5,

p. 343], C_E is the affine symmetry of the affine module $\mathcal{C}(M)$ with respect to the affine submodule $\mathcal{C}_E(M)$, made parallel with the linear submodule $\ker \chi_E$.

The present paper is devoted to a large study of this new connection $C_E(\nabla)$, since all above computations put in evidence its rôle in the geometry of E. Therefore, the aim of our study is to obtain several properties of it in both the general case and Riemannian geometry. The first section is devoted to this scope and after a general result connecting ∇ and $C_E(\nabla)$, we treat two items:

i) the behavior of the product conjugate connections to a linear change of almost product structure,

ii) the introduction in the product geometry of two tensor fields previously considered in the complex geometry.

With respect to i) we arrive at two particular remarkable cases concerning the recurrence of the given almost product structures, while for ii) we derive some useful new identities.

The second part of this paper is directed towards examples and to put in our framework the geometry of (two complementary) distributions. The most important case is when the considered distributions are in a natural relationship with the initial linear connection ∇ or with the almost product structure E.

In the last section we give some generalizations of the results from the first part by adding an arbitrary tensor field of $(1,2)$-type. All generalized product conjugate connections which form a duality with the initial linear connection are determined.

1. Properties of the product conjugate connection

In what follows, for simplification we will denote by a superscript E the product conjugate connection of ∇

$$\nabla^{(E)} := C_E(\nabla) = \nabla + E \circ \nabla E \tag{1.1}$$

and then

$$\nabla^{(E)}_X Y = \nabla_X Y + E(\nabla_X EY - E(\nabla_X Y)) = E(\nabla_X EY). \tag{1.2}$$

The first properties of the product conjugate connection are stated in the next proposition:

Proposition 1.1. *Let E be an almost product structure, ∇ a linear connection and $\nabla^{(E)}$ the product conjugate connection of ∇. Then:*

1. $\nabla^{(E)}E = -\nabla E$; *it results that $\nabla \in \mathcal{C}_E(M)$ if and only if $\nabla^{(E)} \in \mathcal{C}_E(M)$.*
2. ∇ *and* $\nabla^{(E)}$ *are in duality:* $(\nabla^{(E)})^{(E)} = \nabla$.
3. $T_{\nabla^{(E)}} = T_\nabla + E(d^\nabla E)$, *where d^∇ is the exterior covariant derivative induced by ∇, namely $(d^\nabla E)(X,Y) := (\nabla_X E)Y - (\nabla_Y E)X$; it results that for $\nabla \in \mathcal{C}_E(M)$, the connections ∇ and $\nabla^{(E)}$ have the same torsion.*
4. $R_{\nabla^{(E)}}(X,Y,Z) = E(R_\nabla(X,Y,EZ))$; *it results that ∇ is flat if and only if $\nabla^{(E)}$ is so.*
5. *Assume that (M,g,E) is an almost product Riemannian manifold, i.e. $g(EX,EY) = g(X,Y)$. Then $(\nabla^{(E)}_X g)(EY,EZ) = (\nabla_X g)(Y,Z)$; it results that ∇ is a g-metric connection if and only if $\nabla^{(E)}$ is so.*

Proof. 1. The main relations used here are

$$\nabla^{(E)}_X EY = E(\nabla_X Y), \quad E(\nabla^{(E)}_X Y) = \nabla_X EY \tag{1.3}$$

and then

$$(\nabla_X E)Y = \nabla_X EY - E(\nabla_X Y) = E(\nabla^{(E)}_X Y) - \nabla^{(E)}_X EY = -(\nabla^{(E)}_X E)Y. \tag{1.4}$$

2. Although a direct proof can be provided by the formula (0.6), we prefer a proof here, in order to use (1.1):

$$(\nabla^{(E)})^{(E)} = \nabla^{(E)} + E \circ \nabla^{(E)}E = \nabla + E \circ \nabla E + E \circ (-\nabla E) = \nabla.$$

3. A direct computation gives

$$\begin{aligned} T_{\nabla^{(E)}}(X,Y) &:= \nabla^{(E)}_X Y - \nabla^{(E)}_Y X - [X,Y] \\ &= E(\nabla_X EY) - E(\nabla_Y EX) - [X,Y] \\ &= E(\nabla_X EY - \nabla_Y EX) + T_\nabla(X,Y) - \nabla_X Y + \nabla_Y X \\ &:= T_\nabla(X,Y) + E((\nabla_X E)Y - (\nabla_Y E)X). \end{aligned} \tag{1.5}$$

4.

$$\begin{aligned} R_{\nabla^{(E)}}(X,Y,Z) &:= \nabla^{(E)}_X \nabla^{(E)}_Y Z - \nabla^{(E)}_Y \nabla^{(E)}_X Z - \nabla^{(E)}_{[X,Y]} Z \\ &= \nabla^{(E)}_X E(\nabla_Y EZ) - \nabla^{(E)}_Y E(\nabla_X EZ) - E(\nabla_{[X,Y]} EZ) \\ &= E(\nabla_X \nabla_Y EZ) - E(\nabla_Y \nabla_X EZ) - E(\nabla_{[X,Y]} EZ) \\ &:= E(R_\nabla(X,Y,EZ)). \end{aligned} \tag{1.6}$$

5.

$$\begin{aligned} (\nabla^{(E)}_X g)(V,W) &:= X(g(V,W)) - g(\nabla^{(E)}_X V, W) - g(V, \nabla^{(E)}_X W) \\ &= X(g(V,W)) - g(E(\nabla_X EV), W) - g(V, E(\nabla_X EW)) \end{aligned}$$

for any X, V and $W \in \mathfrak{X}(M)$. With $V := EY$ and $W := EZ$, we get

$$\begin{aligned} &(\nabla^{(E)}_X g)(EY,EZ) = X(g(EY,EZ)) \\ &- g(E(\nabla_X Y), EZ) - g(EY, E(\nabla_X Z)) \\ &= X(g(Y,Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) := (\nabla_X g)(Y,Z). \end{aligned} \tag{1.7}$$

The above substitutions hold for $Y = EV$ and $Z = EW$. □

There are some direct consequences of these formulae:

i) if ∇ is the Levi-Civita connection of g, then $\nabla^{(E)}$ is also metric with respect to g,

ii) if ∇ is the Levi-Civita connection of g and in addition $\nabla \in \mathcal{C}_E(M)$, then $\nabla^{(E)} = \nabla$ as the unique symmetric g-metric connection.

More generally, let $f \in Diff(M)$ be an automorphism of the G-structure defined by E, i.e. $f_* \circ E = E \circ f_*$. If f is an affine transformation for ∇, namely $f_*(\nabla_X Y) = \nabla_{f_* X} f_* Y$, then f is also affine transformation for $\nabla^{(E)}$.

Two natural generalizations of the case $\nabla \in \mathcal{C}_E(M)$ are given by:

Proposition 1.2. *Let ∇ be a symmetric linear connection.*

i) Assume that E is ∇-recurrent, i.e. $\nabla E = \eta \otimes E$, where η is a 1-form. Then $\nabla^{(E)}$ is a semi-symmetric connection.

ii) Assume that $\nabla E = \eta \otimes I_{\mathfrak{X}(M)}$. Then $\nabla^{(E)}$ is a quarter-symmetric connection.

Proof. i) We have $\nabla^{(E)} = \nabla + \eta \otimes I$ and from the item 3 of the previous Proposition, we get $T_{\nabla^{(E)}} = \eta \otimes I - I \otimes \eta$.

ii) It results that $\nabla^{(E)} = \nabla + \eta \otimes E$ and, as above, we get $T_{\nabla^{(E)}} = \eta \otimes E - E \otimes \eta$. □

The next subject consists of the behavior of $\nabla^{(\cdot)}$ for families of almost product structures. Let E_1 and E_2 be two almost product structures and consider the pencil of $(1,1)$-tensor fields $E_{\alpha,\beta} := \alpha E_1 + \beta E_2$ with α and $\beta \in \mathbb{R}$. In order that $E_{\alpha,\beta}$ to be an almost product structure there are necessary two conditions:

1) E_1 and E_2 be skew-commuting structures: $E_1 E_2 = -E_2 E_1$,

2) (α, β) belongs to the unit circle S^1: $\alpha^2 + \beta^2 = 1$. Then:

$$\nabla_X^{(E_{\alpha,\beta})} Y = \alpha^2 \nabla_X^{(E_1)} Y + \beta^2 \nabla_X^{(E_2)} Y + \alpha\beta[E_1(\nabla_X E_2 Y) + E_2(\nabla_X E_1 Y)] \tag{1.8}$$

and there can be distinguished two remarkable particular cases:

i) if E_1 and E_2 are recurrent with respect to ∇ with the same 1-form of recurrence: $\nabla E_i = \eta \otimes E_i$, then the product conjugate connections coincide $\nabla^{(E_1)} \equiv \nabla^{(E_2)} =: \nabla^{(E_{12})}$ and it follows the invariance of $\nabla^{(E)}$:

$$\nabla^{(E_{\alpha,\beta})} = \nabla^{(E_{12})}, \tag{1.9}$$

ii) assume that the triple (∇, E_1, E_2) is a mixed-recurrent structure: $\nabla E_i = \eta \otimes E_j$ with $i \neq j$. Then ∇ is the average of the two product conjugate connections, $\nabla = \frac{1}{2}(\nabla^{(E_1)} + \nabla^{(E_2)})$ and

$$\nabla^{(E_{\alpha,\beta})} = \nabla + (\alpha^2 - \beta^2)\eta \otimes E_1 E_2. \tag{1.10}$$

The last subject of this section treats two tensor fields associated to an almost product structure. The paper [7] introduces the structural and virtual tensor fields of an almost complex structure. Turning into our framework, let us consider for a pair (∇, E) the tensor fields of $(1,2)$-type:

1) *the structural tensor field*

$$C_{\nabla}^{E}(X, Y) := \frac{1}{2}[(\nabla_{EX} E)Y + (\nabla_X E)EY]. \tag{1.11}$$

2) *the virtual tensor field*

$$B_{\nabla}^{E}(X, Y) := \frac{1}{2}[(\nabla_{EX} E)Y - (\nabla_X E)EY]. \tag{1.12}$$

From the item 1 of the first Proposition it results that both these tensor fields are skew-symmetric with respect to the product conjugation of connections:

$$C^E_{\nabla^{(E)}} = -C^E_{\nabla}, \quad B^E_{\nabla^{(E)}} = -B^E_{\nabla}. \tag{1.13}$$

Also

$$C^E_{\nabla}(EX, EY) = C^E_{\nabla}(X, Y), \; B^E_{\nabla}(EX, EY) = -B^E_{\nabla}(X, Y). \tag{1.14}$$

The importance of these tensor fields for our study is given by the following straightforward relation:

$$\nabla^{(E)} = \nabla - C^E_{\nabla} + B^E_{\nabla}. \tag{1.15}$$

Recall after [4] that two linear connections are called *projectively equivalent* if there exists a 1-form τ such that:

$$\nabla' = \nabla + \tau \otimes I + I \otimes \tau. \tag{1.16}$$

A straightforward calculus gives that C^E is invariant for projectively changes (1.16) while for B^E we have:

$$(B^E_{\nabla'} - B^E_{\nabla})(X, Y) = \tau(EY)EX - \tau(Y)X. \tag{1.17}$$

2. Invariant distributions

Let $\mathcal{D} \subset TM$ be a fixed distribution considered as a vector subbundle of TM.

Definition 2.1. i) $\mathcal{D}$ is called *E-invariant* if $X \in \Gamma(\mathcal{D})$ implies $EX \in \Gamma(\mathcal{D})$.

ii) ([3, p. 210]) The linear connection ∇ *restricts to* $\mathcal{D}$ if $Y \in \Gamma(\mathcal{D})$ implies $\nabla_X Y \in \Gamma(\mathcal{D})$, for any $X \in \Gamma(TM)$.

If ∇ restricts to $\mathcal{D}$, then ∇ may be considered as a connection in the vector bundle $\mathcal{D}$. From this fact, in [1, p. 7] a connection which restricts to $\mathcal{D}$ is called *adapted to* $\mathcal{D}$.

Proposition 2.2. *If the distribution $\mathcal{D}$ is E-invariant and the linear connection ∇ restricts to $\mathcal{D}$, then $\nabla^{(E)}$ also restricts to $\mathcal{D}$.*

Proof. Fix $Y \in \Gamma(\mathcal{D})$. Then $EY \in \Gamma(\mathcal{D})$ and for any $X \in \Gamma(TM)$ we have $\nabla_X Y \in \Gamma(\mathcal{D})$. Therefore, $\nabla^{(E)}_X Y = E(\nabla_X EY) \in \Gamma(\mathcal{D})$. □

A more general notion like restricting to a distribution is that of geodesically invariance [3, p. 118]. The distribution $\mathcal{D}$ is ∇-*geodesically invariant* if for every geodesic $\gamma : [a, b] \to M$ of ∇ with $\dot{\gamma}(a) \in \mathcal{D}_{\gamma(a)}$ it follows $\dot{\gamma}(t) \in \mathcal{D}_{\gamma(t)}$ for any $t \in [a, b]$. The cited book gives a necessary and sufficient condition for a distribution $\mathcal{D}$ to be ∇-geodesically invariant: for any X and $Y \in \Gamma(\mathcal{D})$, *the symmetric product* $\langle X : Y \rangle := \nabla_X Y + \nabla_Y X$ to belong to $\Gamma(\mathcal{D})$ or equivalently, for any $X \in \Gamma(\mathcal{D})$ to have $\nabla_X X \in \Gamma(\mathcal{D})$.

The following result is a direct consequence of definitions:

Proposition 2.3. *If the distribution $\mathcal{D}$ is E-invariant and the linear connection ∇ restricts to $\mathcal{D}$, then $\mathcal{D}$ is geodesically invariant for $\nabla^{(E)}$.*

Example 2.4. Assume that the tangent bundle TM admits a decomposition

$$TM = V \oplus H \tag{2.1}$$

into *vertical* and *horizontal vectors.* Let $\mathcal{D}_v = \Gamma(V)$ respectively, $\mathcal{D}_h = \Gamma(H)$ and the corresponding projectors v and h. Then $E = h - v$ is an almost product structure and both $\mathcal{D}_*$ are E-invariant. As it is proved in [1], the almost product structures are in a natural relationship with decompositions of (2.1)-type and so, $E = h - v$ is the prototype of all possible almost product structures.

The product conjugate connection of ∇ is

$$\nabla^{(E)}_X Y = h(\nabla_X hY) - h(\nabla_X vY) - v(\nabla_X hY) + v(\nabla_X vY) \tag{2.2}$$

and then we have:

Proposition 2.5. *If $\nabla^{(E)}$ is torsion-free, then E is integrable, which means that $\mathcal{D}_h$ and $\mathcal{D}_v$ are involutive distributions.*

Proof. From (2.2) we get $h[X, Y] + v[X, Y] = \nabla^{(E)}_X Y - \nabla^{(E)}_Y X = h(\nabla_X hY - \nabla_Y hX) + v(\nabla_X vY - \nabla_Y vX)$ and then $h[X, Y] = \nabla_X hY - \nabla_Y hX$, $\quad v[X, Y] = \nabla_X vY - \nabla_Y vX$. With $X \to vX$ and $Y \to vY$ in the first relation above it follows $h[vX, vY] = 0$ and the change $X \to hX$ and $Y \to vY$ in the second relation yields $v[hX, hY] = 0$. □

We have:

1) ∇ restricts to $\mathcal{D}_h$ means $v(\nabla_X hY) = 0$ and $h(\nabla_X hY) = \nabla_X hY$,

2) ∇ restricts to $\mathcal{D}_v$ means $h(\nabla_X vY) = 0$ and $v(\nabla_X vY) = \nabla_X vY$.

A straightforward computation gives that the general $\nabla^{(E)}$ of (2.2) restricts to $\mathcal{D}_h$ and $\mathcal{D}_v$. Moreover, if ∇ restricts to both $\mathcal{D}_*$, then

$$\nabla^{(E)}_X Y = \nabla_X hY + \nabla_X vY = \nabla_X Y \tag{2.3}$$

and so $\nabla \in \mathcal{C}_E(M)$. Let us remark that the above connection (2.3) is exactly *the Schouten connection* of the pair (h, v) [6, p. 10]:

$$\nabla_X Y = h(\nabla_X hY) + v(\nabla_X vY). \tag{2.4}$$

Example 2.6. Let M be a vector bundle over the manifold U by $\pi : M \to U$ and $V = \ker T\pi$ *the vertical bundle* over M. Let also H_1 and H_2 be two *horizontal bundles* in the decomposition (2.1) and denote h_1 respectively, h_2 their projectors. Then the skew-symmetry $E_1E_2 = -E_2E_1$ for the corresponding almost product structures of the previous example means the skew-symmetry $h_1h_2 = -h_2h_1$.

Sometimes, a complementary distribution H to the above vertical subbundle V is called *Ehresmann connection*, while if M is exactly the tangent bundle TU, then H is called *nonlinear connection* [2].

Example 2.7. Let E be the almost product structure of example 2.4. Our next step is to express the Kirichenko tensor fields in terms of the projectors h, v:

Proposition 2.8. *The structural and virtual tensor fields of $E = h - v$ are*:

$$\begin{cases} C^{h-v}_{\nabla}(X,Y) = 2[h(\nabla_{vX} vY) + v(\nabla_{hX} hY)] \\ B^{h-v}_{\nabla}(X,Y) = -2[h(\nabla_{hX} vY) + v(\nabla_{vX} hY)]. \end{cases} \tag{2.5}$$

Proof. From (1.14) we get

$$\begin{cases} C^{h-v}_{\nabla}(hX, vY) = -C^{h-v}_{\nabla}(vX, hY) \\ B^{h-v}_{\nabla}(hX, hY) = -B^{h-v}_{\nabla}(vX, vY). \end{cases} \tag{2.6}$$

By making $X \to vX$ in the first relation and $X \to hX$ in the second one, it results

$$\begin{cases} C^{h-v}_{\nabla}(hX, vY) = 0 = C^{h-v}_{\nabla}(vX, hY) \\ B^{h-v}_{\nabla}(hX, hY) = 0 = B^{h-v}_{\nabla}(vX, vY) \end{cases} \tag{2.7}$$

and then

$$\begin{cases} C_{\nabla}^{h-v}(X,Y) = C_{\nabla}^{h-v}(hX,hY) + C_{\nabla}^{h-v}(vX,vY) \\ B_{\nabla}^{h-v}(X,Y) = B_{\nabla}^{h-v}(hX,hY) + B_{\nabla}^{h-v}(vX,vY). \end{cases} \tag{2.8}$$

But

$$\begin{cases} C_{\nabla}^{h-v}(hX,hY) = 2v(\nabla_{hX}hY) \\ C_{\nabla}^{h-v}(vX,vY) = 2h(\nabla_{vX}vY) \end{cases} \tag{2.9}$$

and

$$\begin{cases} B_{\nabla}^{h-v}(hX,vY) = -2h(\nabla_{hX}vY) \\ B_{\nabla}^{h-v}(vX,hY) = -2v(\nabla_{vX}hY) \end{cases} \tag{2.10}$$

and then we have the conclusion. □

Let us recall the well-known *fundamental tensor fields* of O'Neill-Gray:

$$\begin{cases} T(X,Y) = h(\nabla_{vX}vY) + v(\nabla_{vX}hY) \\ A(X,Y) = v(\nabla_{hX}hY) + h(\nabla_{hX}vY). \end{cases} \tag{2.11}$$

Then, a comparison of last two equations yields

$$\begin{cases} C_{\nabla}^{h-v}(X,Y) = 2[T(X,vY) + A(X,hY)] \\ B_{\nabla}^{h-v}(X,Y) = -2[T(X,hY) + A(X,vY)] \end{cases} \tag{2.12}$$

a fact which justifies the second name of T and A as *invariants* of (2.1) [6, p. 9].

3. Generalized product conjugate connections

In this section we present a natural generalization of the product conjugate connection.

Definition 3.1. A *generalized product conjugate connection* of ∇ is

$$\nabla^{(E,C)} = \nabla^{(E)} + C \tag{3.1}$$

with $C \in T_2^1(M)$ arbitrary.

Since the duality $\nabla \leftrightarrow \nabla^{(E)}$ is a main feature of $\nabla^{(E)}$, let us search for tensor fields C such that $(\nabla^{(E,C)})^{(E,C)} = \nabla$. From

$$(\nabla^{(E,C)})^{(E,C)}_X Y = \nabla_X Y + E(C(X, EY)) + C(X, Y) \tag{3.2}$$

it results that we are interested in finding solutions C to

$$E(C(X, EY)) + C(X, Y) = 0. \tag{3.3}$$

Let us remark that:

i) $C_0 = \nabla E$ is a particular solution of (3.3),

ii) if C is a solution, then $E \circ C$ is also a solution.

So, let us search the duality property for

$$\nabla^{(E,\lambda,\mu)} = \nabla^{(E)} + \lambda \nabla E + \mu E(\nabla E) = (1 + \mu)\nabla^{(E)} + \lambda \nabla E \tag{3.4}$$

with λ and $\mu \in \mathbb{R}$.

Proposition 3.2. *The duality $\nabla \leftrightarrow \nabla^{(E,\lambda,\mu)}$ holds only for the pairs $(\lambda, \mu) \in \{(0, 0), (0, -2), (1, -1), (-1, -1)\}$.*

Proof. From $(\nabla^{(E,\lambda,\mu)})^{(E,\lambda,\mu)}_X Y = [(1+\mu)^2+\lambda^2]\nabla_X Y+2\lambda(1+\mu)E(\nabla_X Y)$ it results the system $\begin{cases} (1 + \mu)^2 + \lambda^2 = 1 \\ \lambda(1 + \mu) = 0 \end{cases}$ which has the above solutions.

Let us point out that $\begin{cases} \nabla^{(E,0,0)} = \nabla^{(E)} \\ \nabla^{(E,0,-2)} = -\nabla^{(E)} \\ \nabla^{(E,1,-1)} = \nabla \\ \nabla^{(E,-1,-1)} = -\nabla \end{cases}$ which confirm our result.□

Returning to the general case (3.1), let us present the generalizations of some relations from Proposition 1.1:

1. $\nabla^{(E,C)}E = -\nabla E + C(\cdot, E\cdot) - E \circ C$. Then $\nabla \in \mathcal{C}_E(M)$ if and only if $\nabla^{(E,\lambda\nabla E+\mu E\circ\nabla E)} \in \mathcal{C}_E(M)$ with λ and μ arbitrary real numbers.

2. the discussion above.

3. $T_{\nabla^{(E,C)}} = T_\nabla + E(d^\nabla E) + 2C_{skew}$, where C_{skew} is the skew-symmetric part of C, i.e. $2C_{skew}(X, Y) = C(X, Y) - C(Y, X)$. So, if C is symmetric and $\nabla \in \mathcal{C}_E(M)$, then ∇ and $\nabla^{(E,C)}$ have the same torsion.

4. $R_{\nabla^{(E,C)}}(X, Y)Z = E(R_\nabla(X, Y)EZ) + C(X, E(\nabla_Y EZ)) - C(Y, E(\nabla_X EZ)) - C([X, Y], Z) + E(\nabla_Y E(C(Y, Z))) - E(\nabla_Y E(C(X, Z)))$.

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