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GOLDEN- AND PRODUCT-SHAPED HYPERSURFACES IN REAL SPACE FORMS

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We define two classes of hypersurfaces in real space forms, golden- and product-shaped, respectively, by imposing the shape operator to be of golden or product type. We obtain the whole families of above hypersurfaces, based on the classification of isoparametric hypersurfaces, as follows: in the golden case all are hyperspheres, plus a hyperbolic space and a generalized Clifford torus, while for the product case we obtain the unit hypersphere, the hyperplane, a hypersphere and its associated Clifford torus, respectively, according to the type of the ambient space form namely parabolic, hyperbolic or elliptic, respectively.

Keywords: Hypersurface; shape operator; principal curvatures; golden structure; almost product structure; isoparametric hypersurface.

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1. Introduction

In [4, 7] the first two authors introduce the notion of *golden structure* on a manifold M as a tensor field of $(1,1)$-type J on M which satisfies the same equation as the well-known Golden Ratio:

$$J^2 = J + I, \tag{1}$$

where I is the usual Kronecker tensor field of M. Several examples of geometric golden structures are given in the cited paper. Also, in [9], the third author studied a special kind of logarithmic spiral — the golden spiral — (which has a straight connection with the golden ratio) and he found all constant slope surfaces in the

Euclidean 3-space, for which the normal direction in a point of the surface makes a constant angle with the position vector of that point.

The present work continues the list of remarkable examples by focusing to a class of well-known objects namely hypersurfaces in real space forms. So, we obtain all hypersurfaces of this type having the shape operator as golden structure. It is interesting to notice that they are hyperspheres, a hyperbolic hyperplane in the hyperbolic case, or a generalized Clifford torus in the spherical space form, and this can be a new motivation to see the sphere as a *golden-shaped* surface. A well-known result of parallelism for the shape operators satisfying a quadratic equation is applied also for the spherical framework. More precisely, all golden (as well as product) hypersurfaces in real space forms are parallel hypersurfaces.

Since in [4] there is derived a natural correspondence between golden structures and almost product structures we finish our study with the *product-shaped* version of the above search. In this setting, there are large differences between results: we obtain the unit hypersphere in the Euclidean case, a hyperplane in the hyperbolic case and another hypersphere together with its associated Clifford torus for the spherical case.

2. Golden-Shaped Hypersurfaces

Let M be an oriented embedded hypersurface of the real space form $M^{n+1}(c)$ and for a certain normal field N, let $A = A_N$ be the associated shape operator (see [11]); then $\lambda_1, \ldots, \lambda_n$ will be the principal curvatures of M (see [3] for the semi-Euclidean case). We will take $c \in \{-1, 0, 1\}$ and consequently $M^{n+1}(0) = \mathbb{R}^{n+1}$, $M^{n+1}(1) = \mathbb{S}^{n+1} = S^{n+1}(1)$, respectively $M^{n+1}(-1) = \mathbb{H}^{n+1}$.

Definition 1. M is called *golden-shaped hypersurface* if A is a golden structure i.e. $A^2 = A + I$ where I is the identity on the tangent bundle of M.

It follows that the principal curvatures of M, as eigenvalues of A, are the golden mean $\varphi = \frac{1+\sqrt{5}}{2} > 0$ and $\Phi = 1 - \varphi = \frac{1-\sqrt{5}}{2} < 0$. According to [2], the manifold M is an *isoparametric hypersurface* and this yields the following theorem.

Theorem 1. *The only golden-shaped hypersurfaces of* $\mathbb{R}^{n+1}$ *are the hyperspheres*:

$$S_1^n = S^n\left(\frac{1}{\varphi}\right) = S^n\left(\frac{\sqrt{5}-1}{2}\right), \quad S_2^n = S^n(\varphi) = S^n\left(\frac{\sqrt{5}+1}{2}\right). \tag{2}$$

Proof. We distinguish the following three cases:

(1) If $\lambda_1 = \cdots = \lambda_n = \varphi$ then M is totally umbilical and one obtains $S^n(\frac{1}{\varphi})$ which represents the first above sphere.
(2) If $\lambda_1 = \cdots = \lambda_n = \Phi$ then M is totally umbilical, too, and one gets $S^n(\frac{1}{|\Phi|})$ which corresponds to the second above sphere.

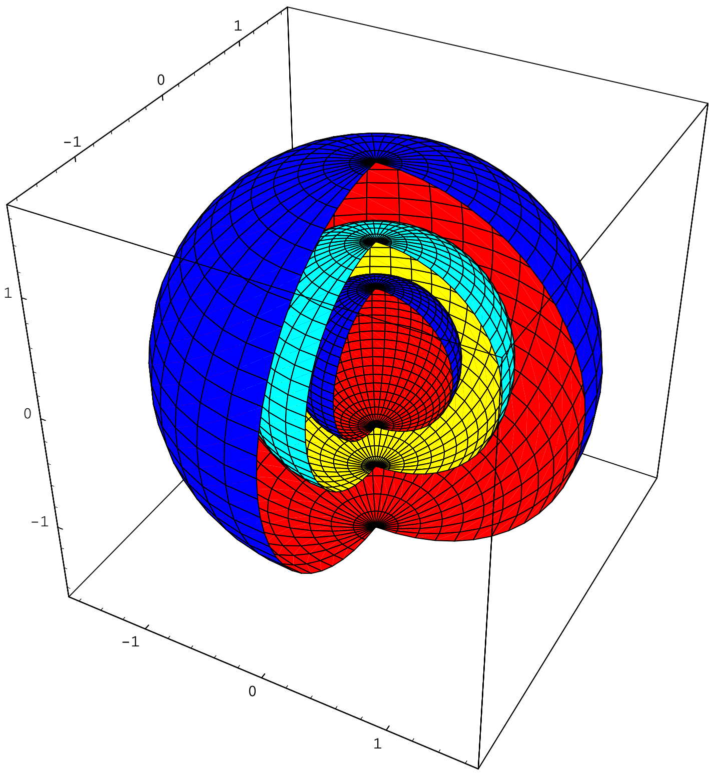

Fig. 1. The spheres S_1^2, S_2^2 and the unit sphere $S^2(1)$ in center.

Please provide Figures 1 to 3 citation in the text

(3) Suppose $\lambda_1 = \cdots = \lambda_k = \varphi$ and $\lambda_{k+1} = \cdots = \lambda_n = \Phi$. Recall that the isoparametric hypersurfaces in $\mathbb{R}^{n+1}$ with non-vanishing principal curvatures are only hyperspheres (for hyperplanes and cylinders, conform the cited paper [2], one principal curvature is zero which is not our case). $\square$

Remark 1. Let us point out that we use both the orientations given by N and $-N$, in order to obtain a large class of golden hypersurfaces. Thus, the sphere $S^n(r)$ is considered with both shape operators $A_N = -\frac{1}{r}I$ as well as $A_{-N} = \frac{1}{r}I$.

Recall that the hyperbolic space in the upper half-space model is defined as:

$$\mathbb{H}^{n+1} = \{x \in \mathbb{R}^{n+2}; (x^1)^2 + \cdots + (x^{n+1})^2 - (x^{n+2})^2 = -1, x^{n+2} > 0\}$$

and the isoparametric hypersurfaces of it are [13, p. 252]:

(i) $M_{h,1,k} = \{x \in \mathbb{H}^{n+1}; x^k = 0\}$, for $1 \leq k \leq n+1$ with $A = O$.

(ii) $M_{h,2,k,r} = \{x \in \mathbb{H}^{n+1}; x^k = r > 0\}$ for $1 \leq k \leq n+1$ with $A = \sqrt{c+1}I$, where $c = -\frac{1}{r^2} \in (-1, 0)$; then $r \in (1, +\infty)$. In this case M is isometric to the hyperbola $H^n(c)$.

(iii) $M_{h,3} = \{x \in \mathbb{H}^{n+1}; x^{n+2} = x^{n+1} + 1\}$ with $A = I$. In this situation M is isometric to $\mathbb{R}^n$.

(iv) $M_{h,4,r} = \{x \in \mathbb{H}^{n+1}; (x^1)^2 + \cdots + (x^{n+1})^2 = r^2\} = \mathbb{H}^{n+1} \cap S^n(r)$ with $A = \sqrt{c+1}I$, where $c = \frac{1}{r^2} > 0$. $M_{h,4,r}$ is isometric to $S^n(r)$.

(v) $M_{h,5,k,r} = \{x \in \mathbb{H}^{n+1}; (x^1)^2 + \cdots + (x^{k+1})^2 = r^2\} = \mathbb{H}^{n+1} \cap S^k$ for $1 \leq k \leq n$ with $A = \lambda I_k \oplus \frac{1}{\lambda} I_{n-k}$ where $\lambda = \frac{\sqrt{r^2+1}}{r} > 0$.

One can state now the following theorem.

Theorem 2. *The only golden-shaped hypersurface of* $\mathbb{H}^{n+1}$ *are as follows*:

(1) *For* $1 \leq k \leq n+1$:

$$M_{h,2,k,\sqrt{\varphi}} = \left\{ x \in \mathbb{H}^{n+1}; x^k = \sqrt{\varphi} = \sqrt{\frac{\sqrt{5}+1}{2}} \right\}, \quad (3)$$

which is isometric to the hyperbola $H^n(\Phi)$.

(2) *The hypersphere* $S_3^n = S^n(\sqrt{\varphi - 1}) = S^n(\sqrt{\frac{\sqrt{5}-1}{2}})$ *via*:

$$M_{h,4,\frac{1}{\sqrt{\varphi}}} = \left\{x \in \mathbb{H}^{n+1}; x^{n+2} = \sqrt{\varphi}\right\}, \quad (4)$$

which is isometric to $S^n(\frac{1}{\sqrt{\phi}})$.

Proof. Again, the third case of the previous proof cannot occur since $\lambda > 0$. Also, for $M_{h,2,k,r}$, with $c + 1 = \varphi^2 = \varphi + 1$, one has $c = \varphi \notin (-1, 0)$. The following situations need to be studied:

(1) $c + 1 = \Phi^2 = 2 - \varphi$ which means $c = 1 - \varphi \in (-1, 0)$ and $-\frac{1}{r^2} = 1 - \varphi$ means $r = \frac{1}{\sqrt{\varphi - 1}} = \sqrt{\varphi}$ and this is (3).

(2) $M_{h,4,r}$ with $r = \frac{1}{\sqrt{\varphi}} = \sqrt{\varphi - 1}$. □

For $n = 1$ let us draw a picture corresponding to the statements of the previous theorem.

In the sequel let M be a hypersurface of $\mathbb{S}^{n+1}$ which is isoparametric of type l, i.e. the constant principal curvatures of M are $\lambda_1 > \cdots > \lambda_l$. It is known (see [12, 13]), that $l \in \{1, 2, 3, 4, 6\}$ and for $l \leq 2$ we obtain:

(i) For $l = 1$ i.e. M is umbilical; $M_{\mathrm{sph},r} = \{x \in \mathbb{S}^{n+1}; x^{n+2} = \sqrt{1 - r^2}\}$ for $r \in (0, 1)$ with $A = \frac{\sqrt{1-r^2}}{r} I$. $M_{\mathrm{sph},r}$ is isometric to $S^n(r)$.

(ii) For $l = 2$, M is *the generalized Clifford torus* $M_{\mathrm{sph},r_1,r_2} = S^m(r_1) \times S^{n-m}(r_2)$ with $r_1^2 + r_2^2 = 1$ and $1 \leq m < n$. From [6, p. 86] we have: $r_1 = \frac{1}{\sqrt{1+\lambda_1^2}}$ and $r_2 = \frac{1}{\sqrt{1+\lambda_2^2}}$ with $\lambda_1 \lambda_2 = -1$ which is exactly our case for $\lambda_1 = \varphi$ and $\lambda_2 = \Phi$. Let θ be the angle given by $r_1 = \cos\theta$ and $r_2 = \sin\theta$; then the shape operator of

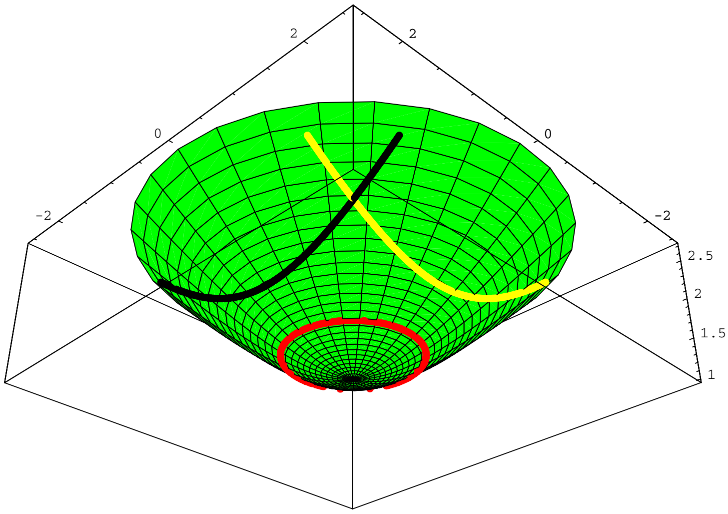

Fig. 2. The circle S^1_3 and the hyperbolas $H^1(\Phi)$ on the hyperbolic plane $\mathbb{H}^2$.

M_{sph,r_1,r_2} is (see [6, p. 80]):

$$A = \begin{pmatrix} -\tan\theta\, I_{n-m} & 0 \\ 0 & \cot\theta\, I_m \end{pmatrix}. \tag{5}$$

Theorem 3. *The only golden-shaped hypersurfaces of $\mathbb{S}^{n+1}$ are:*

(1) *For $l = 1$ the hyperspheres $S^n_4 = S^n(\frac{1}{\sqrt{\varphi+2}}) = S^n(\sqrt{\frac{5-\sqrt{5}}{10}})$ and $S^n_5 = S^n(\frac{1}{\sqrt{3-\varphi}}) = S^n(\sqrt{\frac{5+\sqrt{5}}{10}})$ via:*

$$M_{\mathrm{sph},\frac{1}{\sqrt{\varphi+2}}} = \left\{ x \in \mathbb{S}^{n+1}; x^{n+2} = \frac{\varphi}{\sqrt{\varphi+2}} = \sqrt{\frac{5+\sqrt{5}}{10}} \right\}, \tag{6}$$

$$M_{\mathrm{sph},\frac{1}{\sqrt{3-\varphi}}} = \left\{ x \in \mathbb{S}^{n+1}; x^{n+2} = \sqrt{\frac{2-\varphi}{3-\varphi}} = \sqrt{\frac{5-\sqrt{5}}{10}} \right\}. \tag{7}$$

(2) *For $l = 2$ the generalized Clifford torus:*

$$M_{\mathrm{sph},\frac{1}{\sqrt{2+\varphi}},\frac{1}{\sqrt{3-\varphi}}} = S^m_4 \times S^{n-m}_5. \tag{8}$$

Proof. The case $\frac{\sqrt{1-r^2}}{r} = \varphi$ yields the hypersphere (6), while $\frac{\sqrt{1-r^2}}{r} = |\Phi|$ corresponds to (7). For $r_1 = \frac{1}{\sqrt{1+\varphi^2}} = \frac{1}{\sqrt{2+\varphi}}$ and $r_2 = \frac{1}{\sqrt{1+\Phi^2}} = \frac{1}{\sqrt{3-\varphi}}$ we get $r_1^2 + r_2^2 = 1$ which means the given torus. We have $\tan\theta = \varphi - 1$ and

$$A = \begin{pmatrix} \Phi I_{n-m} & 0 \\ 0 & \varphi I_m \end{pmatrix} \tag{9}$$

is the shape operator of this torus. □

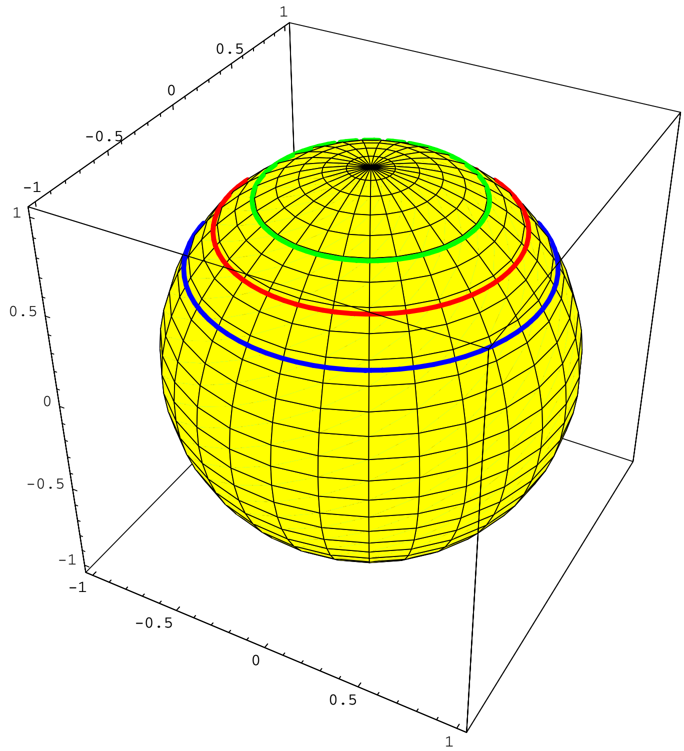

Fig. 3. (Color online) The circles S_4^1 (green) and S_5^1 (blue) on the sphere $\mathbb{S}^2$; in red, the circle of radius $\frac{1}{\sqrt{2}}$.

In order to point out an important feature of golden-shaped hypersurfaces, recall that a hypersurface, is called *parallel* if its second fundamental form (equivalently its shape operator) is covariant constant: $\nabla A = 0$. A classification result for (complete and full) parallel submanifolds (not only hypersurfaces) in real space form appears in [16]. We finish this section with the following proposition.

Proposition 1. *A golden-shaped hypersurface of* $M^{n+1}(c)$ *is parallel.*

Proof. (i) $c = 0$: For S_1^n and S_2^n we have $A = \lambda I$ with $\lambda \in \mathbb{R}$.

(ii) $c = -1$: Again $M_{h,2,k,\sqrt{\varphi}}$ and $M_{h,4,\frac{1}{\sqrt{\varphi}}}$ has $A = \lambda I$.

(iii) For the spherical case we adopt another point of view. In [6, p. 87] it is proved that if M^n is a hypersurface of $M^{n+1}(c)$ with the shape operator satisfying:

$$A^2 = \alpha A + cI, \tag{10}$$

then A is parallel. For $\alpha = 1 = c$ we get the equation provided by (1) and therefore we get the conclusion. □

3. Product-Shaped Hypersurfaces

In [4] it is proved that there exists a natural relationship between golden structures and almost product structures and this naturally leads to the following definition.

Definition 2. M is called product-shaped hypersurface of $M^{n+1}(c)$ if $A^2 = I$.

Since the eigenvalues of A are then ± 1, with similar computations as in Sec. 2 we obtain the following theorem.

Theorem 4. *The product-shaped hypersurfaces are as follows*:

(1) *For* $\mathbb{R}^{n+1}$: $S^n_6 = S^n(1)$.
(2) *For* $\mathbb{H}^{n+1}$: $M_{h,3} \simeq \mathbb{R}^n$.
(3) *For* $\mathbb{S}^{n+1}$: $S^n_7 = S^n(\frac{1}{\sqrt{2}})$ *and the Clifford torus* $S^m_7 \times S^{n-m}_7$.

A product-shaped hypersurface of $M^{n+1}(c)$ *is parallel.*

Remark 2. (i) Let us point out that in [5] it is obtained the classification of Euclidean submanifolds (not necessary hypersurfaces) having an unipotent shape operator which means $A^2_N = k\|N\|^2 I$ with the constant $k > 0$. For unitary N and $k = 1$ we recover our case above.
(ii) It is important to observe that $\mathbb{R}^{n+1}$ and $\mathbb{H}^{n+1}$ have only one product-shaped hypersurface while $\mathbb{S}^{n+1}$ has n such hypersurfaces since in Theorem 4 we have $m \in \{1, \dots, n-1\}$.

4. Conclusions

The golden ratio $\phi = \frac{1+\sqrt{5}}{2} = 1.61803398874989...$ is an irrational number which has many applications in mathematics, computational science, biology, art, architecture, nature, etc. Thus, mathematicians, psychologists, architects, historians, biologists, artists have discovered the beauty of this omnipresent number (see [14]).

We conclude this paper with the following properties for the seven hyperspheres we obtained:

I. Denoting by r_i the radius of S^n_i, $1 \leq i \leq 7$ we get:

(1) Euclidean ambient, golden hypersurface: $r_1 = 0.618$, $r_2 = 1.618$;
(2) hyperbolic ambient, golden hypersurface: $r_3 = 0.786$;
(3) spherical ambient, golden hypersurface: $r_4 = 0.525$, $r_5 = 0.850$;
(4) Euclidean ambient, product hypersurface: $r_6 = 1$;
(5) spherical ambient, product hypersurface: $r_7 = 0.707$.

Hence: $\frac{1}{2} < r_4 < r_1 < r_7 < r_3 < r_5 < r_6 = 1 < r_2 = \phi$.

II. S^n_2 is the inverse of S^n_1 with respect to S^n_6 while S^n_5 is the inverse of S^n_4 with respect to $S^n_8 = S^n(5^{-\frac{1}{4}})$. Also, the stereographic projection, which is a special case of sphere inversion, transforms $M_{h,3}$ of Theorem 4 into S^n_6 except the North Pole.

III. There exists another intimate relationship between the sphere S^3 and the golden ratio. Namely, as it is pointed out in [8, p. 291], the golden ratio yields the set of vertices for *the icosahedron* and its rotation group I yields *the binary icosahedral group* $\tilde{I}$ which is one of the finite subgroups of S^3. So, in analogy with the title of [9] we can rename this paper as: from the icosahedron to spheres.

This paper can be seen as a first step in this theme, the natural following ones are for classes of ambient manifolds with classified isoparametric hypersurfaces: complex space forms ([15]), projective spaces ([1]), Lorentzian products ([10]) and so on.

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