Weighted inequalities in triangle geometry

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Abstract

The paper contains two parts. In the first we point some applications of a weighted inequality and in the second part the equality conditions are obtained.

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In [1] it is proved the following:

**Proposition 1** Let $m, n, p$ be real numbers such that $m + n > 0, n + p > 0, p + m > 0, mn + np + pm > 0$. Then in any triangle $ABC$ the following inequality holds:

\[
ma^2 + nb^2 + pc^2 \geq 4\sqrt{mn} + np + pmS
\]

with standard notations.

Some applications are given in the cited paper:

(2) \[a^2 + b^2 + c^2 \geq 4\sqrt{3}S\] for $m = n = p$

(3) \[a^4 + b^4 + c^4 \geq 4\sqrt{a^2b^2 + b^2c^2 + c^2a^2}S\] for $m = a^2, n = b^2, p = c^2$

and therefore:

(3’) \[a^4 + b^4 + c^4 \geq 16S^2\]

(4) \[9a^2 + 5b^2 - 3c^2 \geq 4\sqrt{3}S\] for $m = 9, n = 5, c = -3$

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Let us point some other applications of (1):

I) the problem O:553 from Gazeta Matematică, no. 5-6(1988), p. 260 (without author):

(7) \[ 3a^2 + 3b^2 - c^2 \geq 4\sqrt{3} \quad \text{for} \quad m = n = 3, c = -1. \]

II) the problem E 3150 proposed by George A. Tsintsifas in American Mathematical Monthly, vol. 93(1986), p. 400:

(8) \[ \frac{m}{n+p}a^2 + \frac{n}{p+m}b^2 + \frac{p}{m+n}c^2 \geq 2\sqrt{3} \]

where \( m, n, p \) are positive real numbers. From (1) we have:

\[ \frac{m}{n+p}a^2 + \frac{n}{p+m}b^2 + \frac{p}{m+n}c^2 \geq \]

\[ \geq 4\sqrt{\frac{mn}{(n+p)(p+m)} + \frac{np}{(p+m)(m+n)} + \frac{pm}{(n+p)(m+n)}} S. \]

Therefore it must be proved that:

\[ \frac{mn}{(n+p)(p+m)} + \frac{np}{(p+m)(m+n)} + \frac{pm}{(m+n)(n+p)} \geq \frac{3}{4} \]

or, equivalent:

(9) \[ mn (m + n) + np (n + p) + pm (p + m) \geq \frac{3}{4} (m + n) (n + p) (p + m). \]

But the left-hand side of (9) is \( m^2n + mn^2 + n^2p + np^2 + p^2m + pm^2 \) and the right-hand side of (9) is \( \frac{3}{4} (2mnp + m^2n + mn^2 + p^2m + pm^2 + n^2p + np^2) \). Then (9) is equivalent with \( m^2n + mn^2 + n^2p + np^2 + p^2m + pm^2 \geq 6mnp \) which is consequence of AM-GM inequality. For others three solutions of (8) see the cited journal, vol. 95(1988), p. 658-659.

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A natural question with respect to (1) is: when equality holds? The aim of this paper is to give the answer. More precisely, we will show:

**Proposition 2** In (1) there is equality if and only if:

\[
\frac{a}{\sqrt{n+p}} = \frac{b}{\sqrt{p+m}} = \frac{c}{\sqrt{m+n}}. 
\]

**Proof** From generalized Pitagora’s theorem for \( c \) and expression \( S = \frac{1}{2}ab\sin C \) it results that in (1) is equality if and only if:

\[
ma^2 + nb^2 + p \left( a^2 + b^2 - 2ab\cos C \right) = 2\sqrt{mn + \ldots}ab\sin C \iff
\]

\[
(m + p) \frac{a}{b} + (n + p) \frac{b}{a} = 2 \left( p\cos C + \sqrt{mn + \ldots}\sin C \right).
\]

From AM-GM inequality we have

\[
(m + n) \frac{a}{b} + (n + p) \frac{b}{a} \geq 2\sqrt{(m + n)(n + p)}
\]

and from Cauchy-Buniakowski-Schwartz inequality we get

\[
\sqrt{(m + n)(n + p)} \geq \left( p\cos C + \sqrt{mn + \ldots}\sin C \right).
\]

From last three relations it results that in (1) is equality if and only if

\[
(m + n) \frac{a}{b} = (n + p) \frac{b}{a} \quad \text{and} \quad \frac{\cos C}{p} = \frac{\sin C}{\sqrt{mn + \ldots}} \quad \text{which means:}
\]

\[
\frac{a}{\sqrt{n+p}} = \frac{b}{\sqrt{m+p}} \quad \text{denote} \quad k
\]

\[
\frac{\cos C}{p} = \frac{\sin C}{\sqrt{mn + \ldots}} = \frac{1}{\sqrt{(m+n)(n+p)}}.
\]

Replacing \( \cos C = \frac{p}{\sqrt{(m+n)(n+p)}} \) from (15_2) and \( b \) from (15_1) in generalized Pitagora’s theorem we have \( c^2 = a^2(k^2 + 1) - 2a^2k\frac{p}{\sqrt{(m+n)(n+p)}} \). But \( k = \ldots \).
\[ \frac{m+n}{n+p} \text{ and then } \left( \frac{x}{a} \right)^2 = 1 + \frac{m+n}{n+p} - 2 \sqrt{\frac{m+n}{n+p} \frac{p}{\sqrt{(m+n)(n+p)}}} = \frac{m+n}{n+p}. \] Therefore \[ \frac{a}{\sqrt{n+p}} = \frac{c}{\sqrt{m+n}} \text{ and this last relation with (15) gives the conclusion.} \] 

Consequences:

\[ a^2 + b^2 + c^2 = 4\sqrt{3}S \iff a = b = c \]

\[ \frac{a^4 + b^4 + c^4}{\sqrt{a^2b^2 + b^2c^2 + c^2a^2}} = 4S \iff \frac{a^2}{b^2 + c^2} = \frac{b^2}{c^2 + a^2} = \frac{c^2}{a^2 + b^2} \iff a = b = c \]

\[ a^4 + b^4 + c^4 = 16S^2 \iff a = b = c \]

\[ 9a^2 + 5b^2 - 3c^2 = 4\sqrt{3}S \iff \frac{a}{\sqrt{2}} = \frac{b}{\sqrt{6}} = \frac{c}{\sqrt{14}} \iff a = \frac{b}{\sqrt{3}} = \frac{c}{\sqrt{7}} \]

\[ 27a^2 + 27b^2 - 13c^2 = 12\sqrt{3}S \iff \frac{a}{\sqrt{14}} = \frac{b}{\sqrt{14}} = \frac{c}{\sqrt{54}} \iff a = \frac{b}{\sqrt{7}} = \frac{c}{\sqrt{27}} \]

\[ 3a^2 - b^2 + 15c^2 = 12\sqrt{3}S \iff \frac{a}{\sqrt{14}} = \frac{b}{\sqrt{18}} = \frac{c}{\sqrt{2}} \iff a = \frac{b}{\sqrt{3}} = c \]

\[ 3a^2 + 3b^2 - c^2 = 4\sqrt{3}S \iff \frac{a}{\sqrt{2}} = \frac{b}{\sqrt{2}} = \frac{c}{\sqrt{6}} \iff a = b = \frac{c}{\sqrt{3}}. \]

References
