NEW TOOLS IN FINSLER GEOMETRY: STRETCH AND RICCI SOLITONS

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Firstly, the notion of stretch from Riemannian geometry is extended to Finsler spaces in relationship with the smoothness function of Ohta and the reversibility function of Rademacher. As an application, the Sphere Theorem of Rademacher is rewritten in terms of stretch for the case of Randers and Matsumoto metrics by pointed out the usual Riemannian pinching constant 1/4. Secondly, one put in evidence a strong relationship, induced by the Zermelo navigation problem, between Randers metrics of constant flag curvature and Ricci solitons.

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1. INTRODUCTION

The notion of *stretch* function, in German *Dehnung*, for a map between two Riemannian manifolds, $f : (M, g_M) \to (N, g_N)$, is introduced in [19] by:

$$\delta_{f,g_M,g_N}(x) = \sup\{\|Tf(x,y)\|_{g_N} : \|y\|_{g_M} = 1\} \tag{1}$$

where $x \in M$ and $y \in T_xM$. The same function is studied sometimes under the name of *dilatation* and is the subject of the very interesting papers [11–13, 21, 27] where is pointed out that the geometrical complexity of f may be measured by its dilatation, just as the topological complexity of f may be measured by its homotopy class or the Brouwer degree.

The first aim of this note is to extend this notion to the Finslerian framework; we prefer to use the word *stretch* since in Finsler geometry there exists already a concept of dilatation, [14]. Let us point out also that in Finsler theory there exists a notion of *stretch curvature* introduced by Berwald [6] and used in [26] for a characterization of Finsler spaces of non-zero scalar curvature; see also ([16], p. 746). Our motivation for the study of stretch function comes from a more simpler form of some remarkable results of Finsler geometry expressed in terms of the reversibility function; for example in the Randers case the fraction $\frac{\lambda}{\lambda+1}$ which appears in some theorems of Rademacher and Wei Wang can be replaced with $\frac{\delta}{2}$ and so we recast the classical Riemannian pinching constant $\frac{1}{4}$.

The second purpose of this paper is the derivation of an intimate relationship between a "best type of Finsler metric" (after the well-known words from the Introduction of [7]), namely Randers metrics of constant flag curvature, and a modern tool of Ricci flow, namely *Ricci solitons* [9]. Since the characterization of the first object is completely described by the *Zermelo navigation problem* it is natural that this approach will be useful in the detection of this remarkable connection.

The paper is structured as follows. The first section is devoted to the introduction of the notion of stretch in Finslerian geometry. This object is used to rephrase the Sphere Theorem of Rademacher as well as the Wei's estimate for the number of closed geodesics on the unit sphere. The Randers case of (α, β)-metrics deserves a special attention and the third section deals with some estimates of the stretch function in terms of non-Riemannian objects. The last section concerns with the Ricci solitons provided by constant flag curvature again in the Randers spaces. An important example is the Funk metric of the unit disk for which we recover the elements of the Zermelo navigation problem and the geodesics.

2. STRETCH (VS. REVERSIBILITY) IN FINSLER GEOMETRY

Let M be a smooth, n-dimensional manifold.

Definition 2.1 ([24]). A *Minkowski norm* on the real vector space V is a function $F : V \to [0, \infty)$, such that:
i) *Regularity*: F is C^∞ on $V \setminus \{0\}$,
ii) *Positive homogeneity*: $F(ay) = aF(y)$ when $a > 0$,
iii) *Strong convexity*: For any $y \in V \setminus \{0\}$ the symmetric bilinear form g_y:

$$g_y(u, v) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t}[F^2(y + su + tv)]|_{s=t=0}$$

is positive definite. Then the pair (V, F) is a *Minkowski space*.
For a manifold M a function $F : TM \to [0, \infty)$, $(x, y) \to F(x, y)$ is a *Finsler fundamental function* if F is smooth on $TM \setminus \{0\} = \{(x, y) \in TM : y \neq 0\}$ and for each $x \in M$ the restriction $F|_{T_xM}$ is a Minkowski norm on T_xM. The pair (M, F) is a *Finsler space* and the tensor field g is a *Finsler metric*.

It results that:

$$g_{ij}(x, y) = \frac{1}{2}\frac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j}. \tag{2}$$

Fix $x \in M$ and $N = M$; also denote $F_M = F_1$ and $F_N = F_2$ two Finsler functions. We shall consider $f = 1_M$ and for this case we denote the stretch by $\delta_{F_1,F_2}(x)$ *i.e.*:

$$\delta_{F_1,F_2}(x) = \sup\left\{F_2(x,y) : F_1(x,y) = 1\right\}. \tag{3}$$

Due to the homogeneity of Finsler functions it results a formula very close to the Riemannian case:

$$\delta_{F_1,F_2}(x) = \sup\left\{\sqrt{g^2_{ij}(x,y)y^iy^j} : g^1_{ij}(x,y)y^iy^j = 1\right\}. \tag{4}$$

This relation can be connected with the *dilatation* (or *Liouville*) *vector field* of TM:
$\Gamma(x,y) = y^i\frac{\partial}{\partial y^i}|_{(x,y)}$ which is a global vector field on the tangent bundle TM. Its norm $\|\Gamma\|^2_{1,x} = F^2_1|_x(\Gamma) = g^1_{ij}(x,y)y^iy^j$ is strictly positive and then $\frac{1}{\|\Gamma\|_1}\Gamma$ is an unitary vector field with respect to F_1 which yields:

$$\delta_{F_1,F_2}(x) \geq F_2\left(\frac{1}{\|\Gamma\|_{1,x}}\Gamma\right) = \frac{\|\Gamma\|_{2,x}}{\|\Gamma\|_{1,x}} \tag{5}$$

another form of the previous equation.

A motivation for the introduction of this notion in the Finslerian framework is provided by the relationship with the concept of *smoothness function* from ([18], p. 680):

$$\mathcal{S}(x) = \sup\left\{\frac{\sqrt{g_w(v,v)}}{F(v)} : v, w \in T_xM\backslash 0\right\} \tag{6}$$

where g_w is the Riemannian metric: $g_w(v_1,v_2) = g_{ij}(x,w)v^i_1v^j_2$.

One use of that last function is a characterization of Riemannian spaces through general Finsler ones by the constancy of $\mathcal{S}$; namely g is a Riemannian metric if and only if $\mathcal{S}(x) = 1$ for every $x \in M$, ([18], p. 680). Other very interesting applications of smoothness in Finsler geometry are included in the cited paper as well as the *uniform convexity function*:

$$\mathcal{C}(x) = \sup\left\{\frac{F(v)}{g_w(v,v)} : v, w \in T_xM\backslash 0\right\}. \tag{7}$$

It follows straightforward that:

$$\mathcal{S}(x) = \sup\left\{\delta_{F,g_w}(x) : w \in T_xM\backslash 0\right\} \tag{8}$$

and then a new inequality holds for every $w \in T_xM\backslash 0$:

$$\mathcal{S}(x) \geq \frac{\|\Gamma\|_{g_w,x}}{\|\Gamma\|_{F,x}}. \tag{9}$$

Let us provide another origin of the stretch function. Namely, $F_1(x,.)$ and $F_2(x,.)$ are equivalent norms on T_xM and so there exists a positive real function $C(x)$ such that:

$$C^{-1}(x)F_1(x,y) \leq F_2(x,y) \leq C(x)F_1(x,y) \tag{10}$$

for every $y \in T_xM$. Then $\delta_{F_1,F_2}(x) = C(x)$ with a similar argument as in ([22], p. 380).

Remark 2.2. 1) The more general notion of stretch can be defined in the framework of Lagrange geometry for a map $f : (M, L_M) \to (N, L_N)$ with L_M and L_N regular Lagrangian on M respectively N, [17]:

$$\delta_{f,L_M,L_N}(x) = \sup\left\{L_N(Tf(x,y)) : L_M(x,y) = 1\right\}. \tag{11}$$

2) For the same Finsler case $M = N$ but with $F_2 = F_1 \circ (-1_{TM})$ the notion of stretch (in the above Lagrangian sense) is exactly the *reversibility function* λ_{F_1} introduced by Rademacher in [22].

3) Using the same argument as in ([23], p. 266–267) for the reversibility function it results the continuity of the stretch function already for the general Finslerian case.

4) Inspired by the same paper let us denote by $\theta : M \times M \to \mathbb{R}$ the pseudo-distance induced by a Finsler function F *i.e.* $\theta(p,q) = \inf\{L(c, \frac{dc}{dt}) | \ c : [0,1] \to M, \text{smooth}, \ c(0) = p, c(1) = q\}$. From (10) it results a comparison formula:

$$\theta_2(x_1,x_2) \leq \delta_{F_1,F_2}(x_1) \leq \theta_1(x_1,x_2). \tag{12}$$

In order to handle some examples we restrict to the class of Finsler metrics called (α,β)-*metrics*.

Let us consider: a Riemannian metric: $a = (a_{ij}(x))_{1\leq i,j\leq n}$ and a 1-form: $b = (b_i(x))_{1\leq i\leq n}$, sometimes called *drift*, both living globally on M and let us associate to these objects the following functions on TM:

- $\alpha(x,y) = \sqrt{a_{ij}y^i y^j}$
- $\beta(x,y) = b_i y^i$
- $F(x,y) = \alpha\phi\left(\frac{\beta}{\alpha}\right)$

with ϕ a C^∞ positive function on some interval $[-r,r]$ big enough such that $r \geq \frac{\beta}{\alpha}$ for all $(x,y) \in TM$.

F is a Finsler fundamental function if the following conditions are satisfied ([25], p. 307, eq. (2–11)) $\phi(s) > 0, \phi(s) - s\phi'(s) > 0, (\phi(s) - s\phi'(s)) + \left(b^2 - s^2\right)\phi''(s) > 0$ for all $|s| \leq b \leq r$ where $s = \frac{\beta}{\alpha}$.

It follows that $\delta_{a,F}(x) = \sup\left\{F(x,y) : \alpha(x,y) = 1\right\}$ which yields:

$$\delta_{a,F}(x) = \sup\left\{\phi(\beta(x,y)) : \alpha(x,y) = 1\right\}. \tag{13}$$

A Cauchy-Schwartz type argument with respect to the Riemannian metric a gives ([4], p. 218):

$$\pm\beta(x,y) \leq |\beta(x,y)| = |b_i(x)y^i| \leq \|b(x)\|\|y\| = \|b(x)\| \tag{14}$$

where $\|b(x)\|$ is the a-norm of 1-form b, $\|b(x)\|^2 = a^{ij}(x)b_i(x)b_j(x)$. The equality holds for $y = \frac{1}{\|b\|}b^{\#}$ with $b^{\#}$ the a-dual of b; therefore we exclude the Riemannian case $b = 0$ for which obviously the stretch is constant equal to 1. So, for an (α,β)-metric with strictly monotone ϕ:

$$|\delta_{a,F}(x)| = \phi(\|b(x)\|). \tag{15}$$

Particular cases:

- *I Randers metrics* $\phi(s) = 1 + s$ are the most used Finsler metrics

$$\delta_{\text{Randers}}(x) = 1 + \|b(x)\|. \tag{16}$$

In [23] it is proved that the reversibility for the Randers case is:

$$\lambda_{\text{Randers}}(x) = \frac{1 + \|b(x)\|}{1 - \|b(x)\|} \tag{17}$$

and then $\lambda_{\text{Randers}} = \frac{\delta_{\text{Randers}}}{2-\delta_{\text{Randers}}}$ which yields the following version of the Rademacher's Sphere Theorem [22]:

Proposition 2.3. *A simply connected and compact Randers manifold of dimension $n \geq 3$ with stretch δ and flag curvature K satisfying $\frac{\delta^2}{4} < K \leq 1$ is homotopy equivalent to the n-sphere.*

Another usefulness of the reversibility function is in providing bounds for the number of (prime) closed geodesics. In order to show again the simplicity of statements in terms of stretch comparing with reversibility we rephrased the Theorem 1.2. of [28] in the framework of Randers metrics:

Proposition 2.4. *On every Randers n-sphere $(\mathbb{S}^n, F)$ satisfying $F < \frac{2}{\delta}\alpha_0$ and $l(\mathbb{S}^n, F) \geq \frac{2\pi}{\delta}$ there always exist at least n prime closed geodesics without self-intersections, where α_0 corresponds to the standard Riemannian metric g_0 on $\mathbb{S}^n$ with constant curvature $+1$ and $l(\mathbb{S}^n, F)$ is the length of a shortest geodesic loop on $(\mathbb{S}^n, F)$.*

- *II Kropina metrics* $\phi(s) = \frac{1}{s}$

$$\delta_{\text{Kropina}}(x) = -\frac{1}{\|b(x)\|}. \tag{18}$$

Conform ([1], p. 92) a Kropina metric is never positive definite.

- *III Matsumoto* (or *slope*) *metrics* ([1], p. 92) $\phi(s) = \frac{1}{1-s}$

$$\delta_{\text{Matsumoto}}(x) = \frac{1}{1 - \|b(x)\|}. \tag{19}$$

In the cited paper it is remarked that a Matsumoto metric is positive definite if and only if $\|b(x)\| < \frac{1}{2}$ for every $x \in M$ and then:

$$\delta_{\text{Matsumoto}}(x) < 2. \tag{20}$$

With a computation similar to the Randers case we derive:

$$\lambda_{\text{Matsumoto}}(x) = \frac{1 - \|b(x)\|}{1 + \|b(x)\|} \tag{21}$$

which yields $\lambda_{\text{Matsumoto}} = \frac{1}{2\delta_{\text{Matsumoto}} - 1}$ and in analogy with Proposition 2.3.:

PROPOSITION 2.5. *A simply connected and compact Matsumoto manifold of dimension $n \geq 3$ with stretch δ and flag curvature K satisfying $\frac{1}{4\delta^2} < K \leq 1$ is homotopy equivalent to the n-sphere.*

Let us remark the presence of classical (Riemannian) pinching constant $\frac{1}{4}$ for both Randers and Matsumoto metrics. So, from this point of view, these Finslerian (α, β) geometries are very closed to Riemannian geometry.

- *IV "Riemann" type (α, β)-metrics* $\phi(s) = \sqrt{1 + s^2}$

$$\delta_{\text{Riemann}}(x) = \sqrt{1 + \|b(x)\|^2}. \tag{22}$$

- V ([15]) $\phi(s) = 1 + s^2$.

$$\delta_{\text{V}}(x) = 1 + \|b(x)\|^2. \tag{23}$$

Example 2.6. Let us consider after ([20], p. 123) the Randers metric on the unit disk D^1 determined by:

$$a_{11} = \frac{1 - (x^2)^2}{\varphi^2(x)}, a_{12} = \frac{x^1 x^2}{\varphi^2(x)}, a_{22} = \frac{1 - (x^1)^2}{\varphi^2(x)}, b_1 = \frac{x^1}{\varphi(x)}, b_2 = \frac{x^2}{\varphi(x)},$$

where φ is the function defining the boundary $\partial D^1 = S^1 : \varphi(x) = 1 - (x^1)^2 - (x^2)^2$. In the cited paper it is proved that the associated Randers-Funk metric is of constant negative flag curvature $(-1/4)$; see also ([2], p. 20).

We have:

$$\|b(x)\| = \sqrt{(x^1)^2 + (x^2)^2} \tag{24}$$

and then, for example:

$$\delta_{\text{Randers}}(x) = 1 + \sqrt{(x^1)^2 + (x^2)^2} \tag{25}$$

$$\lambda_{\text{Randers}}(x) = \frac{(1 + \sqrt{(x^1)^2 + (x^2)^2})^2}{\varphi(x)}. \tag{26}$$

Example 2.7 (Ricci solitons). Let $(a, b^{\#})$ be a Ricci soliton on M *i.e.* [8]:

$$\mathcal{L}_{b^{\#}} a + 2Ric + 2\nu a = 0 \tag{27}$$

with ν a non-vanishing scalar and Ric the Ricci tensor of the metric a. Suppose that:

a) $Ric(b^{\#}, b^{\#})$ is a constant,
b) $\|b^{\#}\|_a = constant < 1$.

Therefore we can associate the Randers metric. From the Ricci equation we get $\|b\|_a = \sqrt{\frac{Ric(b^{\#},b^{\#})}{-\nu}}$ and then $Ric(b^{\#}, b^{\#})$ and ν have opposite signs.

THEOREM 2.8. *A simply connected and compact Ricci soliton in dimension $n \geq 3$ satisfying a) and b) for which the flag curvature of the associated Randers metric belongs to $((\frac{1+\sqrt{\frac{Ric(b^{\#},b^{\#})}{-\nu}}}{2})^2, 1]$ is homotopy equivalent to the n-sphere.*

The vector field $b^{\#}$ of a Ricci soliton is called *the potential* and the Ricci soliton is *shrinking, steady* or *expanding* according as ν is negative, zero or positive. For example, the n-sphere $\mathbb{S}^n$ with the standard Riemannian metric g_0 can be thought of as a Ricci soliton with $b^{\#} = 0$ and is shrinking since $\nu = -(n-1)$.

3. MORE ABOUT THE RANDERS CASE

Since in Randers geometry $\|b(x)\| < 1$ it results that $\delta_{\text{Randers}}(x) \in (1, 2)$ for all $x \in M$. A first natural problem is when the stretch is constant.

PROPOSITION 3.1. *Let $d \in (1, 2)$ and a a Riemannian metric on M. Then there exists an 1-form b such that $\delta_{Randers}(x) = d$ for all $x \in M$.*

Proof. We may assume $a^{11}(x) > 0$ and then we choose b with $b_1 = \frac{d}{\sqrt{a^{11}}}$ and $b_2 = ... = b_n = 0$. □

Let also remark that in [3] is included a drift 1-form of the Riemannian constant norm on S^3. Therefore, on this sphere for every $K > 1$ we get a Randers metric with constant stretch $\delta_{Randers} \equiv 1 + \frac{\sqrt{K-1}}{\sqrt{K}}$ and constant reversibility $\lambda_{Randers} = \frac{\sqrt{K}+\sqrt{K-1}}{\sqrt{K}-\sqrt{K-1}}$.

A second natural problem is to find upper and/or lower bounds for the general, *i.e.* nonconstant, case. Geometrical bounds can be derived using some Finslerian quantities. For example in ([24], p. 108) are defined the *Cartan torsion* $\mathbf{C}$ and the *mean Cartan torsion* $\mathbf{I}$ on Minkowski spaces (V, F) while on the next page are derived two inequalities between the norm of these tensors and $\|b\|$. We obtain:

$$\frac{2\|\mathbf{I}\|}{(n+1)^2}\sqrt{(n+1)^2-\|\mathbf{I}\|^2} \leq \|b\| \tag{28}$$

$$\frac{2\|\mathbf{C}\|}{9}\sqrt{(n+1)^2-\|\mathbf{C}\|^2} \leq \|b\| \tag{29}$$

which yield:

$$\delta_{Minkowski-Randers}(x) \geq 1+\frac{2\|\mathbf{I}\|_x}{(n+1)^2}\sqrt{(n+1)^2-\|\mathbf{I}\|_x^2} \tag{30}$$

$$\delta_{Minkowski-Randers}(x) \geq 1+\frac{2\|\mathbf{C}\|_x}{9}\sqrt{(n+1)^2-\|\mathbf{C}\|_x^2}. \tag{31}$$

In the above formulae the norm is with respect to g *i.e.* $\|I\|_x^2 = g^{ij}(x)I_i(x)I_j(x)$ where $(g^{ij}) = (g_{ij})^{-1}$.

Also, we get $\lambda_{Randers}(x) \in (1, +\infty)$ respectively:

$$\lambda_{Minkowski-Randers}(x) \geq \frac{(n+1)^2+2\|\mathbf{I}\|_x\sqrt{(n+1)^2-\|\mathbf{I}\|_x^2}}{(n+1)^2-2\|\mathbf{I}\|_x\sqrt{(n+1)^2-\|\mathbf{I}\|_x^2}} \tag{32}$$

$$\lambda_{Minkowski-Randers}(x) \geq \frac{9+2\|\mathbf{C}\|_x\sqrt{(n+1)^2-\|\mathbf{C}\|_x^2}}{9-2\|\mathbf{C}\|_x\sqrt{(n+1)^2-\|\mathbf{C}\|_x^2}}. \tag{33}$$

4. RANDERS METRICS WITH CONSTANT FLAG CURVATURE IMPLIES RICCI SOLITONS

Returning to the sphere theorems of the first section it is again natural to study the case of constant flag curvature.

THEOREM 4.1. *Let $(M, F = \alpha + \beta)$ be a Randers space. Its flag curvature is constant if and only if M admits a trivial Ricci soliton (h, W) namely h is a Riemannian metric of constant sectional curvature and the potential W is an infinitesimal homothety of h.*

Proof. Recall after [5] that a Randers metric solves the Zermelo navigation problem on the Riemannian space (M, h) with the "wind" (force vector field) W where $h_{ij} = \varepsilon(a_{ij} - b_ib_j)$ and $W = \frac{-1}{\varepsilon}b^{\#}$ with $\varepsilon = 1 - \|b\|_a^2$. In the same paper it is proved that the flag curvature of the Randers metric is a constant K if and only if:

K1) (M, h) is a space form with the sectional curvature $K + \frac{\sigma^2}{4}$, for some constant σ

K2) $\mathcal{L}_W h = -2\sigma h$. We have then the trivial Ricci soliton (h, W) satisfying the conclusion. $\square$

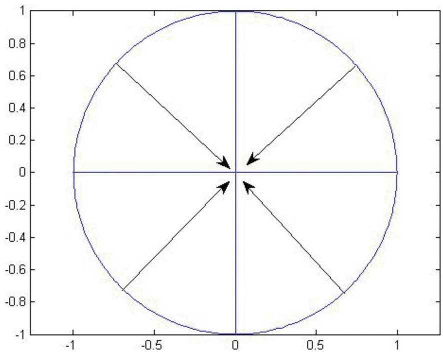
From K1 it results that the Ricci tensor of h is $Ric_h = (n-1)(K + \frac{\sigma^2}{4})h$ and then we have the Ricci equation (27) for the pair (h, W) with:

$$\nu = \sigma - (n-1)(K + \frac{\sigma^2}{4}). \quad (34)$$

Example 2.6 Revisited: In [5] it is pointed out that if $K < 0$ then $\sigma = 4\sqrt{|K|}$. For the Randers-Funk metric we get:

i) $\varepsilon = \varphi$ and then the wind is the radial vector field $W = -(x^1, x^2)$,

ii) the metric h is the Euclidean metric and then $\sigma = \nu = +2$.



The wind for the Funk metric.

Moreover, this Ricci soliton is of gradient type since $W = \nabla f$ with $f(x) = -\frac{1}{2}\|x\|_E^2$, $\|.\|_E$ being the Euclidean norm. Following [8] it results that this Ricci soliton on D^1 is the restriction of *the expanding Gaussian soliton* of $\mathbb{R}^2$.

Let us end with the remark that the same navigation problem is used in [10] in order to determine the geodesics of the Finsler metric. With the computations of the cited paper, we derive that for our Example the unit-speed geodesic through $x \in D^1$ with the tangent vector $y \in \mathbb{R}^2 \setminus \{\bar{0}\}$ is:

$$\gamma(t) = e^{-t}\left(x + \frac{e^t - 1}{(y^1)^2 + (y^2)^2} y\right). \quad (35)$$

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