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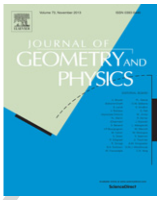

Poisson and Hamiltonian structures on complex analytic foliated manifolds

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ABSTRACT

Poisson and Hamiltonian structures are introduced in the category of complex analytic foliated manifolds endowed with a hermitian metric by analogy with the case of real foliated manifolds studied by Vaisman. A particular case of Hamiltonian structure, called tame, is proved to be induced by a Poisson bracket on the underlying manifold.

1. Introduction

Short time after their introduction in the real case [1,2] the Poisson structures on manifolds were studied in the complex (holomorphic) context in [3–10].

In [11–13], I. Vaisman suggests that it is interesting to study Hamiltonian, Poisson and related structures on foliated manifolds since these may be relevant to the study of physical systems depending on gauge parameters, which are the coordinates along the leaves of a foliation. The aim of our present paper is to extend the study of such structures from the mathematical point of view on complex analytic foliated manifolds endowed with a hermitian metric. This metric allows us to consider an orthogonal complement of the holomorphic tangent bundle of given foliation and therefore yields a graded calculus with vector fields, differential forms and bivectors.

The paper is structured as follows. First, following [14–16] we briefly recall the Vaisman complex of differential forms of mixed type and some preliminary notions about calculus on complex analytic foliated manifolds. Next, we define transversally Poisson and Hamiltonian structures related to our considerations and such a later structure defines a Poisson structure

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on the algebra of complex analytic foliated functions. It is shown that in the so-called tame case, the Hamiltonian structure is induced by an usual Poisson structure of the manifold.

2. Preliminaries on complex analytic foliations

Let us begin our study with a short review of complex analytic foliated manifolds and set up the basic notions and terminology. For more details see [14–16].

Definition 2.1. A *complex analytic foliated* structure $\mathcal{F}$, briefly c.a.f., of complex codimension n on a complex $(n+m)$-dimensional manifold M is given by an atlas $\{\mathcal{U}, (z^a_\alpha, z^u_\alpha)\}$, $a, b, \ldots = 1, \ldots, n$; $u, v, \ldots = n+1, \ldots, n+m$, such that for every $U_\alpha, U_\beta \in \mathcal{U}$ with $U_\alpha \cap U_\beta \neq \phi$, one has, besides analyticity: $\partial z^a_\beta / \partial z^u_\alpha = 0$.

Then the maximal connected submanifolds which can be represented locally by $z^a_\alpha = \text{const.}$ are the *leaves* of $\mathcal{F}$ and the images $\varphi_\alpha(U_\alpha) \subset \mathbb{C}^n$ of the submersions $\varphi_\alpha : U_\alpha \to \mathbb{C}^n$ defined by $\varphi_\alpha(z^a_\alpha, z^u_\alpha) = (z^a_\alpha)$ are called the *local transverse manifolds*.

Let $T'M$ be the holomorphic tangent bundle of M and $T''M = \overline{T'M}$ the antiholomorphic tangent bundle respectively. The tangent vectors of the leaves define the *structural subbundle* $V'\mathcal{F} = T'\mathcal{F}$ of $T'M$ with local bases $\{Z_u = \partial/\partial z^u_\alpha\}$ and the transition functions $(\partial z^u_\beta / \partial z^v_\alpha)$, and $Q'\mathcal{F} = T'M/V'\mathcal{F}$ is the *transversal bundle* with the local bases defined by the equivalence classes $[\partial/\partial z^a_\alpha]$ and the transition functions $(\partial z^a_\beta/\partial z^b_\alpha)$.

Generally, we shall say that the geometrical objects depending only on the leaves are *foliated* and, particularly, c.a.f. For instance, $f : M \to \mathbb{C}$ is foliated if $\partial f/\partial z^u_\alpha = \partial f/\partial \bar{z}^u_\alpha = 0$ and it is c.a.f. if, moreover, $\partial f/\partial \bar{z}^a_\alpha = 0$. A differential form is c.a.f. if it does not contain $d\bar{z}^a_\alpha$, dz^u_α, $d\bar{z}^u_\alpha$ and has local c.a.f. coefficients. A vector bundle on M is c.a.f. if it has c.a.f. transition functions; for instance, the transversal bundle is such.

In the following, let us suppose that M is hermitian with metric h. Then the orthogonal bundle $H'\mathcal{F} = (T'\mathcal{F})^\perp$ of $T'\mathcal{F}$, i.e. $T'M = H'\mathcal{F} \oplus V'\mathcal{F}$, which is differentially isomorphic to $Q'\mathcal{F}$, has local bases of the form:

$$Z_a = \frac{\partial}{\partial z^a} - t^u_a(z^a, z^u)\frac{\partial}{\partial z^u} \tag{2.1}$$

(the index α of the coordinate neighborhood will be omitted) and we shall use in the sequel the bases $\{Z_a, Z_u\}$ to express different vector fields of $\mathcal{X}(M)$. Also, we shall simply denote $H' := H'\mathcal{F}$ and $V' := V'\mathcal{F}$, respectively and by conjugation, we have $V'' = \overline{V'}$ and $H'' = \overline{H'}$ where $V'' = \text{span}\{\overline{Z_u}\}$ and $H'' = \text{span}\{\overline{Z_a}\}$ respectively. Then we have a decomposition of the complexified tangent bundle of $(M, \mathcal{F})$, namely $T_{\mathbb{C}}M = H \oplus V$, where $H = H' \oplus H''$ and $V = V' \oplus V''$. The corresponding dual cobases are given by:

$$\{dz^a\},\ \{\theta^u = dz^u + t^u_a dz^a\},\ \{d\bar{z}^a\},\ \{\overline{\theta}^u = d\bar{z}^u + \overline{t^u_a} d\bar{z}^a\}, \tag{2.2}$$

which span the dual bundles $H'^* = ann\{H'' \oplus V\}$, $H''^* = ann\{H' \oplus V\}$, $V'^* = ann\{H \oplus V''\}$ and $V''^* = ann\{H \oplus V'\}$ respectively. Here, by *ann* we denote the annihilator of a vector bundle. Now, let $\Omega(M)$ and $\mathcal{V}(M)$ be the exterior algebras of differential forms and multivector fields on M respectively.

The cobases from (2.2) allow us to speak of the type (p_1, p_2, q_1, q_2) of a differential form by counting in its expression the number of dz^a, $d\bar{z}^a$, θ^u and $\overline{\theta}^u$, respectively. Thus, we denote by $\Omega^{p_1,p_2,q_1,q_2}(M) = \bigwedge^{p_1,p_2} H \wedge \bigwedge^{q_1,q_2} V$ the set of all (p_1, p_2, q_1, q_2)-differential forms, locally given by:

$$\varphi = \frac{1}{p_1!p_2!q_1!q_2!} \sum \varphi_{A_{p_1}\bar{B}_{p_2}U_{q_1}\bar{V}_{q_2}} dz^{A_{p_1}} \wedge d\bar{z}^{B_{p_2}} \wedge \theta^{U_{q_1}} \wedge \overline{\theta}^{V_{q_2}} \tag{2.3}$$

where $A_{p_1} = (a_1 \ldots a_{p_1})$, $B_{p_2} = (b_1 \ldots b_{p_2})$, $U_{q_1} = (u_1 \ldots u_{q_1})$, $V_{q_2} = (v_1 \ldots v_{q_2})$, $dz^{A_{p_1}} = dz^{a_1} \wedge \cdots \wedge dz^{a_{p_1}}$, $d\bar{z}^{B_{p_2}} = d\bar{z}^{b_1} \wedge \cdots \wedge d\bar{z}^{b_{p_2}}$, $\theta^{U_{q_1}} = \theta^{u_1} \wedge \cdots \wedge \theta^{u_{q_1}}$ and $\overline{\theta}^{V_{q_2}} = \overline{\theta}^{v_1} \wedge \cdots \wedge \overline{\theta}^{v_{q_2}}$ respectively.

These forms can be considered as *complex type* (p_1+q_1, p_2+q_2), as *foliated type* (p_1+p_2, q_1+q_2) and as *mixed type* $(p_1, p_2+q_1+q_2)$, respectively.

Throughout this paper we consider forms of mixed type and we denote by $\Omega^{p,q}_{\text{mix}}(M, \mathcal{F})$ the space of all differential forms of mixed type (p, q) on $(M, \mathcal{F})$. According to above discussion we have:

$$\Omega^{p,q}_{\text{mix}}(M, \mathcal{F}) = \bigoplus_{r,h} \Omega^{p,r,h,q-r-h}(M, \mathcal{F}) = \bigoplus_{k=0}^{q} \bigwedge^{p,k} H \wedge \bigwedge^{q-k}(V, \mathbb{C}).$$

The metric can be locally expressed by:

$$h = h_{a\bar{b}} dz^a \otimes d\bar{z}^b + h_{u\bar{v}} \theta^u \otimes \overline{\theta}^v \tag{2.4}$$

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and the exterior differential d has a corresponding decomposition, [14,15]:

$$d = \mu + \lambda + \nu \tag{2.5}$$

into three parts of the respective mixed types $(1, 0)$, $(0, 1)$ and $(2, -1)$ respectively. Furthermore, the coboundary condition $d^2 = 0$ is equivalent to:

$$\lambda^2 = 0, \qquad \nu^2 = 0, \qquad \mu^2 + \lambda\nu + \nu\lambda = 0, \qquad \mu\lambda + \lambda\mu = 0, \qquad \mu\nu + \nu\mu = 0. \tag{2.6}$$

We also notice that the operator λ is the coboundary of the de Rham type cohomology of forms of mixed type, which computes the cohomology of M with coefficients in the sheaf of germs of λ-closed differential forms of mixed type $(p, 0)$, [14].

Similarly, we consider $\mathcal{V}^{p,q}_{\mathrm{mix}}(M, \mathcal{F})$ the set of multivector fields of mixed type (p, q), that is p vector fields are in $\Gamma(H') := \mathcal{V}^{1,0}_{\mathrm{mix}}(M, \mathcal{F})$ and q vector fields are in $\Gamma(H''^{\perp} \oplus V) := \mathcal{V}^{0,1}_{\mathrm{mix}}(M, \mathcal{F})$, respectively.

Now, let us refer to the Schouten–Nijenhuis bracket. We recall the following general formula of Lichnerowicz, [10]:

$$\imath_{[P,Q]}\, \varphi = (-1)^{q(p+1)}\, \imath_P\, d(\imath_Q\, \varphi) + (-1)^p\, \imath_Q\, d(\imath_P\, \varphi) - \imath_Q(\imath_P\, d\varphi), \tag{2.7}$$

where $P \in \mathcal{V}^p(M)$, $Q \in \mathcal{V}^q(M)$ and $\varphi \in \Omega^{p+q-1}(M)$. The operator $\imath$ is the *interior product*.

If P is of bidegree of mixed type (a, b), $a + b = p$ and Q is of bidegree of mixed type (h, k), $h + k = q$, the component of mixed type $[P, Q]^{uv}$, $u + v = s + t - 1$ is provided by the formula (2.7) where $\varphi \in \Omega^{p,q}_{\mathrm{mix}}(M, \mathcal{F})$. With (2.5) we see that the only possibilities to get non-zero components correspond to the replacement of d by $\mu + \lambda + \nu$ in (2.7), which leads to the cases:

$$\begin{cases} u = a + h - 1, & v = b + k \\ u = a + h, & v = b + k - 1 \\ u = a + h - 2, & v = b + k + 1. \end{cases} \tag{2.8}$$

All the other components vanish because of degree incompatibility.

We will need more precise formulas for a Schouten–Nijenhuis bracket of two bivector fields.

Lemma 2.1 *([12,13]). For any bivector field $P \in \mathcal{V}^2(M)$ one has:*

$$[P, P](\alpha, \beta, \gamma) = 2[d\gamma(P^{\#}(\alpha), P^{\#}(\beta)) - (L_{P^{\#}(\gamma)}P)(\alpha, \beta)], \quad \alpha, \beta, \gamma \in \Omega^1(M), \tag{2.9}$$

where L denotes the Lie derivative and $P^{\#} : T^*_{\mathbb{C}}M \to T_{\mathbb{C}}M$ is defined by $\beta(P^{\#}(\alpha)) = P(\alpha, \beta)$. More generally, for any two bivector fields $P_1, P_2 \in \mathcal{V}^2(M)$ one has:

$$[P_1, P_2](\alpha, \beta, \gamma) = d\gamma(P_1^{\#}(\alpha), P_2^{\#}(\beta)) + d\gamma(P_2^{\#}(\alpha), P_1^{\#}(\beta)) - (L_{P_1^{\#}(\gamma)}P_2)(\alpha, \beta) - (L_{P_2^{\#}(\gamma)}P_1)(\alpha, \beta). \tag{2.10}$$

If the bivector field $P \in \mathcal{V}^2(M)$ is *regular* i.e. $p = \operatorname{rank} P = \dim_{\mathbb{C}} D = \text{const.}$ where $D = \operatorname{im} P^{\#}$ we may compute $[P, P]$ as follows. Choose a decomposition:

$$T_{\mathbb{C}}M = E \oplus D, \qquad T^*_{\mathbb{C}}M = E^* \oplus D^* \quad (E^* = ann\, D,\ D^* = ann\, E). \tag{2.11}$$

Then we have an isomorphism $P^{\#} : D^* \to D$ with an inverse $(P^{\#})^{-1} : D \to D^*$ and there exists a well defined differential 2-form $\theta \in \Gamma(\Lambda^2 D^*)$ of rank p defined by:

$$\theta(X, Y) = P[(P^{\#})^{-1}(p_D X), (P^{\#})^{-1}(p_D Y)], \tag{2.12}$$

where p_D denotes the projection on D. Conversely, (2.12) allows us to reconstruct P from θ such that $\ker P^{\#} = E^*$. We will say that θ is *equivalent to P modulo E*.

In the case of a complex analytic foliated hermitian manifold $(M, \mathcal{F}, h)$ with a normal bundle H if $P \in \mathcal{V}^2(M)$ is an arbitrary bivector field, it has a decomposition:

$$P = P_{20} + P_{11} + P_{02}, \tag{2.13}$$

where the indices denote the bidegree of the components of mixed type $(2, 0)$, $(1, 1)$ and $(0, 2)$ respectively. We can compute the corresponding decomposition of the Schouten–Nijenhuis bracket $[P, P]$ by applying formula (2.9) to 1-forms $\alpha_1, \beta_1, \gamma_1 \in \Omega^{1,0}_{\mathrm{mix}}(M, \mathcal{F})$ and $\alpha_2, \beta_2, \gamma_2 \in \Omega^{0,1}_{\mathrm{mix}}(M, \mathcal{F})$. The results are contained in the following formulas:

$$
\begin{aligned}
&[P_{2,0}, P_{20}]_{30}(\alpha_1, \beta_1, \gamma_1) = 2[\mu\gamma_1(P_{20}^{\#}(\alpha_1), P_{20}^{\#}(\beta_1)) - (L_{P_{20}^{\#}(\gamma_1)}P_{20}^{\#})(\alpha_1, \beta_1)],\\
&[P_{20}, P_{20}]_{21}(\alpha_1, \beta_1, \alpha_2) = 2\nu\alpha_2(P_{20}^{\#}(\alpha_1), P_{20}^{\#}(\beta_1)) = -2\alpha_2([P_{20}^{\#}(\alpha_1), P_{20}^{\#}(\beta_1)]),\\
&[P_{11}, P_{11}]_{21}(\alpha_1, \beta_1, \alpha_2) = 2[\lambda\alpha_2(P_{11}^{\#}(\alpha_1), P_{11}^{\#}(\beta_1)) - (L_{P_{11}^{\#}(\alpha_2)}P_{11})(\alpha_1, \beta_1)],\\
&[P_{11}, P_{11}]_{12}(\alpha_1, \alpha_2, \beta_2) = -2[\mu\beta_2(P_{11}^{\#}(\alpha_2), P_{11}^{\#}(\alpha_1)) + (L_{P_{11}^{\#}(\beta_2)}P_{11})(\alpha_1, \alpha_2)],\\
&[P_{11}, P_{11}]_{03}(\alpha_2, \beta_2, \gamma_2) = 2[\nu\gamma_2(P_{11}^{\#}(\alpha_2), P_{11}^{\#}(\beta_2)) - (L_{P_{11}^{\#}(\gamma_2)}P_{11})(\alpha_2, \beta_2)],\\
&[P_{02}, P_{02}]_{03}(\alpha_2, \beta_2, \gamma_2) = 2[\lambda\gamma_2(P_{02}^{\#}(\alpha_2), P_{02}^{\#}(\beta_2)) - (L_{P_{02}^{\#}(\gamma_2)}P_{02})(\alpha_2, \beta_2)],\\
&[P_{2,0}, P_{11}]_{30}(\alpha_1, \beta_1, \gamma_1) = \lambda\gamma_1(P_{20}^{\#}(\alpha_1), P_{11}^{\#}(\beta_1)) - \lambda\gamma_1(P_{20}^{\#}(\beta_1), P_{11}^{\#}(\alpha_1))\\
&\qquad - (L_{P_{20}^{\#}(\gamma_1)}P_{11})(\alpha_1, \beta_1) - (L_{P_{11}^{\#}(\gamma_1)}P_{20})(\alpha_1, \beta_1),\\
&[P_{20}, P_{11}]_{21}(\alpha_1, \beta_1, \alpha_2) = \mu\alpha_2(P_{20}^{\#}(\alpha_1), P_{11}^{\#}(\beta_1)) - \mu\alpha_2(P_{20}^{\#}(\beta_1), P_{11}^{\#}(\alpha_1)) - (L_{P_{11}^{\#}(\alpha_2)}P_{20})(\alpha_1, \beta_1),\\
&[P_{20}, P_{11}]_{12}(\alpha_1, \alpha_2, \beta_2) = \nu\beta_2(P_{20}^{\#}(\alpha_1), P_{11}^{\#}(\alpha_2)) - (L_{P_{11}^{\#}(\beta_2)}P_{20})(\alpha_1, \alpha_2) - (L_{P_{20}^{\#}(\alpha_1)}P_{11})(\alpha_2, \beta_2),\\
&[P_{20}, P_{02}]_{21}(\alpha_1, \beta_1, \alpha_2) = -(L_{P_{02}^{\#}(\alpha_2)}P_{20})(\alpha_1, \beta_1),\\
&[P_{20}, P_{02}]_{12}(\alpha_1, \alpha_2, \beta_2) = \mu\beta_2(P_{20}^{\#}(\alpha_1), P_{02}^{\#}(\alpha_2)) - (L_{P_{02}^{\#}(\beta_2)}P_{20})(\alpha_1, \alpha_2) - (L_{P_{20}^{\#}(\alpha_1)}P_{02})(\alpha_2, \beta_2),\\
&[P_{11}, P_{02}]_{12}(\alpha_1, \alpha_2, \beta_2) = \lambda\beta_2(P_{11}^{\#}(\alpha_1), P_{03}^{\#}(\alpha_2)) - (L_{P_{11}^{\#}(\beta_2)}P_{02})(\alpha_1, \alpha_2) - (L_{P_{02}^{\#}(\beta_2)}P_{11})(\alpha_1, \alpha_2),\\
&[P_{11}, P_{02}]_{03}(\alpha_2, \beta_2, \gamma_2) = \mu\gamma_2(P_{11}^{\#}(\alpha_2), P_{02}^{\#}(\beta_2)) - \mu\gamma_2(P_{11}^{\#}(\beta_2), P_{02}^{\#}(\alpha_2))\\
&\qquad - (L_{P_{11}^{\#}(\gamma_2)}P_{02})(\alpha_2, \beta_2) - (L_{P_{02}^{\#}(\gamma_2)}P_{11})(\alpha_2, \beta_2)
\end{aligned} \tag{2.14}
$$

all others components being zero.

3. Transversally Poisson and Hamiltonian structures on complex analytic foliated manifolds

As in the case of real foliated manifolds, [11–13], in discussing Poisson geometry on a complex analytic foliated hermitian manifold $(M, \mathcal{F}, h)$ it is natural to look at structures that have the Poisson property in the transversal geometry of $\mathcal{F}$. We notice that $H'\mathcal{F} = \text{span}\{Z_a\}$ is smoothly isomorphic to $Q'\mathcal{F}$ which is holomorphic as $V'\mathcal{F}$, but generally $H'\mathcal{F}$ is not a holomorphic subbundle of $T'M$. The existence of a holomorphic supplementary distribution $H'\mathcal{F}$ is characterized by the vanishing of a certain cohomological obstruction, [16]. However, we shall consider $\mathcal{V}^p_{\text{c.a.f.}}(M, \mathcal{F})$ the set of p-multivector fields that have c.a.f. local components.

Definition 3.1. A *transversally Poisson structure* on the complex analytic foliated hermitian manifold $(M, \mathcal{F}, h)$ is a bivector field $P \in \mathcal{V}^2(M)$ such that:

$$\{f, g\} = P(df, dg), \quad f, g \in C^\infty(M) \tag{3.1}$$

restricts to a Lie algebra bracket on the set of c.a.f. functions on $(M, \mathcal{F})$, denoted by $C^\infty_{\text{c.a.f.}}(M, \mathcal{F})$.

Remark 3.1. It follows by Definition 3.1 that such transversally Poisson structures are those which go the quotient to a holomorphic Poisson structure, whenever the quotient of the manifold by the foliation is a manifold.

An useful characterization is provided by:

Proposition 3.1. *The bivector field $P \in \mathcal{V}^2(M)$ defines a transversally Poisson structure on the complex analytic foliated manifold $(M, \mathcal{F}, h)$ if and only if:*

$$\begin{cases} (L_Z P)|_{H'^*} = 0 & \forall Z \in \mathcal{V}^{0,1}_{\text{mix}}(M, \mathcal{F}) \\ [P, P]|_{H'^*} = 0. \end{cases} \tag{3.2}$$

Proof. If $f, g \in C^\infty_{\text{c.a.f.}}(M, \mathcal{F})$ then for any $Z \in \mathcal{V}^{0,1}_{\text{mix}}(M, \mathcal{F})$ we have $Zf = Zg = 0$. One gets:

$$(L_Z P)(df, dg) = Z(P(df, dg)) = Z\{f, g\},$$

hence, the first condition from (3.2) is equivalent to $\{f, g\} \in C^\infty_{\text{c.a.f.}}(M, \mathcal{F})$. The second condition from (3.2) is a direct consequence of the formula:

$$[P, P](df, dg, dh) = 2 \sum_{\text{cycl}(f,g,h)} \{\{f, g\}, h\}, \tag{3.3}$$

which is a consequence of Lemma 2.1. □

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Remark 3.2. If we choose a normal bundle of $\mathcal{F}$ and write P under the form (2.13) then the first condition (3.2) is equivalent with $(L_Z P_{2,0})|_{H'^*} = 0$. Furthermore, formulas (2.14) show that the second condition from (3.2) is equivalent with $[P_{20}, P_{20}]_{H'^*} = 0$, modulo the first condition from (3.2). Thus, transversally Poisson is a property of the transversal component of mixed type $(2, 0)$, P_{20}, of the bivector field P.

We may define a complex Hamiltonian vector field of a c.a.f. function f with respect to a transversally Poisson structure P by $Z_f = \imath_{df} P$ and, due to (3.2), this will be a vector field in $\mathcal{V}^1_{\text{c.a.f.}}(M, \mathcal{F})$.

Furthermore, we give the following definition:

Definition 3.2. The generalized distribution $\mathcal{D}$ defined by:

$$\mathcal{D}_z = \text{span}\{W(z), Z_f(z) : W \in \mathcal{V}^{0,1}_{\text{mix}}(M, \mathcal{F}),\ f \in C^\infty_{\text{c.a.f.}}(M, \mathcal{F})\}, \quad z \in M$$

is called the *characteristic distribution* of the transversally Poisson structure P on the complex analytic foliated hermitian manifold $(M, \mathcal{F}, h)$.

The most important property of this distribution is:

Proposition 3.2. *The characteristic distribution $\mathcal{D}$ of a transversally Poisson structure is completely integrable.*

Proof. It is easy to see that the Lie brackets of the form $[W_1, W_2]$, $[W, Z_f]$ with $W_1, W_2, W \in \mathcal{V}^{0,1}_{\text{mix}}(M, \mathcal{F})$, $f \in C^\infty_{\text{c.a.f.}}(M, \mathcal{F})$ belong to $\mathcal{D}$ because:

$$[\overline{Z_a}, \overline{Z_b}] = (\overline{Z_b}\, \overline{t^u_a} - \overline{Z_a}\, \overline{t^u_b})\overline{Z_u}, \qquad [Z_u, Z_v] = [Z_u, \overline{Z_v}] = [\overline{Z_u}, \overline{Z_v}] = 0,$$
$$[\overline{Z_a}, Z_u] = Z_u(\overline{t^v_a})\overline{Z_v}, \qquad [\overline{Z_a}, \overline{Z_u}] = \overline{Z_u}(\overline{t^v_a})\overline{Z_v}, \tag{3.4}$$

and since $Z_f \in \mathcal{V}^1_{\text{c.a.f.}}(M, \mathcal{F})$. Then the Jacobi identity for the Poisson structure on $C^\infty_{\text{c.a.f.}}(M, \mathcal{F})$ yields $([Z_f, Z_g] - Z_{\{f,g\}})(h) = 0$ for any $f, g, h \in C^\infty_{\text{c.a.f.}}(M, \mathcal{F})$, whence:

$$[Z_f, Z_g] = Z_{\{f,g\}} + W, \quad W \in \mathcal{V}^{0,1}_{\text{mix}}(M, \mathcal{F}). \tag{3.5}$$

Thus the distribution $\mathcal{D}$ is involutive. □

Now, it is easy to see that the complex Hamiltonian vector fields of c.a.f. functions do not depend on the component P_{02} of mixed type $(0, 2)$ of the decomposition (2.13) and the same holds for the characteristic distribution $\mathcal{D}$.

In the real case, the idea of associating a Hamiltonian vector field to a foliated function is natural and it has a physical meaning. From these considerations the notion of *transversally equivalence* of two transversally Poisson structures of a foliation was given in [13]. In order to discuss about Hamiltonian structures on complex analytic foliated manifolds we consider here the following similar notions.

Definition 3.3. On a complex analytic foliated hermitian manifold $(M, \mathcal{F}, h)$ we say that two transversally Poisson structures P_1, P_2 are *transversally equivalent* if $\#_{P_1}|_{H'^*} = \#_{P_2}|_{H'^*}$. A family of transversally equivalent transversally Poisson structures is a *Hamiltonian structure* of the complex analytic foliation $\mathcal{F}$.

Remark 3.3. We notice that the above transversally equivalence notion of two transversally Poisson structures does not depend on the hermitian structure.

It is easy to see that a Hamiltonian structure of $\mathcal{F}$ is equivalent with a complex vector bundle morphism $\mathcal{H} : H'^* \to T_{\mathbb{C}}M$ such that, if we define the complex Hamiltonian vector fields of c.a.f. functions by $Z_f = \mathcal{H}(df)$, the formula:

$$\{f, g\} = Z_f g \tag{3.6}$$

defines a Poisson algebra structure on $C^\infty_{\text{c.a.f.}}(M, \mathcal{F})$. Namely, $\mathcal{H} = P^\#|_{H'^*}$ for any structure P of the equivalence class that defines the Hamiltonian structure.

A Hamiltonian structure has a characteristic distribution $\mathcal{D}$ which is the common characteristic distribution of all the corresponding equivalent transversally Poisson structures P and $\mathcal{D} = H'' \oplus V + \text{im}\,\mathcal{H}$.

Definition 3.4. A complex subspace $S \subset T_{z,\mathbb{C}}M$ of a complex Poisson manifold (M, P) is called *coisotropic* if for any $\alpha, \beta \in \Omega^1(M)$ such that $\langle \alpha, Z\rangle = \langle \beta, Z\rangle = 0$, $\forall Z \in \Gamma(S)$ we have $\langle P, \alpha \wedge \beta\rangle = 0$ where $\langle \alpha, Z\rangle = \langle Z, \alpha\rangle := i(Z)\alpha$, for every $Z \in \mathcal{V}^1(M)$, $\alpha \in \Omega^1(M)$. In other words, $ann\, S \subset (ann\, S)^\perp_P$, where $(ann\, S)^\perp_P = \{\beta \in T^*_{z,\mathbb{C}}M : \langle P, \alpha \wedge \beta\rangle = 0,\ \forall \alpha \in ann\, S\}$ is the "Poisson orthogonal" of $ann\, S = \{\alpha \in T^*_{z,\mathbb{C}}M : \langle \alpha, Z\rangle = 0,\ \forall Z \in \Gamma(S)\}$. A complex submanifold N of a complex Poisson manifold is called *coisotropic* if its tangent spaces are coisotropic.

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Now, similarly to [11–13], we have:

Example 3.1. Let $\mathcal{D}$ be a coisotropic foliation of complex dimension $n+k$ ($k \leq n$) of a holomorphic symplectic manifold (M, ω) of complex dimension $2n$, with the holomorphic symplectic $(2, 0)$-form ω, e.g. [4,5,17,18]. It is well known that the ω-orthogonal distribution of $\mathcal{D}$ is tangent to a complex analytic foliation $\mathcal{F}$ and that for any $z \in M$ there exist local coordinates (z^a, z^u, w^i) around z such that $a = 1, \ldots, n-k, u = n-k+1, \ldots, n,\ i = 1, \ldots, n,\ z^a =$ const. are local equations of $\mathcal{D}$, and the holomorphic symplectic $(2, 0)$-form has the canonical expression:

$$\omega = \sum_{a=1}^{n-k} dz^a \wedge dw^a + \sum_{u=n-k+1}^{n} dz^u \wedge dw^u.$$

This result is a holomorphic version of a theorem of Lie [19]. The local equations of the complex analytic foliation $\mathcal{F}$ are $z^a =$ const., $z^u =$ const., $w^i =$ const., and the calculation of the Hamiltonian vector field Z_f^ω of an $\mathcal{F}$-c.a.f. function by the relation $\imath_{Z_f^\omega}\omega = df$ yields that Z_f^ω is an c.a.f. vector field tangent to the leaves of $\mathcal{D}$. Therefore, $\mathcal{H} = \omega^\#|_{H'^*}$ is a Hamiltonian structure of the complex analytic foliation $\mathcal{F}$ with the holomorphic presymplectic foliation $\mathcal{D}$. Moreover in this case we have $H'' \oplus V \subseteq \operatorname{im}\mathcal{H}$. The holomorphic Poisson structure defined by the original holomorphic symplectic structure ω, e.g [4,9], is one of the $\mathcal{F}$-transversally Poisson structures that defines $\mathcal{H}$.

Definition 3.5. A Hamiltonian structure $\mathcal{H}$ of a complex analytic foliated hermitian manifold $(M, \mathcal{F}, h)$ is said *transversal* (to $\mathcal{F}$) if $\operatorname{im}\mathcal{H} \subseteq H'$. The transversal distribution H' will be called an *image extension* of $\mathcal{H}$. (It is possible to have more than one image extension). A transversal Hamiltonian structure of $\mathcal{F}$ is a *tame* structure if all the brackets of complex vector fields that belong to $\operatorname{im}\mathcal{H}$ are contained in an image extension H'; we will say also that $\mathcal{H}$ is H'-tame. (In the tame case, only such image extensions will be used.)

Let us recall that if $p_D : TM \to D$ is the projector associated to a given distribution D on a manifold M, then the vanishing of the Nijenhuis tensor $N_D = [p_D, p_D]$, where

$$[p_D, p_D](X, Y) = [p_D X, p_D Y] - p_D[p_D X, Y] - p_D[X, p_D Y] + p_D^2[X, Y]$$

leads to the integrability of the distribution D.

Proposition 3.3. *Let $\mathcal{H}$ be a transversal Hamiltonian structure of the complex analytic foliated hermitian manifold $(M, \mathcal{F}, h)$ with image extension H'. Then $\mathcal{H}$ is H'-tame if and only if the Nijenhuis tensor $N_{H'}$ of the projection $p_{H'} : T_{\mathbb{C}}M \to H'$ satisfies the condition:*

$$N_{H'}(\mathcal{H}(\alpha), \mathcal{H}(\beta)) = 0, \quad \forall\, \alpha, \beta \in \Omega^{1,0}_{\text{mix}}(M, \mathcal{F}). \tag{3.7}$$

Proof. Since $p_{H'}^2 = p_{H'}$ the required Nijenhuis tensor is:

$$N_{H'}(X, Y) = [p_{H'}X, p_{H'}Y] - p_{H'}[p_{H'}X, Y] - p_{H'}[X, p_{H'}Y] + p_{H'}[X, Y], \tag{3.8}$$

where $X, Y \in \mathcal{V}(M)$. Generally, $\mathcal{H}$ has local equations:

$$\mathcal{H}(dz^a) = \mathcal{H}^{ab}Z_b + \mathcal{H}^{a\bar{b}}\overline{Z_b} + \mathcal{K}^{au}Z_u + \mathcal{K}^{a\bar{u}}\overline{Z_u}, \tag{3.9}$$

and, if H' is an image extension, $\mathcal{H}^{a\bar{b}} = \mathcal{K}^{au} = \mathcal{K}^{a\bar{u}} = 0$. In view of (3.5), (3.6), (3.9) and Definition 3.3., $\mathcal{H}$ is H'-tame if and only if:

$$\mathcal{H}(d\mathcal{H}^{ab}) = [\mathcal{H}(dz^a), \mathcal{H}(dz^b)],$$

which is equivalent to:

$$\mathcal{H}^{ac}\mathcal{H}^{bd}R^u_{cd} = 0, \qquad R^u_{cd} = Z_d t^u_c - Z_c t^u_d. \tag{3.10}$$

But by a straightforward local calculus it is easy to see that the condition (3.7) is equivalent to (3.10) which end's the proof. □

Remark 3.4. The above proposition says that if H' is integrable then every transversal Hamiltonian structure of complex analytic foliated hermitian manifold $(M, \mathcal{F}, h)$ with image extension H' is tame. Also the integrability of H' would mean that we have a Poisson structure on leaves of this new foliation (defined by H').

Example 3.2 (*Product Structure*). Let $(M_i, P_i),\ i = 1, 2$ be two holomorphic complex Poisson manifolds. Then the product manifold $M_1 \times M_2$ carries two complementary complex analytic foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ by copies of M_1 and M_2, respectively. Then the Hamiltonian structure induced by the Poisson structure P_1 is transversal to $\mathcal{F}_1$ and with image extension $T'\mathcal{F}_2$ is tame. Similarly, we get that the Hamiltonian structure induced by the Poisson structure P_2 is tame with image extension $T'\mathcal{F}_1$.

Example 3.3. Let us consider (M, F) a n-dimensional complex Finsler manifold; for necessary definitions see [20–22]. We use the notations from [22]. If $\eta = \eta^k \frac{\partial}{\partial z^k} \in T'_z M$ is a holomorphic tangent vector then we can consider (z^k, η^k), $k = 1, \ldots, n$ as local coordinates on the complex manifold given by the total space of the holomorphic tangent bundle $\pi : T'M \to M$. Also it is well known that $T'M$ together with the Sasaki lift G of the fundamental metric tensor $g_{j\bar{k}}(z, \eta) = \partial^2 F^2 / \partial \eta^j \partial \overline{\eta}^k$ is a hermitian manifold. Considering the holomorphic vertical foliation $\mathcal{V}$ defined by the fibers of the projection π with leaves characterized by $z^k =$ const. we obtain the complex analytic foliated hermitian manifold $(T'M, \mathcal{V}, G)$. Here the vertical distribution of this foliation is generated by $\{\frac{\partial}{\partial \eta^k}, \frac{\partial}{\partial \overline{\eta}^k}\}$ and we choose a horizontal distribution $H' = \mathrm{span}\{\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N^j_k \frac{\partial}{\partial \eta^j}\}$ (and its conjugates H''), where $N^j_k = g^{\overline{m}j} \frac{\partial g_{l\overline{m}}}{\partial z^k} \eta^l$ are the local coefficients of the Chern–Finsler complex nonlinear connection (briefly c.n.c). This complex nonlinear connection has an important property, namely:

$$\left[\frac{\delta}{\delta z^j}, \frac{\delta}{\delta z^k}\right] = 0 \tag{3.11}$$

see for instance Lemma 2.1 from [20] (or also [21,22]), which means that H' is integrable. Also we can consider the particular case when the complex Finsler structure is the norm of a hermitian metric, namely $F(z, \eta) = g_{j\bar{k}}(z)\eta^j \overline{\eta}^k$. Thus by Proposition 3.3 every transversal Hamiltonian on the complex analytic foliated hermitian manifold $(T'M, \mathcal{V}, G)$ with the image extension H' is tame. Such transversal Hamiltonian can be easily obtained from the horizontal lift with respect to Chern–Finsler c.n.c of a holomorphic Poisson structure on the base manifold M using an argument similar to [23]. Indeed, if $P = \frac{1}{2} P^{ij}(z) \frac{\partial}{\partial z^i} \wedge \frac{\partial}{\partial z^j}$ is a holomorphic Poisson bivector field on the base manifold M then we define $P_{H'} = \frac{1}{2} P^{ij}(z) \frac{\delta}{\delta z^i} \wedge \frac{\delta}{\delta z^j}$ the horizontal lift of P with respect to Chern–Finsler c.n.c. By the integrability of H' a similar calculus as in Proposition 4.3 from [23] says that $P_{H'}$ is a Poisson bivector on $T'M$ which is of mixed type $(2, 0)$. Then a transversal Hamiltonian structure of $(T'M, \mathcal{V}, G)$ with image extension H' is given by $\mathcal{H}(df) = \imath_{df} P_{H'}$ which according to above discussion is tame.

In the following we present an example of a tame Hamiltonian structure on a complex analytic foliated hermitian manifold when the horizontal distribution H' is not integrable.

Example 3.4. Consider the complex Heisenberg group:

$$G = \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \mid z_1, z_2, z_3 \in \mathbb{C} \right\},$$

with multiplication. The complex Iwasawa manifold is the compact quotient space $M = G/\Gamma$ formed from the right cosets of the discrete group Γ given by the matrices whose entries z_1, z_2, z_3 are Gaussian integers, see for details [24].

A basis of holomorphic left invariant 1-forms on M is given by $\{\alpha_1 = dz^1, \alpha_2 = dz^2, \alpha_3 = dz^3 - z^2 dz^1\}$. The dual vector fields $\{Z_1 = \frac{\partial}{\partial z^1} + z^2 \frac{\partial}{\partial z^3}, Z_2 = \frac{\partial}{\partial z^2}, Z_3 = \frac{\partial}{\partial z^3}\}$ are left-invariant holomorphic vector fields.

Consider now the Hermitian manifold (M, h, J), where J is the natural complex structure on M arising from the complex coordinates z_1, z_2, z_3 on G and the metric h determined by $h = \sum_{i=1}^3 \alpha_i \otimes \overline{\alpha}_i$. On this manifold we consider the vertical holomorphic foliation $\mathcal{V}$ given by fibers of natural projection $\pi : M \to \mathbb{C}^2$ defined by $\pi(z_1, z_2, z_3) = (z_1, z_2)$. For this complex analytic foliated hermitian manifold we have: $H' = \mathrm{span}\{Z_1, Z_2\}$, $H'' = \mathrm{span}\{\overline{Z_1}, \overline{Z_2}\}$, $V = \mathrm{span}\{Z_3, \overline{Z_3}\}$ and their dual $H'^* = \mathrm{span}\{\alpha_1, \alpha_2\}$, $H''^* = \mathrm{span}\{\overline{\alpha_1}, \overline{\alpha_2}\}$, $V^* = \mathrm{span}\{\alpha_3, \overline{\alpha_3}\}$. Then $t^3_1 = -z^2$, $t^3_2 = 0$ and by direct calculus we get $R^3_{11} = R^3_{22} = 0$, $R^3_{12} = -1$, $R^3_{21} = 1$. Thus every transversal Hamiltonian structure on the complex analytic foliated hermitian manifold $(M, \mathcal{V}, h)$ with H' an image extension verifies the Eq. (3.10) so it is tame. For instance, an usual holomorphic Poisson bracket on the manifold M is given by:

$$\{z^1, z^3\} = z^1 z^3 - 2z^2, \qquad \{z^3, z^2\} = z^3 z^2 - 2z^1, \qquad \{z^2, z^1\} = z^2 z^1 - 2z^3$$

and a transversal Hamiltonian structure on the complex analytic foliated hermitian manifold $(M, \mathcal{V}, h)$ with H' an image extension can be locally defined by $\mathcal{H}(dz^1) = \mathcal{H}^{12} Z_2$ and $\mathcal{H}(dz^2) = \mathcal{H}^{21} Z_1$, where $\mathcal{H}^{12} = \{z^1, z^2\} = 2z^3 - z^1 z^2 = -\mathcal{H}^{21}$ which is tame.

In the end of this section we notice that we can relate our study with complex analytic foliations defined by complex Poisson–Dirac submanifolds of a holomorphic Poisson manifold. As in the real case [25,26] a complex submanifold N of a holomorphic Poisson manifold (M, P) is called a *complex Dirac submanifold* if the holomorphic tangent bundle of M along N admits a vector bundle decomposition:

$$T'M|_N = T'N \oplus H'N \tag{3.12}$$

with the following properties:

(i) $P^\#(ann\, H'N) \subseteq T'N$;
(ii) For every $z \in N$ there exists an open neighborhood U of z in M such that for any holomorphic functions f, g on U which satisfy the conditions $df|_{H'N} = dg|_{H'N} = 0$ one has $d\{f, g\}|_{H'N} = 0$, where $\{f, g\}$ is the holomorphic Poisson bracket defined by P.

In this case we call $H'N$ a *complex Dirac complement* to N which is not unique in general.

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Example 3.5. Let us consider the product $M = S \times \mathbb{C}^n$, where each S-slice is a complex Poisson submanifold. Namely the holomorphic Poisson tensor at each point $(z, t) \in S \times \mathbb{C}^n$ is of the form $P(z, t) = P_t(z)$, where $P_t(z),\ t \in \mathbb{C}^n$ is a family of t-dependent holomorphic Poisson structures on S. Consider a particular S-slice $N = S \times \{t_0\}$ which is a complex Poisson submanifold. We will investigate when N is a complex Dirac submanifold.

We consider a local coordinate system $(t^1, \dots, t^n)$ on $\mathbb{C}^n$. If N is a complex Dirac submanifold then the complex Dirac complement $H'N$ must be of the form:

$$H'N = \mathrm{span}\left\{\frac{\partial}{\partial t^i} + Z_i \mid i = 1, \dots, n\right\},$$

where $Z_i,\ i = 1, \dots, n$, are complex vector fields on S. Using a similar argument as in Example 2.17 from [26] we can obtain that N is a complex Dirac submanifold of M if and only if the map $T'_{t_0}\mathbb{C}^n \mapsto H^2_{P_{t_0}}(S)$ explicitly given by $Z \mapsto [Z(P_t)]$ vanishes. Here $H^\bullet_{P_{t_0}}(S)$ denotes the holomorphic Lichnerowicz cohomology groups of S with respect to P_{t_0}, see [4].

4. An extended Hamiltonian formalism

As in the case of real foliated manifolds, [11], for the case of a transversal Hamiltonian structure $\mathcal{H}$ on a complex analytic foliated manifold $(M, \mathcal{F})$ it is possible to extend the Hamiltonian formalism in a way similar to what was done for presymplectic manifolds in [27].

We fix an image extension H' of $\mathcal{H}$ and use the decomposition (2.5) of the exterior differential d. Then $\mathcal{H}$ is defined for any differential form $\alpha \in \Omega^{1,0}_{\mathrm{mix}}(M, \mathcal{F})$ and $\mathcal{H}(\alpha) \in \Gamma(H')$. Accordingly, for any $f \in C^\infty(M)$, we get a complex Hamiltonian vector field:

$$Z'_f = \mathcal{H}(\mu f), \tag{4.1}$$

and for every $f, g \in C^\infty(M)$, we get an *extended Poisson bracket*:

$$\{f, g\}' := Z'_f g = \langle \mathcal{H}(\mu f), dg\rangle = \langle \mathcal{H}(\mu f), \mu g\rangle = -\langle \mu f, \mathcal{H}(\mu g)\rangle = -\{g, f\}'. \tag{4.2}$$

On the other hand, for $Z \in \Gamma(H')$ and $\alpha \in \Omega^{1,0}_{\mathrm{mix}}(M, \mathcal{F})$, we can decompose the Lie derivative as:

$$L_Z\alpha = L'_Z\alpha + L''_Z\alpha, \tag{4.3}$$

where:

$$L'_Z = i(Z)\mu + \mu i(Z), \qquad L''_Z = i(Z)\lambda + \lambda i(Z), \tag{4.4}$$

and we can extend the Gelfand–Dorfman–Schouten–Nijenhuis bracket, [28], to arbitrary forms $\alpha, \beta, \gamma \in \Omega^{1,0}_{\mathrm{mix}}(M, \mathcal{F})$ by putting:

$$[\mathcal{H}, \mathcal{K}]'(\alpha, \beta, \gamma) := \sum_{\mathrm{Cycl}(\alpha,\beta,\gamma)} \{\langle \mathcal{K}L'_{\mathcal{H}(\alpha)}\beta, \gamma\rangle + \langle \mathcal{H}L'_{\mathcal{K}(\alpha)}\beta, \gamma\rangle\}, \tag{4.5}$$

where $\mathcal{H}, \mathcal{K} : H'^* \to H'$ are skew-symmetric morphisms. This extended bracket is trilinear over $C^\infty(M)$. Hence, for a Hamiltonian structure $\mathcal{H}$, we have:

$$[\mathcal{H}, \mathcal{H}]'(\alpha, \beta, \gamma) = 0, \quad \forall \alpha, \beta, \gamma \in \Omega^{1,0}_{\mathrm{mix}}(M, \mathcal{F}),$$

since this is true for c.a.f. forms. Using (4.2) and (4.4) we see that the previous property implies that for every $f, g, h \in C^\infty(M)$ one has:

$$\sum_{\mathrm{Cycl}(f,g,h)} [\{\{f, g\}', h\}' + \mu^2 f(Z'_g, Z'_h)] = \frac{1}{2}[\mathcal{H}, \mathcal{H}]'(\mu f, \mu g, \mu h) = 0. \tag{4.6}$$

Proposition 4.1. *If $\mathcal{H}$ is a H'-tame Hamiltonian structure on the complex analytic foliated hermitian manifold $(M, \mathcal{F}, h)$, then the biderivation $\{,\}'$ is a Poisson bracket on M.*

Proof. For any normal bundle $H = H' \oplus H''$ of the complex analytic foliation $\mathcal{F}$ one claims:

$$\mu^2 f(Z, W) = \langle \lambda f, N_{H'}(Z, W)\rangle, \quad \forall f \in C^\infty(M),\ \forall Z, W \in \Gamma(H'), \tag{4.7}$$

where $N_{H'}$ is the Nijenhuis tensor from (3.8). Indeed, if $Z, W \in \Gamma(H')$, (3.7) yields:

$$N_{H'}(Z, W) = p_{V'}[Z, W], \tag{4.8}$$

where $p_{V'}$ is the projection from $T_{\mathbb{C}}M$ onto V'. On the other hand:

$$\mu^2 f(Z, W) = d(\mu f)(Z, W) = ZWf - WZf - \langle \mu f, [Z, W]\rangle$$
$$= [Z, W]f - (p_{H'}[Z, W])f = \langle df, [Z, W]\rangle = \langle \lambda f, p_{V'}[Z, W]\rangle.$$

Thus (4.7) holds and the conclusion follows from the characterization (3.7) of the tame Hamiltonian structures and formula (4.6). □

The above discussion yields

Theorem 4.1. *A tame Hamiltonian structure is defined by an usual Poisson structure P on the manifold M. The complex Hamiltonian vector fields of c.a.f. functions with respect to $\mathcal{H}$ coincide with those with respect to P, $P^\#|_{H'^*} = \mathcal{H}$ and $P^\#|_{H''^* \oplus V^*} = 0$.*

Now, returning to a general transversal Hamiltonian structure $\mathcal{H}$ of the complex analytic foliation $\mathcal{F}$, with the image extension H', we can use the component L'_Z, $(Z \in \mathcal{V}^{1,0}_{\text{mix}}(M, \mathcal{F}))$, in order to define a bracket of 1-forms $\alpha, \beta \in \Omega^{1,0}_{\text{mix}}(M, \mathcal{F})$ similar to that encountered on a Poisson manifold. Namely:

$$\{\alpha, \beta\}' = L'_{\mathcal{H}(\alpha)}\beta - L'_{\mathcal{H}(\beta)}\alpha - \mu\langle \mathcal{H}(\alpha), \beta\rangle. \tag{4.9}$$

The above formula implies:

$$\{f\alpha, g\beta\}' = fg\{\alpha, \beta\}' + f(\mathcal{H}(\alpha)g)\beta - g(\mathcal{H}(\beta)f)\alpha, \quad f, g \in C^\infty(M), \tag{4.10}$$

whence we see that the bracket (4.9) is skew-symmetric because it is such for c.a.f. 1-forms (forms of mixed type (1, 0) with local components c.a.f. functions).

The notion of Complex Lie Algebroid was introduced in [29] as a complex vector bundle E over a (real) manifold M and a complex bundle map $\rho : E \to T_{\mathbb{C}}M$ satisfying the complex version of the two axioms in the definition of a Lie algebroid.

We have

Proposition 4.2. *Let $\mathcal{H}$ be a H'-tame Hamiltonian structure of the complex analytic foliated manifold $(M, \mathcal{F})$ and P' the Poisson structure defined by $\{,\}'$. Then the triple $(H'^*, \{,\}', \mathcal{H})$, with the bracket (4.9), is a Complex Lie sub-Algebroid of the Complex cotangent Lie Algebroid $(T^*_{\mathbb{C}}M, \{,\}_{P'}, P'^\#)$.*

Proof. The bracket $\{,\}_{P'}$ is given by (4.9) without primes and with $\mathcal{H}$ replaced by $P'^\#$. Since $P'^\#|_{H'^*} = \mathcal{H}$ we have:

$$\{\alpha, \beta\}' = \{\alpha, \beta\}_{P'}, \quad \alpha, \beta \in \Omega^1_{\text{c.a.f.}}(M, \mathcal{F}).$$

Then (4.10) implies:

$$\{f\alpha, g\beta\}' = \{f\alpha, g\beta\}_{P'}, \quad f, g \in C^\infty(M), \ \alpha, \beta \in \Omega^1_{\text{c.a.f.}}(M, \mathcal{F}). \quad \square$$

We notice that there exist an inclusion and a splitting morphism of Complex Lie Algebroids:

$$i : H'^* \hookrightarrow T^*_{\mathbb{C}}M, \qquad \pi = p_{H'^*} : T^*_{\mathbb{C}}M \to H'^* \quad (\pi \circ i = id), \tag{4.11}$$

where $p_{H'^*}$ is the projection onto H'^* in the decomposition $T^*_{\mathbb{C}}M = H'^* \oplus H''^* \oplus V^*$.

For the de Rham cohomology of (Complex) Lie Algebroids we refer to [30] and we remind that the Lichnerowicz–Poisson cohomology is the de Rham cohomology of the cotangent bundle (Complex) Lie Algebroid $T^*_{\mathbb{C}}M$ of the Poisson manifold (M, P), [10]. These definitions show the existence of homomorphisms:

$$H^\bullet_{\text{dR}}(M, \mathcal{F}) \xrightarrow{j_1^*} H^\bullet_{\text{LP}}(M, \mathcal{F}, P') \xrightarrow{i^*} H^\bullet(H'^*),$$
$$H^\bullet_{\text{dR}}(M, \mathcal{F}) \xrightarrow{j_2^*} H^\bullet(H'^*) \xrightarrow{\pi^*} H^\bullet_{\text{LP}}(M, \mathcal{F}, P').$$

Here $j_1 = P'^\#$ and $j_2 = \mathcal{H}$. Arguments similar to those of [11] leads to the following result:

Proposition 4.3. *Under the hypotheses of Proposition 4.2., the projection π induces an injection π^* of the de Rham cohomology of the Complex Lie sub-Algebroid H'^* into the Lichnerowicz–Poisson cohomology of $(M, \mathcal{F}, P')$. For any complex vector bundle S over the complex analytic foliated manifold $(M, \mathcal{F})$ the Lichnerowicz–Poisson Chern classes $c_k^{\text{LP}}(S) := j_1^* c_k(S)$ belong to the image of the injection π^*.*

Finally we notice that in similar manner with the real case [11,13] we can consider the notion of *distinguished function* associated to a non-tame Hamiltonian structure of a complex analytic foliated hermitian manifold $(M, \mathcal{F}, h)$ and some cohomology spaces related to non-tame Hamiltonian structure can be obtained.

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