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Second order parallel tensors and Ricci solitons in 3-dimensional normal paracontact geometry

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Abstract The aim of this paper is to study two problems in the framework of paracontact geometry of dimension 3, namely, the class of parallel symmetric tensor fields of (0, 2)-type and possible Lorentz Ricci solitons. We search for two types of Ricci solitons: the first when its potential vector field is exactly the characteristic vector field ξ of the paracontact structure and the second when the potential vector field is a paracontact-holomorphic one. For the former case we find all variants of Ricci solitons, expanding, steady and shrinking, and the fact that ξ is a conformal Killing vector field. A class of examples is completely discussed.

Keywords Normal almost paracontact manifold · K-paracontact manifold · Lorentz Ricci soliton · Riccati equation · Paracontact-holomorphic vector field

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1 Introduction

The paracontact geometry appears as a natural counter-part of the contact geometry in [12]. Comparing with the huge literature in almost contact geometry, it seems that there are necessary new studies in almost paracontact geometry; a very interesting paper connecting these fields is [7]. The present work is another step in this direction, more precisely from the point of view of parallel symmetric tensor fields of (0, 2)-type and Ricci solitons. The importance of the latter subject is given by the rôle of Ricci solitons as self-similar solutions of the Ricci flow as it is pointed out in [8, p. 153].

Dedicated to the memory of Professor Dr. Mircea Craioveanu.

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We make two restrictions for the paracontact metrics studied here. Firstly, the hypothesis of normality is taken into account in order to use the features of the integrability of an associated paracomplex structure. Secondly, we restrict to dimension 3 due to the fact that in this case we have an explicit formula for the covariant derivative of the characteristic vector field ξ, necessary for the problem of Ricci solitons. Our dimension implies the Lorentz character of the metric and let us note that another paper treating Ricci solitons in (any dimensional) paracontact geometry is [6].

Our work is structured as follows: the first section is a very brief review of (normal) almost paracontact geometry with a special view toward the dimension 3 and Ricci solitons. The next section is focused on the study of conditions under which a given parallel second order tensor field is a constant multiple of the metric tensor. The third section is concerned with Ricci solitons and we obtain the necessary conditions for their existence. We study separately the quasi-para-Sasakian and para-Kenmotsu geometries; the former admits only expanding Ricci solitons while the latter admits Ricci solitons of all types: expanding, steady and shrinking. A main tool for the latter case is a Riccati differential equation which we solve completely. The last section is devoted to examples and we obtain a class of normal paracontact structures which generalizes several previous geometries from the literature.

2 Normal almost paracontact geometry and Ricci solitons

Let M be a $(2n+1)$-dimensional smooth manifold, φ a tensor field of $(1,1)$-type called *the structural endomorphism*, ξ a vector field called *the characteristic vector field*, η a 1-form called *the paracontact form* and g a semi-Riemannian metric on M of signature $(n+1,n)$. We say that (φ,ξ,η,g) defines an *almost paracontact metric structure* on M if [20, p. 38]:

1. $\varphi(\xi)=0, \eta\circ\varphi=0$,
2. $\eta(\xi)=1, \varphi^2=I-\eta\otimes\xi$,
3. φ induces on the $2n$-dimensional distribution $\mathcal{D}:=\ker\eta$ an almost paracomplex structure P, i.e., $P^2=1$ and the eigensubbundles T^+, T^-, corresponding to the eigenvalues 1, -1 of P respectively, have equal dimension n; hence $\mathcal{D}=T^+\oplus T^-$,
4. $g(\varphi\cdot,\varphi\cdot)=-g+\eta\otimes\eta$.

For a list of examples of almost paracontact metric structures see [11, p 84]. From the definition it follows that η is the g-dual of ξ, i.e., $\eta(X)=g(X,\xi)$ and ξ is an unitary vector field: $g(\xi,\xi)=1$; more precisely ξ is space-like. Let ∇ be the Levi-Civita connection of g and with respect to the orthogonal splitting of the tangent bundle $TM=\mathcal{D}\oplus span\{\xi\}$ let v_ξ and h_ξ be the corresponding projectors; thus $v_\xi(X)=X-\eta(X)\xi$. An interesting open problem is to study this splitting according to the approach of Craioveanu and Slesar from [9].

The Nijenhuis tensor field with respect to the tensor field φ, denoted by N_φ, is given by:

$$N_\varphi(X,Y)=[\varphi(X),\varphi(Y)]+\varphi^2([X,Y])-\varphi([\varphi(X),Y]) \\ -\varphi([X,\varphi(Y)]),\ \forall X,Y\in\Gamma(TM). \tag{2.1}$$

Definition 2.1 The almost paracontact metric manifold (M,φ,ξ,η,g) is said to be *normal* if the almost paracomplex structure J on the manifold $M\times\mathbb{R}$, given by:

$$J\left(X,\lambda\frac{d}{dt}\right)=\left(\varphi(X)+\lambda\xi,\eta(X)\frac{d}{dt}\right),\quad X\in\Gamma(TM),\ \ t\in\mathbb{R}, \tag{2.2}$$

is integrable, where λ is a real valued function on $M\times\mathbb{R}$.

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The condition above is equivalent to:

$$N_{\varphi} - 2d\eta \otimes \xi = 0. \tag{2.3}$$

In the following, we restrict to the dimension 3 for which the metric is a Lorentz one and the normality is equivalent with, [18, p. 379]:

$$\begin{cases} \nabla_X \xi = \alpha(X - \eta(X)\xi) + \beta\varphi(X), \\ (\nabla_X \varphi)Y = \alpha(g(\varphi X, Y)\xi - \eta(Y)\varphi X) + \beta(g(X, Y)\xi - \eta(Y)X). \end{cases} \tag{2.4}$$

where $\alpha = \frac{1}{2} div\xi$ and $\beta = \frac{1}{2} trace(\varphi\nabla\xi)$. An important consequence of $(1.4a)$ is that ξ is *a geodesic vector field*:

$$\nabla_{\xi}\xi = 0. \tag{2.5}$$

For arbitrary $X \in \Gamma(\mathcal{D})$ we have that:

$$d\eta(\xi, X) = 0 \tag{2.6}$$

Indeed, from the metric condition of ∇:

$$0 = (\nabla_{\xi} g)(\xi, X) = \xi(\eta(X)) - \eta(\nabla_{\xi} X) = \xi(\eta(X)) - \eta(\nabla_X \xi + [\xi, X])$$

but $\nabla_X \xi \in \Gamma(\mathcal{D})$ also.

A straightforward computation yields the curvature:

$$\begin{aligned} R(X, Y)\xi =\ & X(\alpha)(Y - \eta(Y)\xi) - Y(\alpha)(X - \eta(X)\xi) + X(\beta)\varphi Y \\ & -Y(\beta)\varphi X - 2\alpha d\eta(X, Y)\xi + 2\alpha\beta[g(\varphi X, Y) + \eta(X)\varphi Y \\ & -\eta(Y)\varphi X] + (\alpha^2 + \beta^2)[\eta(X)Y - \eta(Y)X]. \end{aligned} \tag{2.7}$$

The ξ-sectional curvature K_{ξ} of (M, g) is the sectional curvature of a plane spanned by ξ and a unitary vector field $X \in \Gamma(\mathcal{D})$. Since from (2.6) it results:

$$R(X, \xi)\xi = -\left(\xi(\alpha) + \alpha^2 + \beta^2\right)(X - \eta(X)\xi) - (\xi(\beta) + 2\alpha\beta)\,\varphi X \tag{2.8}$$

we get:

$$K_{\xi} = g(R(X, \xi)\xi, X) = -\left(\xi(\alpha) + \alpha^2 + \beta^2\right). \tag{2.9}$$

It follows that the ξ-sectional curvature does not depend on X and this fact explains the notation.

We recall now the main examples of three-dimensional paracontact geometry:

Definition 2.2 The almost paracontact metric manifold $(M^3, \varphi, \xi, \eta, g)$ is called:

1. *quasi-para-Sasakian* if $\alpha = 0$ and $\beta \neq 0$. In particular, for $\beta = -1$ the manifold is *para-Sasakian* (for another point of view of this notion see [11, p. 82]);
2. *para-Kenmotsu* if $\beta = 0$ and $\alpha \neq 0$.

In the last part of this section we recall the notion of Ricci solitons from [8]. On the manifold M, a *Ricci soliton* is a triple (g, V, λ) with g a (semi-) Riemannian metric, V a vector field (called *the potential vector field*) and λ a real scalar such that the tensor field $Ric_{(\xi,\lambda)}$ of $(0, 2)$-type:

$$Ric_{(\xi,\lambda)} := \mathcal{L}_V g + 2S + 2\lambda g = 0 \tag{2.10}$$

where $\mathcal{L}_V$ is the Lie derivative with respect to V and S is the Ricci tensor of g. The Ricci soliton is said to be *shrinking*, *steady* or *expanding* according as λ is negative, zero or positive.

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Denote by Q the Ricci operator determined by $S(X, Y) = g(QX, Y)$ and by r the scalar curvature of the metric g.

3 Second order parallel tensors in 3-dimensional normal almost paracontact geometry

The purpose of this section is to prove one of the main results of the paper.

Theorem 3.1 *Let $(M^3, \varphi, \xi, \eta, g)$ be a normal almost paracontact metric manifold with non-vanishing ξ-sectional curvature and $\alpha^2 \neq \beta^2$. Suppose M is endowed with a tensor field $\rho \in \Gamma(T_2^0(M))$ which is symmetric and φ-skew-symmetric:*

$$\rho(\varphi X, Y) + \rho(X, \varphi Y) = 0 \tag{3.1}$$

for all $X, Y \in \Gamma(TM)$. If ρ is parallel with respect to ∇ then it is a constant multiple of the metric tensor g.

Proof Applying the Ricci commutation identity [8, p. 14] and $\nabla^2_{X,Y}\rho(Z, W) - \nabla^2_{X,Y}\rho(W, Z) = 0$, we obtain the relation (2.1) of [16, p. 787]:

$$\rho(R(X, Y)Z, W) + \rho(Z, R(X, Y)W) = 0 \tag{3.2}$$

which is fundamental in all papers treating this subject. Replacing $Z = W = \xi$ and using (2.7) it results, by the symmetry of ρ:

$$\begin{aligned} &\rho(X(\alpha)Y - Y(\alpha)X) + X(\beta)\varphi Y - Y(\beta)\varphi X + 2\alpha\beta[\eta(X)\varphi Y - \eta(Y)\varphi X] \\ &\quad + (\alpha^2 + \beta^2)[\eta(X)Y - \eta(Y)X], \xi) \\ &\quad = [X(\alpha)\eta(Y) - Y(\alpha)\eta(X) + 2\alpha d\eta(X, Y) - 2\alpha\beta g(\varphi X, Y)]\rho(\xi, \xi). \end{aligned} \tag{3.3}$$

The skew-symmetry condition (3.1) reduces this relation to:

$$\begin{aligned} &\rho(X(\alpha)Y - Y(\alpha)X + (\alpha^2 + \beta^2)[\eta(X)Y - \eta(Y)X], \xi) \\ &\quad = [X(\alpha)\eta(Y) - Y(\alpha)\eta(X) + 2\alpha d\eta(X, Y) - 2\alpha\beta g(\varphi X, Y)]\rho(\xi, \xi). \end{aligned} \tag{3.4}$$

With $X = \xi$ and (2.6) we get:

$$[\xi(\alpha) + \alpha^2 + \beta^2][\rho(Y, \xi) - \eta(Y)\rho(\xi, \xi)] = 0 \tag{3.5}$$

which, with the hypothesis on K_ξ, yields:

$$\rho(Y, \xi) = \eta(Y)\rho(\xi, \xi). \tag{3.6}$$

The parallelism of ρ and (3.6) imply that $\rho(\xi, \xi)$ is a constant:

$$X(\rho(\xi, \xi)) = 2\rho(\nabla_X \xi, \xi) = 2\eta(\nabla_X \xi)\rho(\xi, \xi) = 0 \cdot \rho(\xi, \xi) = 0. \tag{3.7}$$

Applying X to (3.6) and using (3.7) we have:

$$X(\rho(Y, \xi)) = X(g(Y, \xi))\rho(\xi, \xi) = [g(\nabla_X Y, \xi) + g(Y, \nabla_X \xi)]\rho(\xi, \xi)$$

which means via the parallelism of ρ that:

$$\rho(Y, \alpha X + \beta\varphi X) = g(Y, \alpha X + \beta\varphi X)\rho(\xi, \xi). \tag{3.8}$$

In the last relation we make $X \to \varphi X$ and then:

$$\rho(Y, \beta X + \alpha\varphi X) = g(Y, \beta X + \alpha\varphi X)\rho(\xi, \xi). \tag{3.9}$$

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Multiplying (3.8) with α, respectively, (3.9) with β and making the difference of the resulting equations we get:

$$(\alpha^2 - \beta^2)[\rho(Y, X) - g(Y, X)\rho(\xi, \xi)] = 0$$

which gives the conclusion. □

Particular cases 3.2 1. (*quasi-para-Sasakian*) For $\alpha = 0$ and $\beta \neq 0$ we get the *quasi-para-Sasakian case* ([18, p. 380]) and since $K_\xi = -\beta^2$ we have a non-vanishing ξ-sectional curvature as well as the condition $\alpha^2 \neq \beta^2$. From the general formula:

$$\mathcal{L}_\xi g = 2\alpha g - 2\alpha\eta \otimes \eta \tag{3.10}$$

one obtains that a quasi-para-Sasakian manifold is a *K-paracontact* one, i.e., ξ is a Killing vector field.
In particular, for $\beta = -1$ we have the *para-Sasakian case* while for a constant $\beta \neq 0$ the manifold is called *β-para-Sasakian*. The Theorem 3.1 becomes:

Proposition 3.3 *Let $(M^3, \varphi, \xi, \eta, g)$ be a quasi-para-Sasakian manifold endowed with a tensor field $\rho \in \Gamma(T^0_2(M))$ which is symmetric and φ-skew-symmetric. If ρ is parallel with respect to ∇ then it is a constant multiple of the metric tensor g.*

2. For $\alpha \neq 0$ and $\beta = 0$ we get the para-Kenmotsu case. In particular, for a constant α the manifold is called α-para-Kenmotsu, [19]. *From* (3.3) *it follows that the condition* (3.1) *is not necessary when $\beta = 0$.* □

Remark 3.4 The reduction of a covariant second order tensor field to a multiple of the metric holds generally under the hypothesis of irreducibility of the holonomy group/algebra, see for example the theorem 57 of [15, p. 254]. A classification of Lorentzian holonomy groups is given in [13], but our result 2.1 implies weaker conditions for the Lorentz metric in the paracontact 3-dimensional framework.

4 Ricci solitons in 3-dimensional normal paracontact geometry

Following the approach of [10] we determine by a direct computation the following expressions:

$$\begin{cases} S(X, \xi) = [-2(\alpha^2 + \beta^2) - \xi(\alpha)]\eta(X) - X(\alpha) + \varphi X(\beta) \\ Q\xi = [-2(\alpha^2 + \beta^2) - \xi(\alpha)]\xi - \nabla\alpha - \varphi(\nabla\beta) \end{cases} \tag{4.1}$$

(here ∇u denotes the gradient of the function u) which implies:

$$QX = \left[\alpha^2 + \beta^2 + \xi(\alpha) + \frac{r}{2}\right]X - \left[3(\alpha^2 + \beta^2) + \xi(\alpha) + \frac{r}{2}\right]\eta(X)\xi$$
$$-[\nabla\alpha + \varphi(\nabla\beta)]\eta(X) + [\varphi X(\beta) - X(\alpha)]\xi - [\xi(\beta) + 2\alpha\beta)]\varphi X.$$

Since Q is a symmetric operator with respect to g the last term of above equation vanishes:

$$\xi(\beta) + 2\alpha\beta = 0 \tag{4.2}$$

and consequently the Ricci operator is:

$$QX = \left[\alpha^2 + \beta^2 + \xi(\alpha) + \frac{r}{2}\right]X - \left[3(\alpha^2 + \beta^2) + \xi(\alpha) + \frac{r}{2}\right]$$
$$\times\eta(X)\xi - [\nabla\alpha + \varphi(\nabla\beta)]\eta(X) + [\varphi X(\beta) - X(\alpha)]\xi. \tag{4.3}$$

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Based on formula (4.3) we derive the expression of the Ricci tensor field:

$$\begin{aligned} S(X, Y) = g(QX, Y) = \left[\alpha^2 + \beta^2 + \xi(\alpha) + \frac{r}{2}\right] g(X, Y) \\ -\left[3(\alpha^2 + \beta^2) + \xi(\alpha) + \frac{r}{2}\right] \eta(X)\eta(Y) \\ +[\varphi X(\beta) - X(\alpha)]\eta(Y) + [\varphi Y(\beta) - Y(\alpha)]\eta(X) \end{aligned} \tag{4.4}$$

and then, for constant α, β it results that (M^3, g) is an *eta-Einstein manifold*. The main result of this section is:

Theorem 4.1 *Let $(g, \xi, \lambda = \lambda_\xi)$ be a Ricci soliton in (M^3, φ). Then ξ is a conformal Killing vector field:*

$$\mathcal{L}_\xi g = -4\xi(\alpha) g \qquad (cKilling)$$

the scalar λ and the scalar curvature are:

$$\lambda = 2[\xi(\alpha) + \alpha^2 + \beta^2] = -2K_\xi, \quad r = -6[\xi(\alpha) + \alpha^2 + \beta^2] - 2\alpha = 6K_\xi - 2\alpha, \tag{4.5}$$

and for all $X \in \Gamma(\mathcal{D})$ we have:

$$(X + \varphi X)(\alpha - \beta) = 0. \tag{4.6}$$

Thus K_ξ and $r + 2\alpha$ are constants.

Proof From $\lambda = -S(\xi, \xi)$ and (4.4) we derive the first part of (4.5). In the Ricci equation (2.10) we make three choices:

I. $X = Y = \xi$; we get a trivial identity;
II. $X = Y \in \Gamma(\mathcal{D})$ of unit norm: it follows the second part of (4.5). Returning with the expression of S and λ in (2.10) it follows the relation $(cKilling)$;
III. $Y = \xi$ and $X \in \Gamma(\mathcal{D})$ again of unit norm: then $\varphi X(\beta) = X(\alpha)$. In this last relation we replace also X with φX and derive $X(\beta) = \varphi X(\alpha)$. By adding these last two equations we get the condition (4.6).

Since λ is a constant it follows from the first part of (4.5) that K_ξ is a constant and from the last part of (4.5) we get the constancy of $r + 2\alpha$. □

Particular cases 4.2 1. (*quasi-para-Sasakian*) From $\alpha = 0$ and the constancy of K_ξ it results that β is a (non-vanishing) constant. From (4.5) it follows that (g, ξ) is an expanding ($\lambda = 2\beta^2$) Ricci soliton with constant scalar curvature: $r = -6\beta^2 < 0$. The expanding character of Ricci solitons in this case was already obtained in [1, p. 240] as Example 3.7.

Precisely, (4.4) yields $S = -2\beta^2 g$ which means that (M^3, g) is an Einstein manifold, a fact implied also by the K-paracontact framework. The Einstein condition is equivalent to the constancy of the sectional curvature and we have $K_\xi = -\beta^2 < 0$. In the para-Sasakian case we get $K_\xi = -1$ as in [4].

2. (para-Kenmotsu) From $\beta = 0$ and (4.6) we get that $(X + \varphi X)(\alpha) = 0$ for all $X \in \Gamma(\mathcal{D})$. A Ricci soliton (g, ξ) has $\lambda = 2[\xi(\alpha) + \alpha^2]$ and is also expanding for a constant α. Its scalar curvature is: $r = -6[\xi(\alpha) + \alpha^2] - 2\alpha$ and therefore:

$$\begin{aligned} S(X, Y) = -[2\alpha^2 + 2\xi(\alpha) + \alpha] g(X, Y) + [2\xi(\alpha) + \alpha]\eta(X)\eta(Y) \\ -X(\alpha)\eta(Y) - Y(\alpha)\eta(X) \end{aligned} \tag{4.7}$$

with two consequences:

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i. the Ricci tensor field S restricted to $\mathcal{D}$ becomes $S|_{\mathcal{D}} = -[2\alpha^2 + 2\xi(\alpha) + \alpha]g$ which means that $(M, \mathcal{D}, g|_{\mathcal{D}})$ is an Einstein sub-Riemannian manifold;

ii. if α is a constant then $S = -[2\alpha^2 + \alpha]g + \alpha\eta \otimes \eta$ and $r = -6\alpha^2 - 2\alpha$ is a constant.

In order to obtain a constant λ let us consider $\xi = \frac{\partial}{\partial z}$ and search for corresponding α. Note that the differential equation:

$$\alpha' + \alpha^2 = \text{constant}$$

is a Riccati equation. With $\alpha(x, y, z) = c\tanh(cz)$ for a constant number $c \neq 0$ we have:

$$\xi(\alpha) + \alpha^2 = c^2 \tag{4.8}$$

and then $\lambda = 2c^2 > 0$. The scalar curvature is $r(z) = -6c^2 - 2c\tanh(cz)$ with $\lim_{z\to+\infty} r(z) = -6c^2 - 2c = -2c(3c+1)$.

Note that the expanding character is according to the results of [6]. For the same ξ we take consequently: (a) a shrinking case: $\alpha(x, y, z) = c\tan(-cz)$. We have:

$$\xi(\alpha) + \alpha^2 = -c^2 \tag{4.9}$$

and the scalar curvature is $r(z) = 6c^2 - 2c\tan(-cz)$ with $\lim_{z\to\pm\frac{\pi}{c}} r(z) = 6c^2 > 0$. (b) a steady case: $\alpha(x, y, z) = \frac{1}{z+C}$ with C a real constant and the corresponding scalar curvature: $r(z) = \frac{-2}{z+C}$.

3. The functions α, β are constants. From (4.3) we have:

$$Q\xi = -2(\alpha^2 + \beta^2)\xi, \quad QX = \left(\alpha^2 + \beta^2 + \frac{r}{2}\right)X, \qquad X \in \Gamma(\mathcal{D}) \tag{4.10}$$

which means that the Ricci operator has the eigenvalues $\lambda_1 = -2(\alpha^2 + \beta^2)$, $\lambda_2 = \lambda_3 = \alpha^2 + \beta^2 + \frac{r}{2}$ and therefore it has the Segre type $\{11, 1\}$ according to [5]. In the Ricci soliton case, from (4.5) it follows that $\lambda_1 = \lambda_2 = \lambda_3 = -2(\alpha^2 + \beta^2) = -\lambda$ and then $Q = -\lambda I_3$ which means that (M, g) is Einstein. The case of non-trivial Lorentzian Ricci solitons with Ricci operator having 3 equal eigenvalues is resolved by Theorem 2 of [3, p. 388]. □

Returning now to the approach of Sect. 3, let us consider the tensor field inspired by the Ricci Eq. (2.10):

$$\rho = \mathcal{L}_\xi g + 2S \tag{4.11}$$

It is obviously symmetric and satisfies (3.1). Its detailed expression is:

$$\begin{aligned}\rho(X, Y) = {} & [2\alpha^2 + 2\beta^2 + 2\xi(\alpha) + 2\alpha + r]g(X, Y) \\ & -[6\alpha^2 + 6\beta^2 + 2\xi(\alpha) + 2\alpha + r]\eta(X)\eta(Y) \\ & +2[\varphi Y(\beta) - Y(\alpha)]\eta(X) + 2[\varphi X(\beta) - X(\alpha)]\eta(Y)\end{aligned} \tag{4.12}$$

where we do not use the expression (4.5) of r. Since this general formula is too complicated we suppose from now that α and β are constant and then:

$$\rho(X, Y) = [2\alpha^2 + 2\beta^2 + 2\alpha + r]g(X, Y) - [6\alpha^2 + 6\beta^2 + 2\alpha + r]\eta(X)\eta(Y) \tag{4.13}$$

for which the parallelism hypothesis means:

$$\begin{aligned}0 = \nabla_X\rho(Y, Z) = {} & -[6\alpha^2 + 6\beta^2 + 2\alpha + r][\alpha\eta(Z)g(X, Y) + \beta\eta(Z)g(Y, \varphi X) \\ & -\alpha\eta(Y)g(X, Z) - \beta\eta(Y)g(Z, \varphi X)]\end{aligned} \tag{4.14}$$

for all vector fields X, Y, Z on M. We make the choices:

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1. $Y = \xi$ and $X = Z \in \Gamma(\mathcal{D})$ an unitary vector field; then:

$$\alpha[6\alpha^2 + 6\beta^2 + 2\alpha + r] = 0 \tag{4.15}$$

2. $Y = \xi$ and $\varphi X = Z \in \Gamma(\mathcal{D})$ an unitary vector field; then:

$$\beta[6\alpha^2 + 6\beta^2 + 2\alpha + r] = 0. \tag{4.16}$$

Under the hypothesis $\alpha^2 \neq \beta^2$ it results:

$$r = -6(\alpha^2 + \beta^2) - 2\alpha$$

as we expect from (4.5). Returning with this expression in (4.13) we get: $\rho = -4(\alpha^2 + \beta^2)g$.

Until now we search for Ricci solitons with $V = \xi$. Now, we extend the class of possible Ricci solitons and for this aim we introduce inspired by [2, 196]:

Definition 4.3 The vector field $X \in \Gamma(TM)$ is called *paracontact-holomorphic* if its Lie derivative of the structural endomorphism vanishes:

$$v_\xi \circ \mathcal{L}_X \varphi = 0. \tag{4.17}$$

Let $\mathfrak{phol}(M)$ be the set of all paracontact-holomorphic vector fields.

The condition (4.17) says that for any vector field Y we have that $(\mathcal{L}_X \varphi)Y$ is collinear with ξ; let us denote $\alpha_X(Y)$ the collinearity factor. Then:

$$\alpha_X(Y) = g([X, \varphi Y] - \varphi([X, Y]), \xi) = \eta([X, \varphi Y]). \tag{4.18}$$

The next result shows the invariance of the above defined holomorphicity and its proof is exactly as in [2, p. 197].

Proposition 4.4 *Let X be a paracontact-holomorphic vector field on the normal almost paracontact metric manifold $(M, \varphi, \xi, \eta, g)$. Then φX is also a paracontact-holomorphic vector field. Moreover, $\mathfrak{phol}(M)$ is a Lie subalgebra in the Lie algebra of vector fields of M.*

Remark 4.5 i. Fix X a paracontact-holomorphic vector field. Then computing $\alpha_X(\xi)$ with (4.18) we get:

$$\alpha_X(\xi) = 0 \tag{4.19}$$

which means that $[X, \xi]$ is collinear with ξ, i.e., $v_\xi([X, \xi]) = 0$.

ii. The vanishing of the tensor field $N^{(3)} = 2h = \mathcal{L}_\xi \varphi$ means that ξ is a paracontact-holomorphic vector field with $\alpha_\xi = 0$. From [20, p. 40] we have that this vanishing is equivalent with the K-paracontact condition.

iii. The paracontact-holomorphicity of a fixed X implies for any vector field Y:

$$\mathcal{L}_X Y = \eta(\mathcal{L}_X Y)\xi + \varphi(\mathcal{L}_X \varphi Y), \quad \mathcal{L}_X \varphi Y = \alpha_X(Y)\xi + \varphi([X, Y]). \tag{4.20}$$

In both relations, the first term in the right hand-side is from $span\xi$ while the second is from $\mathcal{D}$.

We apply these computations for studying paracontact-holomorphic Ricci solitons:

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Proposition 4.6 *Let* $(g, V, \lambda = \lambda_V)$ *be a Ricci soliton in* (M^3, φ) *with* $V \in \mathfrak{phol}(M)$. *Then:*

$$\lambda = 2[\xi(\alpha) + \alpha^2 + \beta^2] + \eta(\mathcal{L}_V \xi) = \lambda_\xi + \eta(\mathcal{L}_V \xi). \tag{4.21}$$

It follows that if ξ *and* $V \in \mathfrak{phol}(M)$ *are both Ricci solitons then* $\eta(\mathcal{L}_V \xi)$ *is a constant and* V *is also a conformal Killing vector field with:*

$$\mathcal{L}_V g = -2[\eta(\mathcal{L}_V \xi) + 2\xi(\alpha)]g. \tag{4.22}$$

Proof The Ricci Eq. (2.10) on the pair (ξ, ξ) gives:

$$\lambda = -g(\nabla_\xi V, \xi) - S(\xi, \xi) = -g(\nabla_\xi V, \xi) + 2[\xi(\alpha) + \alpha^2 + \beta^2]. \tag{4.23}$$

From (4.20_1) we have (see also Remark 3.6i):

$$\mathcal{L}_V \xi = \eta(\mathcal{L}_V \xi)\xi. \tag{4.24}$$

and then:

$$\nabla_\xi V = \nabla_V \xi - \mathcal{L}_V \xi = \alpha(V - \eta(V)\xi) + \beta\varphi(V) - \eta(\mathcal{L}_V \xi)\xi \tag{4.25}$$

which plugging into (4.23) yields the conclusion. □

5 Examples

Let N be an open connected subset of $\mathbb{R}^2$, $I = (a, b)$ an open interval in $\mathbb{R}$ and let us consider the manifold $M = N \times I$. Let (x, y) be the coordinates on N induced from the cartesian coordinates on $\mathbb{R}^2$ and let z be the coordinate on I induced from the cartesian coordinate on $\mathbb{R}$. Thus (x, y, z) are the coordinates on M. Now we choose the functions:

$$\omega_1, \omega_2 : N \to \mathbb{R}, \quad \sigma : M \to \mathbb{R}^*_+ \tag{5.1}$$

and following the idea from [17] we define a normal almost paracontact metric structure (φ, ξ, η, g) on M as follows:

$$\varphi\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial y} - \omega_2 \frac{\partial}{\partial z}, \quad \varphi\left(\frac{\partial}{\partial y}\right) = \frac{\partial}{\partial x} - \omega_1 \frac{\partial}{\partial z}, \quad \varphi\left(\frac{\partial}{\partial z}\right) = 0, \xi = \frac{\partial}{\partial z},$$

$$\eta = dz + \omega_1 dx + \omega_2 dy, \tag{5.2}$$

$$g = [g_{ij}] = \begin{bmatrix} \omega_1^2 - \sigma^2 & \omega_1\omega_2 & \omega_1 \\ \omega_1\omega_2 & \omega_2^2 + \sigma^2 & \omega_2 \\ \omega_1 & \omega_2 & 1 \end{bmatrix}. \tag{5.3}$$

It follows that:

$$\alpha = \frac{1}{2\sigma^2}\frac{\partial \sigma^2}{\partial z} = \frac{\sigma_z}{\sigma}, \quad \beta = \frac{1}{2\sigma^2}\left(-\frac{\partial \omega_1}{\partial y} + \frac{\partial \omega_2}{\partial x}\right) \tag{5.4}$$

which satisfies the condition (4.2). It follows that $\beta = \beta(x, y)$ if and only if $\sigma = \sigma(x, y)$ which means the quasi-para-Sasakian case $\alpha = 0$. Also, $\beta = 0$ if and only if η is closed and in this case ξ is the gradient of the function $f(x, y, z) = z$. In the particular case when $\alpha = \beta = 0$, we have that g is a flat Einstein metric.

A basis in $\mathcal{D}$ is $\{U, \varphi U\}$ with:

$$U = \frac{\partial}{\partial x} - \omega_1 \xi, \quad \varphi U = \frac{\partial}{\partial y} - \omega_2 \xi \tag{5.5}$$

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and for $X \in \Gamma(TM)$ with the decomposition $X = X^1 U + X^2 \varphi U + X^3 \xi$ the paracontact-holomorphicity means: $X^1 = X^1(x, y)$, $X^2 = X^2(x, y)$ and the para-Cauchy–Riemann equations:

$$\frac{\partial X^1}{\partial x} = \frac{\partial X^2}{\partial y}, \quad \frac{\partial X^1}{\partial y} = \frac{\partial X^2}{\partial x} \tag{5.6}$$

which justifies the name. A fundamental example is:

$$X^1 = \frac{1}{2}\left(x^2 + y^2\right), \quad X^2 = xy \tag{5.7}$$

while $\mathcal{L}_X \xi = \frac{\partial X^3}{\partial z} \xi$ and the constancy of $\eta(\mathcal{L}_X \xi) = c$ means that $X^3 = cz$. Hence $V_c \in \mathfrak{phol}(M)$ has the ordinary expression:

$$V_c = \left(\frac{1}{2}(x^2 + y^2), xy, cz - \frac{\omega_1}{2}(x^2 + y^2) - \omega_2 xy\right) \tag{5.8}$$

an its square norm is independent of ωs:

$$g(V_c, V_c) = c^2 z^2 - \frac{\sigma^2}{4}(x^2 - y^2)^2. \tag{5.9}$$

The first Example from [18, p. 384] is recovered with $N = \mathbb{R}^2$, $(a, b) = (0, +\infty)$ and:

$$\omega_1 = 0, \quad \omega_2 = 2x, \sigma = \sqrt{2z} \tag{5.10}$$

which yields:

$$\alpha = \beta = \frac{1}{2z}. \tag{5.11}$$

This $(M, \varphi, \xi, \eta, g)$ is not quasi-para-Sasakian but does not satisfies the hypothesis $\alpha^2 \neq \beta^2$. The Example from [18, p. 385] is obtained for the same ωs but with $\sigma = |x|$. Therefore:

$$\alpha = 0, \quad \beta = \frac{1}{x^2} \tag{5.12}$$

and then $(M, \varphi, \xi, \eta, g)$ is quasi-para-Sasakian and not para-Sasakian. Since β is not constant we have that ξ is not a Ricci soliton in these geometries.

Fix now $\omega_1 = \omega_2 = 0$ and derive all the three cases of para-Kenmotsu geometry studied above; remark that the metric is $g = \sigma^2(-dx^2 + dy^2) + dz^2$, the Ricci soliton is a gradient one (with the potential function $f(x, y, z) = z$) and for $\alpha = \alpha(z)$ the manifold is a Lorentzian warped product $I \times_\sigma \mathbb{R}^2_1$. Firstly, let us note that the last example of [18, p. 386] is of this form with $\sigma = \sqrt{2z}$ on $I = (0, +\infty)$. Secondly, for $\sigma(z) = \cos(-cz)$ on $I = (-\frac{\pi}{c}, \frac{\pi}{c})$ we get:

$$\alpha(z) = c \tan(-cz)$$

which is the shrinking case of previous section. For $\sigma(z) = \cosh(cz)$ on $I = \mathbb{R}$ we get:

$$\alpha(z) = c \tanh(cz)$$

which is the expanding case of the previous section. Finally, the function $\sigma(z) = z + C$ on $I = (-C, +\infty)$ yields the steady case.

For $\omega_1 = 0$, $\omega_2 = x$ and $\sigma = 1$ we obtain the metric g_1 of [14, p. 315] which is a left-invariant Lorentzian metric on the Heisenberg group H_3. The Theorem 3 of the cited paper

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(at page 316) gives that g_1 is a shrinking non-gradient Ricci soliton which is not Einstein. With our formalism we have a non-gradient expanding Ricci soliton since $\eta = xdy$ is not closed and:

$$\alpha = 0, \quad \beta = \frac{1}{2}, \quad \lambda = \frac{1}{2}. \tag{5.13}$$

The difference between the characters (shrinking versus expanding) is explained in the same paper in pages 317–318.

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