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ADAPTED METRICS AND WEBSTER CURVATURE ON THREE CLASSES OF 3-DIMENSIONAL GEOMETRIES

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Abstract. The Chern-Hamilton notion of adapted metric and the corresponding Webster curvature W is discussed for 3-dimensional unimodular Lie groups, Bianchi-Cartan-Vranceanu metrics and warped metrics. For the first two metrics, we control the value of W by means of two parameters: A and B provided by the Milnor frame in the former case respectively l and m in the latter case. For warped metrics, the value of W depends on the derivatives of the warping function up to order two. Bi-warped 3-metrics are introduced and studied from the point of view of Webster curvature.

1. Introduction

Among the Riemannian manifolds of non-constant sectional curvature a special rôle is played by the homogeneous spaces with a large isometry group. Due to the recent approach of Hamilton-Perelman to the Poincaré conjecture (by means of Ricci flow, [8]), a great interest is in dimension 3; for a recent survey on this dimension see [12]. The present note aims to discuss two topics in 3-dimensional Riemannian geometry, both considered by Chern and Hamilton in [5]: adapted metric to a given differential 1-form and the Webster curvature W. In fact, we generalize the concept of adapted metric by modifying the original condition of Chern-Hamilton from the scalar 2 to a general $\rho \in \mathbb{R}$ in order to cover all possibilities. The above notions are considered in three settings: unimodular Lie groups, Bianchi-Cartan-Vranceanu geometries and warped metrics.

The Webster scalar curvature W was introduced already into the framework of 3-dimensional geometry of Lie groups by Domenico Perrone [16] in order to classify homogeneous contact metrics on 3-manifolds. More precisely, in the cited paper was proven that a simply connected Riemannian homogeneous contact 3-manifold is a Lie group with a left-invariant contact Riemannian structure and the classification is given in terms of the (non-)unimodularity of the group, its Webster curvature and the torsion invariant τ introduced also by Chern and Hamilton.

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We propose here a new approach for computing W in geometry of a unimodular Lie group (G, g). In fact, we use another formula of [5] which is based on structural equations (see Section 1) through the dual of the Milnor frame ([13]) of G. The advantage of our result from Section 3 is that we can control the value of W by means of two parameters, denoted A and B in Section 2, provided by the orthonormality of metric g with respect to Milnor frame. As example of this possibility of control, since A and B are strictly positive, a look at our formula (3.1) yields that the vanishing of W implies that λ and μ, the scalars given by Lie brackets of Milnor frame, are of opposite signs.

Regarding the second setting it is well-known that the maximum dimension of the isotropy group of a 3-dimensional manifold is 6 and the fact that there is no metric with 5-dimensional group. The Bianchi-Cartan-Vranceanu spaces are certain 3-dimensional homogeneous Riemannian manifolds with 4-dimensional isometry group. They form a two parameters family containing, among others, some remarkable three-manifolds: $\mathbb{R}^3$, $\mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$ and the 3-dimensional Heisenberg group Nil_3. Recently, several studies are devoted to special submanifolds in these spaces: parallel surfaces [1], biharmonic curves [3] and [6], constant angle surfaces [10], graphs of constant mean curvature [11], biharmonic surfaces [15], higher order parallel and totally umbilical surfaces [17].

Let us pointed out that two motivations for a common presentation of the unimodular Lie groups and Bianchi-Cartan-Vranceanu spaces are provided by:
-there exist remarkable examples belonging to both classes: some of them are listed above,
-in both situations W is controlled by two parameters: in the latter case these are denoted l and m and they have arbitrary signs.
We note that on this way we recover some well-known results but there are also some new computations.

2. Webster scalar curvature: the Chern-Hamilton formalism

Fix (M^3, g) a 3-dimensional Riemannian manifold and consider $\{\omega_1, \omega_2, \omega_3\}$ an orthonormal basis of 1-forms on M; let $\{e_1, e_2, e_3\}$ be the corresponding orthonormal basis of vector fields. There exists a unique skew-symmetric matrix of 1-forms:

$$\begin{pmatrix} 0 & \varphi_3 & -\varphi_2 \\ -\varphi_3 & 0 & \varphi_1 \\ \varphi_2 & -\varphi_1 & 0 \end{pmatrix}$$

such that the structural equations:

$$\begin{cases} d\omega_1 = \varphi_2 \wedge \omega_3 - \varphi_3 \wedge \omega_2 \\ d\omega_2 = \varphi_3 \wedge \omega_1 - \varphi_1 \wedge \omega_3 \\ d\omega_3 = \varphi_1 \wedge \omega_2 - \varphi_2 \wedge \omega_1 \end{cases} \tag{2.1}$$

hold on M. Making one step further we derive the existence of the functions $\{K_{ij}; 1 \leq i, j \leq 3\}$ such that $K_{ij} = K_{ji}$ and:

$$\begin{cases} d\varphi_1 = \varphi_2 \wedge \varphi_3 + K_{11}\omega_2 \wedge \omega_3 + K_{12}\omega_3 \wedge \omega_1 + K_{13}\omega_1 \wedge \omega_2 \\ d\varphi_2 = \varphi_3 \wedge \varphi_1 + K_{21}\omega_2 \wedge \omega_3 + K_{22}\omega_3 \wedge \omega_1 + K_{23}\omega_1 \wedge \omega_2 \\ d\varphi_3 = \varphi_1 \wedge \varphi_2 + K_{31}\omega_2 \wedge \omega_3 + K_{32}\omega_3 \wedge \omega_1 + K_{33}\omega_1 \wedge \omega_2. \end{cases} \tag{2.2}$$

Recall that the subject of [5] consists in *adapted metrics* for a contact 1-form ω i.e. Riemannian metrics satisfying:

$$\|\omega\| = 1, \quad d\omega = 2 * \omega. \tag{2.3}$$

If g is adapted to ω_3 then *the Webster scalar curvature* W of the triple (M, g, ω_3) is defined as:

$$W(M, g, \omega_3) = \frac{1}{8}\left(K_{11} + K_{22} + 2K_{33} + 4\right) \tag{2.4}$$

and in the cited paper of Chern and Hamilton is computed for three examples: the unit sphere $\mathbb{S}^3$, the unit tangent bundle of a compact orientable surface of genus $g \neq 1$ (for $g = 0$ it results $W = 1$) and the Heisenberg group Nil_3. In fact: $W(\mathbb{S}^3) = 1$ and $W(Nil_3) = 0$.

A second formula on Webster curvature is in [2, p. 212] and our relation (4.7) below. Another interpretation of Webster curvature is as the scalar curvature of the Tanaka-Webster connection which is recently studied in arbitrary dimension in [7].

3. 3-dimensional unimodular Lie groups and adapted metrics

Let G be a 3-dimensional Lie group and $\pi : TG \to G$ its tangent bundle. Suppose that G is *unimodular* i.e. its volume form is bi-invariant, with a left-invariant metric g. Then on TG there exists a left-invariant frame field $\{f_1, f_2, f_3\}$ with dual co-frame $\{\eta^1, \eta^2, \eta^3\}$ such that there exist positive constants A, B, C making g a diagonal metric, [8, p. 170]:

$$g = A\eta^1 \otimes \eta^1 + B\eta^2 \otimes \eta^2 + C\eta^3 \otimes \eta^3 \tag{3.1}$$

and the Lie brackets are:

$$[f_i, f_j] = c^k_{ij} f_k. \tag{3.2}$$

where $c^k_{ij} \in \{-2, 0, 2\}$ and $c^k_{ij} = 0$ unless i, j, k are distinct. This special frame is usually called *Milnor frame*. In fact, we work with the associate orthonormal frame:

$$e_1 = \frac{f_1}{\sqrt{A}}, \quad e_2 = \frac{f_2}{\sqrt{B}}, \quad e_3 = \frac{f_3}{\sqrt{C}} \tag{3.3}$$

and denoting $\lambda = c^1_{23}$, $\mu = c^2_{31}$, $\nu = c^3_{12}$ we have [8, p. 170]:

$$[e_i, e_j] = \frac{\lambda_k c^k_{ij}}{\sqrt{\lambda_1\lambda_2\lambda_3}} e_k \tag{3.4}$$

where $\lambda_1 = A$, $\lambda_2 = B$ and $\lambda_3 = C$. Let as in previous Section $\{\omega^1, \omega^2, \omega^3\}$ be the dual co-frame of $\{e_i\}$. More precisely, we have:

$$[e_1, e_2] = \frac{C\nu}{\sqrt{ABC}} e_3, \quad [e_2, e_3] = \frac{A\lambda}{\sqrt{ABC}} e_1, \quad [e_3, e_1] = \frac{B\mu}{\sqrt{ABC}} e_2 \tag{3.5}$$

which gives the structural equations:

$$\begin{cases} d\omega_1 = -\frac{A\lambda}{\sqrt{ABC}}\omega_2 \wedge \omega_3 \\ d\omega_2 = -\frac{B\mu}{\sqrt{ABC}}\omega_3 \wedge \omega_1 \\ d\omega_3 = -\frac{C\nu}{\sqrt{ABC}}\omega_1 \wedge \omega_2 \end{cases} \tag{3.6}$$

and then g is adapted to ω_3 if and only if:

$$\sqrt{C}\nu = -2\sqrt{AB}. \tag{3.7}$$

In order to enlarge the class of suitable metrics we consider the following notion which appears (with a factor 2 in RHS) in [14]:

Definition 3.1. Fix a 1-form ω on a general (M^3, g) and the real number ρ. The Riemannian metric g on M is called *ρ-adapted to ω* if $d\omega = \rho * \omega$.

We conclude from (3.6) that:

Proposition 3.1. *The metric g is: i) $\frac{-A\lambda}{\sqrt{ABC}}$-adapted to the ω_1, ii) $\frac{-B\mu}{\sqrt{ABC}}$-adapted to ω_2, iii) $\frac{-C\nu}{\sqrt{ABC}}$-adapted to ω_3 in the general case.*

Therefore, if the triples (A, B, C), (λ, μ, ν) are inverse proportional i.e. $A\lambda = B\mu = C\nu = \alpha$ then g is $\frac{-\alpha}{\sqrt{ABC}}$-adapted to all ω's. As example we have the 3-sphere S^3 where $A = B = C = 1$ and $\lambda = \mu = \nu = -2$.

4. Webster curvature

We are ready for the first main result of this note:

Proposition 4.1. *If the Riemannian metric g is adapted to ω_3 then $\nu = -2$ and ω_3 is a contact form with e_3 its Reeb vector field. The Webster curvature is:*

$$W(G, g, \omega_3) = -\frac{1}{4}\left(\frac{\lambda}{B} + \frac{\mu}{A}\right). \tag{4.1}$$

Proof. From (3.7) it results that $\nu < 0$ with the only possible variant $\nu = -2 \neq 0$ and we apply the discussion of [2, p. 223] to conclude the first part. Also, it result:

$$AB = C. \tag{4.2}$$

Regarding the second part we search for scalars U, V, W such that:

$$\varphi_1 = U\omega_1, \quad \varphi_2 = V\omega_2, \quad \varphi_3 = W\omega_3 \tag{4.3}$$

and it results the system:

$$\begin{cases} V + W = -\frac{A\lambda}{C} \\ W + U = -\frac{B\mu}{C} \\ U + V = 2. \end{cases} \tag{4.4}$$

The solution is:

$$\begin{cases} U = 1 + \frac{A\lambda - B\mu}{2C} \\ V = 1 - \frac{A\lambda - B\mu}{2C} \\ W = -1 - \frac{A\lambda + B\mu}{2C}. \end{cases} \tag{4.5}$$

We have also:

$$\begin{cases} K_{11} = -\frac{UA\lambda}{C} - VW \\ K_{22} = -\frac{VB\mu}{C} - WU \\ K_{33} = 2W - UV \end{cases} \tag{4.6}$$

which gives:

$$8W(G, g, \omega_3) = -\frac{UA\lambda + VB\mu}{C} - 2W + 4W - 2UV + 4 = -\frac{2}{C}(A\lambda + B\mu)$$

which gives the final conclusion. $\square$

Example 4.1. I) $Nil_3 : \lambda = \mu = 0$, $\nu = -2$. We reobtain $W(Nil_3) = 0$.

II) $SU(2) : \lambda = \mu = \nu = -2$. Then: $W(SU(2), A, B) = \frac{1}{2}\left(\frac{1}{A} + \frac{1}{B}\right)$ and since for the usual metric $A = B = 1$ we recast $W(S^3) = 1$.

III) $SL(2,\mathbb{R}) : \lambda = -2, \mu = +2, \nu = -2$. It results: $W(SL(2,\mathbb{R}), A, B) = \frac{1}{2}\left(\frac{1}{B} - \frac{1}{A}\right)$. Therefore $W(SL(2,\mathbb{R}), 1, 1) = 0$. The importance of these metrics is connected with Corollary 3.3 of [16, p. 247] that: "the Heisenberg group and the Lie group $\widetilde{SL}(2,\mathbb{R})$ are the only simply connected 3-manifolds which admit an unimodular homogeneous contact Riemannian structure with Webster scalar curvature $W = 0$."

IV) $E(2) = Isom(\mathbb{E}^2) : \lambda = -2, \mu = 0, \nu = -2$. Then: $W(E(2), A, B) = \frac{1}{2B} > 0$. For $B = 1$ we reobtain the result of [16, p. 252] that $W(\widetilde{E}(2)) = \frac{1}{2}$.

V) $E(1,1) = Sol : \lambda = +2, \mu = 0, \nu = -2$. It follows: $W(Sol, A, B) = -\frac{1}{2B} < 0$. □

A second and third formula for the Webster scalar formula holds if the pair (g, e_3) belongs to an almost contact structure [2, p. 213], [16, p. 245], [9, p. 222]:

$$W(M, g, \omega_3) = \frac{1}{8}(r - Ric(e_3) + 4) = \frac{1}{8}\left(r + 2 + \frac{\|\tau_3\|^2}{4}\right) \tag{4.7}$$

where r is the scalar curvature of the metric g, $Ric(e_3)$ is the Ricci curvature in the direction of e_3 and $\tau_3 = \mathcal{L}_{e_3} g$. Note also, that conform [2, p. 214], we have:

$$r = 2K(\mathcal{D}) + 2Ric(e_3) \tag{4.8}$$

where $K(\mathcal{D})$ is the sectional curvature of the 2-plane $\mathcal{D} = \ker \omega_3$.

For our manifold it results, in the general almost contact case:

$$\begin{cases} r = -2 + \frac{\lambda\mu}{C} + \frac{\mu\nu}{A} + \frac{\nu\lambda}{B} - \frac{1}{2ABC}\left[A^2\lambda^2 + B^2\mu^2 + C^2\nu^2\right] \\ Ric(e_3) = \frac{C^2\nu^2 - (A\lambda - B\mu)^2}{2ABC} \\ K(\mathcal{D}) = \frac{(A\lambda - B\mu)^2 + C\nu(2A\lambda + 2B\mu - 3C\nu)^2}{4ABC} \\ \|\tau_3\| = 2\left[|2 - \frac{C^2\nu^2 - (A\lambda - B\mu)^2}{2ABC}|\right]^{\frac{1}{2}}. \end{cases} \tag{4.9}$$

In the hypothesis of Proposition 4.1 it results:

$$\begin{cases} r = -2 - 2(\frac{\lambda}{B} + \frac{\mu}{A}) - \frac{1}{2}\left(\frac{\lambda}{B} - \frac{\mu}{A}\right)^2 \\ Ric(e_3) = 2 - \frac{1}{2}(\frac{\lambda}{B} - \frac{\mu}{A})^2 \\ K(\mathcal{D}) = -3 - (\frac{\lambda}{B} + \frac{\mu}{A}) + \frac{1}{4}(\frac{\lambda}{B} - \frac{\mu}{A})^2 \\ \|\tau_3\| = \sqrt{2}|\frac{\lambda}{B} - \frac{\mu}{A}|. \end{cases} \tag{4.10}$$

and, on this way we reobtain (4.1). It results that the adapted metric g is K-contact, in fact Sasakian ([4]), if and only if $A\lambda = B\mu$.

5. Bianchi-Cartan-Vranceanu metrics

Fix l and m two real numbers and denotes by M^3_m the manifold $\{(x, y, z) \in \mathbb{R}^3; F(x, y, z) = 1 + m(x^2 + y^2) > 0\}$. We shall consider on M^3_m *the Bianchi-Cartan-Vranceanu metric*, [17, p. 343]:

$$g_{l,m} = \frac{1}{F^2}dx^2 + \frac{1}{F^2}dy^2 + \left(dz + \frac{ly}{2F}dx - \frac{lx}{2F}dy\right)^2. \tag{5.1}$$

An important feature of these metrics is their S^1-invariance i.e. the invariance with respect to transformations:

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{5.2}$$

Also, we note the invariance:

$$g_{-l,m}(x, y, z) = g_{l,m}(-y, -x, z). \tag{5.3}$$

For other remarks concerning these metrics see [17].

An orthonormal basis in $\Omega^1(M^3_m)$ is:

$$\omega_1 = \frac{dx}{F}, \quad \omega_2 = \frac{dy}{F}, \quad \omega_3 = dz + \frac{ly}{2F}dx - \frac{lx}{2F}dy \tag{5.4}$$

and then:

$$d\omega_3 = -\frac{l}{F^2}dx \wedge dy, \quad *\omega_3 = \omega_1 \wedge \omega_2 = \frac{1}{F^2}dx \wedge dy. \tag{5.5}$$

Since the $g_{l,m}$-dual of ω_3 is the vertical vector field $E_3 = \frac{\partial}{\partial z}$ we may call *vertical adapted* a metric adapted to ω_3. Therefore the only vertical adapted Bianchi-Cartan-Vranceanu metrics are given by $l = -2$; in particular those of the sphere S^3 and the Heisenberg group Nil_3. With respect to the general definition 3.1 the Bianchi-Cartan-Vranceanu metrics are vertical $(-l)$-adapted.

A straightforward computation gives:

$$\varphi_1 = -\frac{l}{2F}dx, \quad \varphi_2 = -\frac{l}{2F}dy, \quad \varphi_3 = \frac{l}{2}dz + \frac{(l^2 - 8m)y}{4F}dx - \frac{(l^2 - 8m)x}{4F}dy. \tag{5.6}$$

Also, we derive the matrix of K's:

$$diag\left(\frac{l^2}{4}, \frac{l^2}{4}, \frac{16m - 3l^2}{4}\right) \tag{5.7}$$

which yields the second main result:

Proposition 5.1. *The Webster scalar curvature of the triple* $(M^3_m, g_{l,m}, \omega_3)$ *is:*

$$W = m + \frac{4 - l^2}{8}. \tag{5.8}$$

Remark 5.1. Recall the formula (4.7). The Ricci tensor field of $(0, 2)$-type for Bianchi-Cartan-Vranceanu metrics is computed in [3, p. 124]:

$$Ric_{11} = Ric_{22} = 4m - \frac{l^2}{2}, \quad Ric_{33} = Ric(\xi) = \frac{l^2}{2}. \tag{5.9}$$

Then an almost contact Bianchi-Cartan-Vranceanu metric has:

$$r = Ric_{11} + Ric_{22} + Ric_{33} = 8m - \frac{l^2}{2}, \quad K(\mathcal{D}) = 4m - \frac{3l^2}{4}, \quad \|\tau_3\| = 2\sqrt{2 - \frac{l^2}{2}} \tag{5.10}$$

which yields again (5.8) and implies that l must be considered only in the interval $[-2, 2]$. Then the triple $(M^3_m, g_{l,m}, \omega_3)$ is a Sasakian manifold (i.e. K-contact manifold since in dimension 3 these notions coincides) if and only if $l \in \{-2, 2\}$.

Example 5.1. 1) $m = 0, l = -2$ is Nil_3 and then we recast: $W(Nil_3) = 0$.

2) if $4m = l^2$ (e.q. $m = 1, l = -2$) then $(M_3, g_{l,m})$ is $S^3(m) \setminus \{\infty\}$ and then: $W(\mathbb{S}^3(m)) = \frac{m+1}{2}$. We recover: $W(S^3) = 1$.

3) $m = 0 = l$ is the Euclidean $\mathbb{R}^3$, thus: $W(\mathbb{E}^3) = \frac{1}{2}$.

4) if $m > 0$ and $l = 0$ then we have $M^3_m = (S^2(4m) \setminus \{\infty\}) \times \mathbb{R}$ and thus: $W(S^2(m) \times \mathbb{R}) = \frac{m}{4} + \frac{1}{2}$.

5) if $m < 0$ and $l = 0$ then we have $M^3_m = H^2(4m) \times \mathbb{R}$ where $H^2(k)$ is the hyperbolic plane of constant Gaussian curvature $k < 0$. Then: $W(H^2(m) \times \mathbb{R}) = \frac{m}{4} + \frac{1}{2}$.

6) if $m > 0$ and $l \neq 0$ we get $SU(2) \setminus \{\infty\}$.

7) if $m < 0$ and $l \neq 0$ we have $\widetilde{SL}(2, \mathbb{R})$. In conclusion, for $l \in (0, 2)$ we get: $W\big(\widetilde{SL}(2, \mathbb{R}), g_{l, \frac{l^2-4}{8}}\big) = 0$. $\square$

Obviously, an important problem is to obtain metrics with prescribed Webster scalar curvature. We conclude with:

Proposition 5.2. *Fix $m \in \mathbb{R}$ and let $c \in (-\infty, m + \frac{1}{2})$. Then the Bianchi-Cartan-Vranceanu metrics $g_{\pm l,m}$ with:*

$$l = \sqrt{4 + 8(m - c)} \tag{5.11}$$

have the Webster scalar curvature equal to c. In particular, we can obtain Bianchi-Cartan-Vranceanu metrics with vanishing Webster scalar curvature only for $m \geq -\frac{1}{2}$.

6. Warped metrics

Let B and N be two smooth manifolds endowed with the Riemannian metrics g_B and g_N of dimension b and n respectively. Let $f : B \to \mathbb{R}^*_+$ be a smooth and strictly positive function. The *warped product* of B and F with *warping function* f is the Riemannian manifold:

$$B \times_f N = (M_{b+n}, g) = (B \times N, g_B + f^2 g_N) \tag{6.1}$$

where in the right-hand-side of above equation the function f is in fact $f \circ \pi$ with $\pi : B \times N \to B$ the projection on the first factor.

In the following we restrict to the case: $B = I$ is an open real interval with the Euclidean metric $g_B(z) = dz^2$ and $N = \mathbb{E}^2$ the Euclidean plane. We use the classical coordinates (x, y) on $\mathbb{E}^2$ and z on I; therefore the main vector field considered below on $(M_3, g) = I \times_f \mathbb{E}^2$, namely $\frac{\partial}{\partial z} = \partial_z$ will be called *the vertical vector field*. The warping function is then $f = f(z)$ and for further use we consider the function $F : B \to \mathbb{R}$:

$$F = \ln f. \tag{6.2}$$

Since the warped metric g is:

$$g = f^2(z) \left(dx^2 + dy^2\right) + dz^2 \tag{6.3}$$

we have the orthonormal basis:

$$\omega_1 = f(z)dx, \quad \omega_2 = f(z)dy, \quad \omega_3 = dz \tag{6.4}$$

with the derivatives:

$$d\omega_1 = -f'(z)dx \wedge dz, \quad d\omega_2 = -f'(z)dy \wedge dz, \quad d\omega_3 = 0 \tag{6.5}$$

and hence:

$$\varphi_1 = f'(z)dy, \quad \varphi_2 = -f'(z)dx, \quad \varphi_3 = 0. \tag{6.6}$$

Also, we derive the matrix of K's:

$$diag\left(-\frac{f''}{f}, -\frac{f''}{f}, -\left(\frac{f'}{f}\right)^2\right) \tag{6.7}$$

which yields the another main result:

Proposition 6.1. *The warped metric* (6.3) *is not vertically adapted but formally the Webster scalar curvature of the triple* $(I \times_f \mathbb{E}^2, g, \omega_3)$ *is:*

$$W(\mathbb{E}^2, f) = \frac{1}{4}\left(2 - F''(z) - 2\left(F'(z)\right)^2\right). \tag{6.8}$$

Proof. Since $d\omega_3 = 0 \neq 2 * \omega_3 = 2\omega_1 \wedge \omega_2 = 2f^2 dx \wedge dy$ we have the first part of conclusion; in fact the metric is 0-adapted to ω_3. The matrix (6.7) implies:

$$W(\mathbb{E}^2, f) = \frac{1}{4}\left[2 - \frac{f''}{f} - \left(\frac{f'}{f}\right)^2\right]. \tag{6.9}$$

and a straightforward computation of F' and F'' gives the claimed formula (6.8). □

As in the previous section we obtain for the almost contact case:

$$r = -4\frac{f''}{f} - 2\left(\frac{f'}{f}\right)^2, \quad K(\mathcal{D}) = -\left(\frac{f'}{f}\right)^2, \quad \|\tau_3\| = 2\sqrt{2\left(1 + \frac{f''}{f}\right)} \tag{6.10}$$

which yields the necessary condition regarding the warping function:

$$f''(z) + f(z) \geq 0. \tag{6.11}$$

Example 6.1. For $f(z) = 1$ we get the Euclidean $\mathbb{R}^3$ and (6.8) yields again: $W(\mathbb{E}^3) = \frac{1}{2}$. The relations (6.10) give:

$$r = K(\mathcal{D}) = 0, \quad \|\tau_3\| = 2\sqrt{2} \tag{6.12}$$

similar to (5.10) for $m = l = 0$ and then the Euclidean 3-geometry with $e_3 = \frac{\partial}{\partial z}$ is not an almost contact geometry.

Example 6.2. Let $A, B \in \mathbb{R}$ such that the function:

$$f(z) = A\cos z + B\sin z \tag{6.13}$$

is strictly positive on I. We have the equality case of (6.11) and then $e_3 = \frac{\partial}{\partial z}$ is a Killing vector field for the warped metric g. The functions r and $K(\mathcal{D})$ are non-constant and $r < -4$, $K(\mathcal{D}) < 0$.

Example 6.3. In order to find a constant Webster curvature we have to solve the differential equation:

$$F'' + 2\left(F'\right)^2 = C = constant \tag{6.14}$$

For $C = 0$ we have the 1-parameter family of solutions: $f_c(z) = \sqrt{2z + c}$ for c a real constant. Hence, with $c = 0$ and condition (6.11) we obtain $I = (\frac{1}{2}, +\infty)$ and:

$$W(\mathbb{E}^2, f(z) = \sqrt{z}) = \frac{1}{2}, \quad g = z\left(dx + dy\right)^2 + dz^2, \quad r(z) = \frac{1}{2z^2} = -2K(\mathcal{D})(z). \tag{6.15}$$

For $C=\lambda>0$ we have the 1-parameter family of solutions:

$$f_c(z)=\sqrt{\cosh\left(\sqrt{2\lambda}z\right)+c}. \tag{6.16}$$

For $c=0$ the condition (6.11) means $\tanh^2(\sqrt{2\lambda}z)\leq 2(1+\frac{1}{\lambda})$ and since the range of tanh is $(-1,1)$ it follows that $I=\mathbb{R}$ and:

$$\begin{cases} W(\mathbb{E}^2, f(z)=\sqrt{\cosh\left(\sqrt{2\lambda}z\right)})=\frac{2-\lambda}{4}<\frac{1}{2}, g=\cosh\left(\sqrt{2\lambda}z\right)\left(dx^2+dy^2\right)+dz^2 \\ r(z)=-4\lambda+\lambda\tanh^2(\sqrt{2\lambda}z)<0, \quad K(\mathcal{D})=-\frac{\lambda}{2}\tanh^2(\sqrt{2\lambda}z)\end{cases}. \tag{6.17}$$

For $C=-\lambda<0$ we get again a 1-parameter family of solutions:

$$f_c(z)=\sqrt{\cos\left(\sqrt{2\lambda}z\right)+c}. \tag{6.18}$$

For $c=0$ the condition (6.11) means $z\geq\frac{1}{\sqrt{2\lambda}}\arctan\left(2(1-\frac{1}{\lambda})\right)$ and then $\lambda\in(0,1)$. With $I=\left(0,\frac{1}{\sqrt{2\lambda}}\arctan\left(2(1-\frac{1}{\lambda})\right)<\frac{\pi}{2\sqrt{2\lambda}}\right)$ we have:

$$\begin{cases} W(\mathbb{E}^2, f(z)=\sqrt{\cos\left(\sqrt{2\lambda}z\right)})=\frac{2+\lambda}{4}\in(\frac{1}{2},\frac{3}{4}), g=\cos\left(\sqrt{2\lambda}z\right)\left(dx^2+dy^2\right)+dz^2 \\ r(z)=-4\lambda-3\lambda\tan^2(\sqrt{2\lambda}z)<0, \quad K(\mathcal{D})=-\frac{\lambda}{2}\tan^2(\sqrt{2\lambda}z)\end{cases}. \tag{6.19}$$

Let us remark that for all three metrics above we have $K(\mathcal{D})<0$. □

In order to enlarge the class of metrics we consider a more general 2-manifold instead of $\mathbb{E}^2$:

$$g=f^2(z)\left(dx^2+u^2(x)dy^2\right)+dz^2 \tag{6.20}$$

which can be called a *bi-warped metric*. Then the orthonormal basis is:

$$\omega_1=f(z)dx, \quad \omega_2=f(z)u(x)dy, \quad \omega_3=dz \tag{6.21}$$

and hence:

$$\varphi_1=f'(z)u(x)dy, \quad \varphi_2=-f'(z)dx, \quad \varphi_3=-u'(x)dy. \tag{6.22}$$

The matrix of K's is:

$$diag\left(-\frac{f''}{f},-\frac{f''}{f},-\frac{(f')^2u+u''}{f^2u}\right) \tag{6.23}$$

which yields:

Proposition 6.2. *The pair (bi-warped metric* (6.20)*, $\omega_3=dz$) has the Webster scalar curvature:*

$$W(u,f)=\frac{1}{4}\left(2-\frac{f''}{f}-\frac{(f')^2u+u''}{f^2u}\right)=W(\mathbb{E}^2,f)-\frac{u''}{4f^2u}. \tag{6.24}$$

Example 6.4. Let us consider the three 2-dimensional geometries of constant curvature:

i) Euclidean: $u(x)=x$. We reobtain (6.8).

ii) Elliptic i.e. $N=S^2$: $u(x)=\sin x$. We have $W(S^2,f)=W(\mathbb{E}^2,f)+\frac{1}{4f^2}$.

iii) Hyperbolic i.e. $N=H^2$: $u(x)=\sinh x$. We derive $W(H^2,f)=W(\mathbb{E}^2,f)-\frac{1}{4f^2}$.

For the almost contact case we get the same $\|\tau_3\|$ but:

$$r = -4\frac{f''}{f} - 2\frac{(f')^2u + u''}{f^2u}, \quad K(\mathcal{D}) = -\frac{(f')^2u + u''}{f^2u} \tag{6.25}$$

and it follows:

$$\begin{cases} r(S^2, f) = r(\mathbb{E}^2, f) + \frac{2}{f^2}, \quad K(\mathcal{D})(S^2, f) = K(\mathcal{D})(\mathbb{E}^2, f) + \frac{1}{f^2} \\ r(H^2, f) = r(\mathbb{E}^2, f) - \frac{2}{f^2}, \quad K(\mathcal{D})(H^2, f) = K(\mathcal{D})(\mathbb{E}^2, f) - \frac{1}{f^2}. \end{cases} \tag{6.26}$$

References

[1] Belkhelfa, M., Dillen, F. and Inoguchi, J.-I., Surfaces with parallel second fundamental form in Bianchi-Cartan-Vranceanu spaces, in "PDEs, submanifolds and affine differential geometry" (Warsaw, 2000), 67-87, Banach Center Publ., 57, Polish Acad. Sci., Warsaw, 2002.

[2] Blair, D. E., Riemannian geometry of contact and symplectic manifolds, Second edition. Progress in Mathematics, 203. Birkhäuser Boston, Inc., Boston, MA, 2010.

[3] Caddeo, R., Montaldo, S., Oniciuc, C. and Piu, P., The classification of biharmonic curves of Cartan-Vranceanu 3-dimensional spaces, in "Modern trends in geometry and topology", 121-131, Cluj Univ. Press, Cluj-Napoca, 2006.

[4] Calvaruso, G., Three-dimensional homogeneous almost contact metric structures, J. Geom. Phys., 69(2013), 60-73.

[5] Chern S. S. and Hamilton, R. S., On Riemannian metrics adapted to three-dimensional contact manifolds, With an appendix by Alan Weinstein. Lecture Notes in Math., 1111, Workshop Bonn 1984, 279-308, Springer, Berlin, 1985.

[6] Cho, J. T., Inoguchi J.-I. and Lee, J.-E., On slant curves in Sasakian 3-manifolds, Bull. Austral. Math. Soc., 74(2006), no. 3, 359-367.

[7] Cho, J. T. and Kon, M., The Tanaka-Webster connection and real hypersurfaces in a complex space form, Kodai Math. J., 34(2011), no. 3, 474-484.

[8] Chow, B., Lu, P. and Ni, L., Hamilton's Ricci flow, Graduate Studies in Mathematics, 77, American Mathematical Society, Providence, RI; Science Press, New York, 2006.

[9] Dragomir, S. and Perrone, D., Harmonic vector fields. Variational principles and differential geometry, Amsterdam, Elsevier, 2012.

[10] Fastenakels, J., Munteanu, M. I. and Van Der Veken, J., Constant angle surfaces in the Heisenberg group, Acta Math. Sin. (Engl. Ser.), 27(2011), no. 4, 747-756.

[11] Lee, H., Extensions of the duality between minimal surfaces and maximal surfaces, Geom. Dedicata, 151(2011), 373-386.

[12] McMullen, C. T., The evolution of geometric structures on 3-manifolds, Bull. Amer. Math. Soc., 48(2011), no. 2, 259-274.

[13] Milnor, J., Curvatures of left invariant metrics on Lie groups, Advances in Math., 21(1976), no. 3, 293-329.

[14] Nicolaescu, L. I., Adiabatic limits of the Seiberg-Witten equations on Seifert manifolds, Comm. Anal. Geom., 6(1998), no. 2, 331-392.

[15] Ou, Y.-L. and Wang, Z.-P., Constant mean curvature and totally umbilical biharmonic surfaces in 3-dimensional geometries, J. Geom. Phys., 61(2011), no. 10, 1845-1853.

[16] Perrone, D., Homogeneous contact Riemannian three-manifolds, Illinois J. Math., 42 (1998), no. 2, 243-256.

[17] Van der Veken, J., Higher order parallel surfaces in Bianchi-Cartan-Vranceanu spaces, Results Math., 51(2008) no. 3-4 339-359.

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