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Para-CR structures of codimension 2 on tangent bundles in Riemann-Finsler geometry

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Abstract We determine a 2-codimensional para-CR structure on the slit tangent bundle T_0M of a Finsler manifold (M, F) by imposing a condition regarding the almost paracomplex structure P associated to F when restricted to the structural distribution of a framed para-f-structure. This condition is satisfied when (M, F) is of scalar flag curvature (particularly constant) or if the Riemannian manifold (M, g) is of constant curvature.

Keywords para-CR structure, metric framed para-f-structure, Finsler geometry, scalar flag curvature, space form

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1 Introduction

The Finsler geometry is very rich in remarkable tensor fields F of $(1, 1)$-type and associated structures. More precisely, there are: an (almost) tangent structure ($F^2 = 0$), an almost complex one ($F^2 = -I$) and also an almost product structure ($F^2 = I$). In fact, the almost product structure is a special one, namely paracomplex, since its $\pm$-eigenspaces have the same rank, namely the dimension of the base manifold. In [9] another well-known type of structures, namely $f(3, -1)$-structure ($F^3 - F = 0$) is obtained in this geometry. In fact, this $f(3, -1)$-structure belongs to a very interesting particular case which is called *framed para-f-structure* and have, in addition to F, a set of vector fields and differential 1-forms interrelated. Moreover, a conformal deformation of the Sasaki type metric can be added in order to obtain a *metric framed para-f-structure*.

The present note is concerning with another kind of structures, namely the *para-CR structures*, with an important rôle at the border between differential geometry and paracomplex analysis, as it is pointed out in [8]. More precisely, based on a relationship between framed para-f-structures and para-CR structure we found a para-CR structure on the slit tangent bundle T_0M of a Finsler manifold (M, F). This para-CR structure is constructed with the above

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almost paracomplex structure denoted P_F in Section 2 and its existence is constrained by one condition expressing the vanishing of the Nijenhuis tensor of P_F on the structural distribution of the framed para-f-structure. The above condition is expressed as a relation between the curvature of the Cartan nonlinear connection and the Jacobi endomorphism and is satisfied in dimension two or if (M, F) is of scalar flag curvature which in the particular case of Riemannian manifold (M, g) means that the metric g has a constant curvature. Several important classes of Finsler manifolds with scalar flag curvature are discussed in Chapter 7 of [7]. It seems that these results are new even for Riemannian geometry.

2 Para-CR-structures from framed para-f-structures

Let N be a smooth $(2n+s)$-dimensional manifold with $n, s \geq 1$ and fix D a distribution on N of dimension $2n$. Considering D as a vector bundle over N let $\Gamma(D)$ be the module of its sections. Supposing D is endowed with a morphism $K : D \to D$ of vector bundles satisfying $K^2 = I$ and $K \neq \pm I$ where I is the identity (Kronecker) morphism on D; then the pair (D, K) is called *weak almost para-CR structure*. The first main notion is given by [1, p. 3], see also [2]:

Definition 2.1 *If for all $X, Y \in \Gamma(D)$ we have:*

$$\begin{cases} [KX, KY] + [X, Y] \in \Gamma(D) \\ N_K(X, Y) := [KX, KY] + [X, Y] - K([X, KY] + [KX, Y]) = 0 \end{cases} \tag{2.1}$$

then (D, K) is a weak para-CR structure on N. If, in addition, the sub-distributions of D corresponding to the eigenvalues ± 1 have the same rank then the pair (D, K) is a para-CR structure.

A second main notion is, [5, p. 196]:

Definition 2.2 *Let φ be a tensor field of $(1,1)$-type and s pairs (ξ_a, η^a), $1 \leq a \leq s$ of vector fields and 1-forms on N. If:*
i) $\varphi^3 - \varphi = 0$, $rank\varphi = 2n$,
ii) $\varphi^2 = I - \sum_{a=1}^{s} \eta^a \otimes \xi_a$, $\varphi(\xi_a) = 0$, $\eta^a(\xi_b) = \delta^a_b$, $\eta^a \circ \varphi = 0$,
then the data (φ, ξ_a, η^a) is called a framed para-f-structure.

To a framed para-f-structure we associate:
1) *the torsion tensor field* S of $(1,2)$-type:

$$S = N_\varphi - 2\sum_{a=1}^{s} d\eta^a \otimes \xi_a. \tag{2.2}$$

2) *the structural distribution* D:

$$D = \{X \in \Gamma(TM); \eta^1(X) = \ldots = \eta^s(X) = 0\} = \cap_{a=1}^{s} \ker \eta^a. \tag{2.3}$$

For a 1-form η we use the differential:

$$2d\eta(X, Y) = X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y]). \tag{2.4}$$

These notions lead to:

Definition 2.3 *The framed para-f-structure is called D-normal if S vanishes on D i.e. $S(X, Y) = 0$ for all $X, Y \in \Gamma(D)$.*

The relationship between the above structures is given by:

Proposition 2.4 *If (φ, ξ_a, η^a) is a D-normal framed para-f-structure then $(D, K = \varphi|_D)$ is a weak para-CR structure.*

Proof The restriction K of φ to D is obviously an weak almost para-CR structure. The conditions (2.1) result from the fact that for $X, Y \in \Gamma(D)$ we have:

$$S(X,Y) = 0 = [KX, KY] + \varphi^2([X,Y]) - \varphi([X,KY] + [KX,Y]) - \sum_{a=1}^{s} \eta^a([X,Y])\xi_a. \quad (2.5)$$

The details are similar to that of Proposition 1.1 of [3, p. 130] where the complex version of this result is considered. □

3 A metric framed para-f-structure on the tangent bundle of a Finsler manifold

Let M be now a smooth m-dimensional manifold with $m \geq 2$ and $\pi : TM \to M$ its tangent bundle. Let $x = (x^i) = (x^1, ..., x^m)$ be the local coordinates on M and $(x, y) = (x^i, y^i) = (x^1, ..., x^m, y^1, ..., y^m)$ the induced local coordinates on TM. Denote by O the null-section of π.

Recall after [7] or [10] that a *Finsler fundamental function* on M is a map $F : TM \to \mathbb{R}_+$ with the following properties:

F1) F is smooth on the slit tangent bundle $T_0M := TM \setminus O$ and continuous on O,

F2) F is positive homogeneous of degree 1: $F(x, \lambda y) = \lambda F(x, y)$ for every $\lambda > 0$,

F3) the matrix $(g_{ij}) = \left(\frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}\right)$ is invertible and its associated quadratic form is positive definite.

The tensor field $g = \{g_{ij}(x, y); 1 \leq i, j \leq\}$ is called *the Finsler metric* and the homogeneity of F implies:

$$F^2(x, y) = g_{ij} y^i y^j = y_i y^i \quad (3.1)$$

where $y_i = g_{ij} y^j$. The pair (M, F) is called *Finsler manifold*.

On T_0M we have two distributions:

i) $V(TM) := \ker \pi_*$, called *the vertical distribution* and not depending of F. It is integrable and has the basis $\{\frac{\partial}{\partial y^i}; 1 \leq i \leq m\}$. A remarkable section of it is *the Liouville vector field* $\Gamma = y^i \frac{\partial}{\partial y^i}$.

ii) $H(TM)$ with the basis $\{\frac{\delta}{\delta x^i} := \frac{\partial}{\partial y^i} - N^j_i \frac{\partial}{\partial y^j}\}$ where:

$$N^i_j = \frac{1}{2}\frac{\partial \gamma^i_{00}}{\partial y^j} \quad (3.2)$$

with $\gamma^i_{00} = \gamma^i_{jk} y^j y^k$ built from the usual Christoffel symbols:

$$\gamma^i_{jk} = \frac{1}{2} g^{ia} \left(\frac{\partial g_{ak}}{\partial x^j} + \frac{\partial g_{ja}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^a}\right). \quad (3.3)$$

$H(TM)$ is often called the *Cartan* (or canonical) *nonlinear connection* of the geometry (M, F) and a remarkable section of it is *the geodesic spray*:

$$S_F = y^i \frac{\delta}{\delta x^i}. \quad (3.4)$$

In particular, if g does not depends on y we recover the Riemannian geometry.

The dual basis of the above local basis $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}$ of $\Gamma(T_0M)$ is $(dx^i, \delta y^i = dy^i + N^i_j dx^j)$. On T_0M we have a Riemannian metric of Sasaki type:

$$G_F = g_{ij}dx^i \otimes dx^j + g_{ij}\delta y^i \otimes \delta y^j. \tag{3.5}$$

Another Finslerian object is the tensor field of $(1,1)$-type $P_F : \Gamma(T_0M) \to \Gamma(T_0M)$:

$$P_F\left(\frac{\delta}{\delta x^i}\right) = \frac{\delta}{\delta x^i}, \quad P_F\left(\frac{\partial}{\partial y^i}\right) = -\frac{\partial}{\partial y^i}. \tag{3.6}$$

Let us remark that P_F is a global geometric object although it is defined using a fixed coordinate chart. It results that P_F is an almost paracomplex structure and the pair (P_F, G_F) is an almost para-Kähler structure on T_0M.

In order to obtain a framed para-f-structure on T_0M associated to the Finslerian function F, the following objects are considered in [9]:

$$\begin{cases} \xi_1 = S_F, \xi_2 = \Gamma \\ \eta^1 = \frac{1}{F^2}y_i dx^i, \eta^2 = \frac{1}{F^2}y_i\delta y^i \\ \varphi = P_F - \eta^1 \otimes \xi_2 + \eta^2 \otimes \xi_1 \\ G = \frac{1}{F^2}G_F. \end{cases} \tag{3.7}$$

Then the main result of [9] is that the data $(\varphi, \xi_1, \xi_2, \eta^1, \eta^2)$ is a framed para-f-structure on T_0M with η^a the G-dual of ξ_a, $1 \leq a \leq 2$ and, moreover:

$$G(\varphi\cdot, \varphi\cdot) = G - \eta^1 \otimes \eta^1 - \eta^2 \otimes \eta^2. \tag{3.8}$$

Also, ξ_a are unitary vector fields with respect to G and $(G, \varphi, \xi_a, \eta^a)$ is a *metric framed para-f-structure.*

4 Putting all together

The last paragraph of the previous Section provides the ingredients of the first Section with $N = T_0M$, $s = 2$ and $n = m-1$ which motivates our choice $m \geq 2$. The structural distribution is then:

$$D_F = \ker\eta^1 \cap \ker\eta^2 = \{\xi_1\}^{\perp G} \cap \{\xi_2\}^{\perp G} = \{\xi_1\}^{\perp G_F} \cap \{\xi_2\}^{\perp G_F} \tag{4.1}$$

where $\{X\}^{\perp G}$ is the G-orthogonal complement of $span\{X\}$. We have $D_F = (span\{\xi_1, \xi_2\})^{\perp G_F}$ and this implies that D_F has the dimension $2m-2$. For a geometrical meaning of the distribution $span\{\xi_1, \xi_2\}$ we define the differential 2-form ω_F, naturally associated to the metric framed para-f-structure:

$$\omega_F = G(\cdot, \varphi\cdot) \tag{4.2}$$

and it follows that $span\{\xi_1, \xi_2\}$ is the kernel of ω_F. Also, the homogeneity of F implies the homogeneity of $S_F = \xi_1$ which means:

$$[\Gamma, S_F] = [\xi_2, \xi_1] = \xi_1 \tag{4.3}$$

and thus $span\{\xi_1, \xi_2\}$ is an integrable distribution; see also Theorem 3.15 of [4, p. 236].

A concrete expression of D_F appears in [6, p. 11]. More precisely, consider after the cited paper:

i) the horizontal vector fields:

$$h_i = \frac{\delta}{\delta x^i} - \frac{1}{F^2} y_i S_F \tag{4.4}$$

and the corresponding $(m-1)$-distribution $\mathcal{H}_{m-1} = span\{h_i; 1 \leq i \leq m\}$,
ii) the vertical vector fields:

$$v_i = \frac{\partial}{\partial y^i} - \frac{1}{F^2} y_i \Gamma \tag{4.5}$$

and also the corresponding $(m-1)$-distribution $\mathcal{V}_{m-1} = span\{v_i; 1 \leq i \leq m\}$; see also [11, p. 473].
We have:

$$D_F = \mathcal{H}_{m-1} \oplus \mathcal{V}_{m-1} \tag{4.6}$$

and the same Theorem 3.15 of [4, p. 236] proves the integrability of $\mathcal{V}_{m-1}$; see also [6, p. 12].

Regarding the integrability of the nonlinear connection $H(TM)$ we have:

$$\left[\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k}\right] = R^i_{jk} \frac{\partial}{\partial y^i} \tag{4.7}$$

where:

$$R^i_{jk} = \frac{\delta N^i_j}{\delta x^k} - \frac{\delta N^i_k}{\delta x^j}. \tag{4.8}$$

The tensor field $R = \{R^i_{jk}(x, y); 1 \leq i, j, k \leq m\}$ is called *the curvature* of the Cartan nonlinear connection and:

$$R^i_j := R^i_{kj} y^k \tag{4.9}$$

are the components of *the Jacobi endomorphism* $\Phi = R^i_j \frac{\partial}{\partial y^i} \otimes dx^j$, [6, p. 5]. We are ready now for the main result:

Theorem 4.1 *If the curvature tensor of (M, F) has the form:*

$$R^i_{jk} = \lambda(X^i_k y_j - X^i_j y_k) \tag{4.10}$$

with λ a smooth function on T_0M and the tensor field $\{X^i_j(x, y); 1 \leq i, j \leq m\}$ satisfying:

$$y_i X^i_j = y_j \tag{4.11}$$

for all $i, j \in \{1, ..., m\}$ then the pair $(D_F, K_F = P_F|_{D_F})$ is a weak para-CR structure on T_0M.

Proof We express the Nijenhuis tensor field of P_F as:

$$N_{P_F}(X, Y) = [P_F X, P_F Y] + [X, Y] - P_F(A(X, Y)) = B(X, Y) - P_F(A(X, Y)) \tag{4.12}$$

with $A(X, Y) := [X, P_F Y] + [P_F X, Y]$ and $B(X, Y) = [P_F X, P_F Y] + [X, Y]$. It follows that $B(X, Y) = A(P_F X, Y)$ and then:

$$N_{P_F}(X, Y) = A(P_F X, Y) - P_F \circ A(X, Y). \tag{4.13}$$

We prove firstly that A is a D_F-valued $(0, 2)$-tensor field. From (4.7) and:

$$\left[\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^k}\right] = \frac{\partial N^i_j}{\partial y^k} \frac{\partial}{\partial y^i} = \frac{\partial^2 \gamma^i_{00}}{\partial y^j \partial y^k} \frac{\partial}{\partial y^i} \tag{4.14}$$

we obtain:

$$A\left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k}\right) = 2R^i_{jk} \frac{\partial}{\partial y^i}, \quad A\left(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k}\right) = A\left(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^k}\right) = 0 \tag{4.15}$$

which means that $\eta^1 \circ A = 0$ and:

$$A = R^i_{jk} dx^j \wedge dx^k \otimes \frac{\partial}{\partial y^i}. \tag{4.16}$$

A main identity in Finsler geometry is:

$$y_i R^i_{ab} = 0 \tag{4.17}$$

and then $\eta^2 \circ A = 0$ which conclude the first part of the proof.

Secondly, we search for the framework of Proposition 2.4. The torsion tensor S on D_F is:

$$S(X,Y) = N_\varphi(X,Y) + \eta^1([X,Y])\xi_1 + \eta^2([X,Y])\xi_2$$

with:

$$N_\varphi(X,Y) = [P_F X, P_F Y] + \varphi^2([X,Y]) - \varphi \circ A(X,Y).$$

Since φ is an element of a framed para-f-structure we get:

$$N_\varphi(X,Y) = [P_F X, P_F Y] + [X,Y] + \eta^1([X,Y])\xi_1 + \eta^2([X,Y])\xi_2 - \varphi \circ A(X,Y)$$

and from the definition (3.7) of φ it follows:

$$S(X,Y) = [P_F X, P_F Y] + [X,Y] - (P_F - \eta^1 \otimes \xi_2 + \eta^2 \otimes \xi_1) \circ A(X,Y) = N_{P_F}(X,Y). \tag{4.18}$$

In local coordinates we have:

$$N_{P_F} = 4R^i_{jk} dx^j \wedge dx^k \otimes \frac{\partial}{\partial y^i} \tag{4.19}$$

and then N_{P_F} has components only when applied on the pair (h_a, h_b). A long but straightforward computation yields:

$$N_{P_F}(h_a, h_b) = 8\left[R^i_{ab} + \frac{1}{F^2}(R^i_a y_b - R^i_b y_a)\right]\frac{\partial}{\partial y^i} \tag{4.20}$$

and therefore the normality condition is:

$$F^2 R^i_{ab} = R^i_b y_a - R^i_a y_b \tag{4.21}$$

which can be expressed as:

$$N_{P_F} = 4\eta^1 \wedge \left(R^i_k dx^k \otimes \frac{\partial}{\partial y^i}\right). \tag{4.22}$$

The relation (4.10) yields:

$$R^i_k = \lambda(F^2 X^i_k - y^a X^i_a y_k) \tag{4.23}$$

and then, both sides of (4.21) are equal with $\lambda F^2(X^i_k y_j - X^i_j y_k)$ which gives the final conclusion. The condition (4.11) corresponds to the relation (4.17).

Let us also point out that the condition (4.10) gives the following expression for the Nijenhuis tensor:

$$N_{P_F} = 8\lambda F^2 \eta^1 \wedge \left(X^i_j dx^j \otimes \frac{\partial}{\partial y^i}\right) \tag{4.24}$$

which yields again the vanishing of N_{P_F} on D_F due to the presence of η^1. Concerning the tensor field A we have:

$$A = 2\lambda F^2 \eta^1 \wedge (X^i_k dx^k \otimes \frac{\partial}{\partial y^i}) \tag{4.25}$$

which proves the relations: $\eta^1 \circ A = \eta^2 \circ A = 0$. □

Example 4.2 In dimension 2 the Nijenhuis tensor field of any almost paracomplex structure vanishes. Then every 2-dimensional Finsler manifold (M_2, F) satisfies the condition of Theorem 4.1. Let $V(TM)$ be spanned by the vector fields Γ and V respectively $H(TM)$ be spanned by the vector fields S_F and H. Then D_F is spanned by V and H and:

$$K_F(H) = H, \quad K_F(V) = -V. \tag{4.26}$$

We have that H is a linear combination of h_1 and h_2 while V is a linear combination of v_1 and v_2. □

In order to consider examples in any dimension we remark that a solution of condition (4.11) is:

$$X^i_j = \mu\delta^i_j + (1-\mu)\frac{y^i y_j}{F^2} \tag{4.27}$$

again with μ a smooth function on T_0M. It follows:

Example 4.3 If $\mu = 1$ then $X^i_j = \delta^i_j$ and the Finsler manifold (M, F) is *of scalar flag curvature* λ ([12]) since:

$$R^i_{jk} = \lambda(\delta^i_k y_j - \delta^i_j y_k) \tag{4.28}$$

and then:

$$R^i_k = \lambda(\delta^i_k F^2 - y^i y_k). \tag{4.29}$$

Corollary 4.4 *If (M, F) is of scalar flag curvature then $(D_F = (span\{S_F, \Gamma\})^{\perp G_F}, K_F)$ is a para-CR structure on T_0M.*

Remark also that the hypothesis of scalar flag curvature yields:

$$N_{P_F} = 8\lambda F^2 \eta^1 \wedge \pi_{V(TM)} \tag{4.30}$$

where $\pi_{V(TM)}$ is the projector on the vertical part in the G_F-orthogonal decomposition $T(T_0M) = H(TM) \oplus V(TM)$ i.e $\pi_{V(TM)} = \delta y^i \otimes \frac{\partial}{\partial y^i}$. However, P_F is integrable only in the flat case (i.e. $\lambda = 0$) since $N_{P_F}(S_F, h_a) = 8\lambda F^2 v_a$. The integrability of P_F as a tensor field of $(1,1)$-type which is equivalent with the integrability of the Cartan nonlinear connection of (M, F) and then (T_0M, P_F, G_F) is a para-Kähler manifold.

Particular case 4.5 (Riemannian geometry) Let $g = (g_{ij}(x))$ be a Riemannian metric on M. Then $\gamma^i_{jk}(x, y) = \Gamma^i_{jk}(x)$ the Riemannian Christoffel symbols and:

$$R^i_{jk}(x, y) = R^i_{jka}(x) y^a \tag{4.31}$$

with $R_g = (R^i_{jka})$ the Riemannian curvature tensor of g. It results that a Riemannian geometry $(M, F = (g_{ij}(x) y^i y^j)^{\frac{1}{2}})$ is of scalar flag curvature if and only if g is of constant curvature. Therefore on the slit tangent bundle of a space form (M, g) there exists a para-CR structure on the distribution complementary (with respect to the Sasaki lift of g) to the distribution generated by the Liouville vector field and the geodesic spray S_g. □

Example 4.5 Returning to the general non-Riemannian case (4.27) with $\mu = 0$ we get:

$$X^i_j = \frac{y^i y_j}{F^2} \tag{4.32}$$

and then $R^i_{jk} = 0$ which means that (M, F) is flat, a situation belonging also to the Example 4.3 for vanishing scalar curvature. □

For the general μ we have:

$$N_{P_F} = 2\lambda F^2 \eta^1 \wedge [\mu \pi_{V(TM)} + (1-\mu)\eta^1 \otimes \Gamma] = 2\lambda\mu F^2 \eta^1 \wedge \mu \pi_{V(TM)}. \qquad (4.33)$$

Remark 4.6 In [1, p. 4] it is pointed out that conditions (2.1) for K are equivalent with the integrability of its ± 1-eigenspaces. Therefore, the integrability of $\mathcal{H}_{m-1}$ is characterized by equation (4.21) and it holds under the condition (4.10), particularly on Finsler manifolds of scalar flag curvature or Riemannian space forms.

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